7. RELATIVISTIC COSMOLOGY

Matthew Baring – Lecture Notes for ASTR 360, Spring 2025

1 Robertson-Walker Metric

• The metric of simplest form consistent with the cosmological principle is the **Robertson-Walker (RW) metric** for proper-times $ds \equiv d\tau$: **C & O, Sec. 29.3**

$$ds^2 \;=\; dt^2 - a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right\} \quad . \tag{1}$$

Here the $d\Omega = d\theta^2 + \sin^2\theta \, d\phi^2$ constitutes the angular part, with (r, θ, ϕ) being spherical polar coordinates from any point in the universe:

$$\begin{aligned} x &= a(t)\, r \, \sin\theta \sin\phi \quad , \\ y &= a(t)\, r \, \sin\theta \cos\phi \quad , \\ z &= a(t)\, r \, \cos\theta \quad . \end{aligned} \tag{2}$$

We call r the *comoving coordinate*. If r is dimensionless, then $a(t)$ represents the physical scale of the universe, and we call it the **scale factor**.

Contrast: the RW metric must match homogeneity, whereas the Schwarzschild solution is intrinsically inhomogeneous due to its central force.

∗ Isotropy ⇒ the angular part is the same in $g_{\rm RW}$ and $g_{\rm SCH}$.

∗ Homogeneity ⇒ (i) the coefficient of dt^2 is independent of time, (ii) the space and time portions of the dr^2 coefficient are separable, and (iii) the constant k is independent of the location in space.

If $k > 0$, the surface resembles a hypersphere, and is of positive curvature with radius $R = 1/\sqrt{k}$. If $k < 0$, the hypersurface is like a saddle and is of negative curvature (i.e. imaginary radius). If $k = 0$, there is no curvature and the hypersurface is a hyperplane.

• An alternative form for the Robertson-Walker is obtained by setting $r = \sin\chi, \chi, \sinh\chi$ according to

$$\frac{d\chi}{dr} = \frac{1}{\sqrt{1-kr^2}} \quad ; \quad \chi = \begin{cases} \arcsin r & , \quad k = 1, \\ r & , \quad k = 0, \\ \sinh^{-1} r & , \quad k = -1, \end{cases} \tag{3}$$

so that the RW metric becomes

$$ds^2 = dt^2 - a^2(t)\left\{d\chi^2 + S_k^2(\chi)\, d\Omega\right\} \quad , \tag{4}$$

where

$$S_k(\chi) = \begin{cases} \sin\chi, & k = 1, \\ \chi, & k = 0, \\ \sinh\chi, & k = -1 \ . \end{cases} \tag{5}$$

Note that other values of k can be scaled to these without loss of generality. Note that we will eventually express $\chi \to \Theta$ as the development angle.

This form leads intuitively to the following attributions of curvature: **C & O, pp. 1184-9**

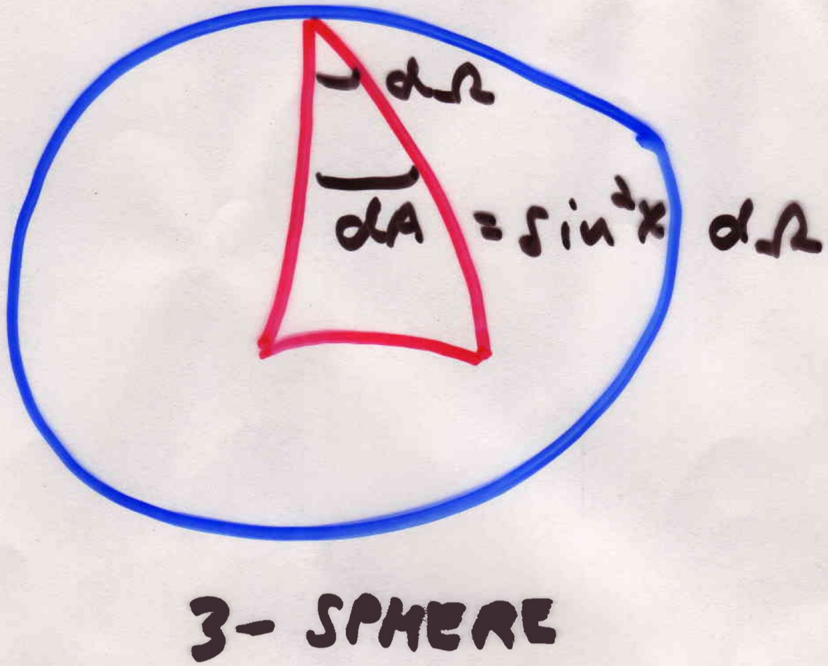
∗ $k = 1 \Rightarrow \sin^2\chi < 1 \Rightarrow$ solid angles are less than $d\Omega$, i.e. a hyperspherical universe of positive curvature;

∗ $k = -1 \Rightarrow \sinh^2\chi > 1 \Rightarrow$ solid angles are greater than $d\Omega$, i.e. a hypersaddle universe of negative curvature.

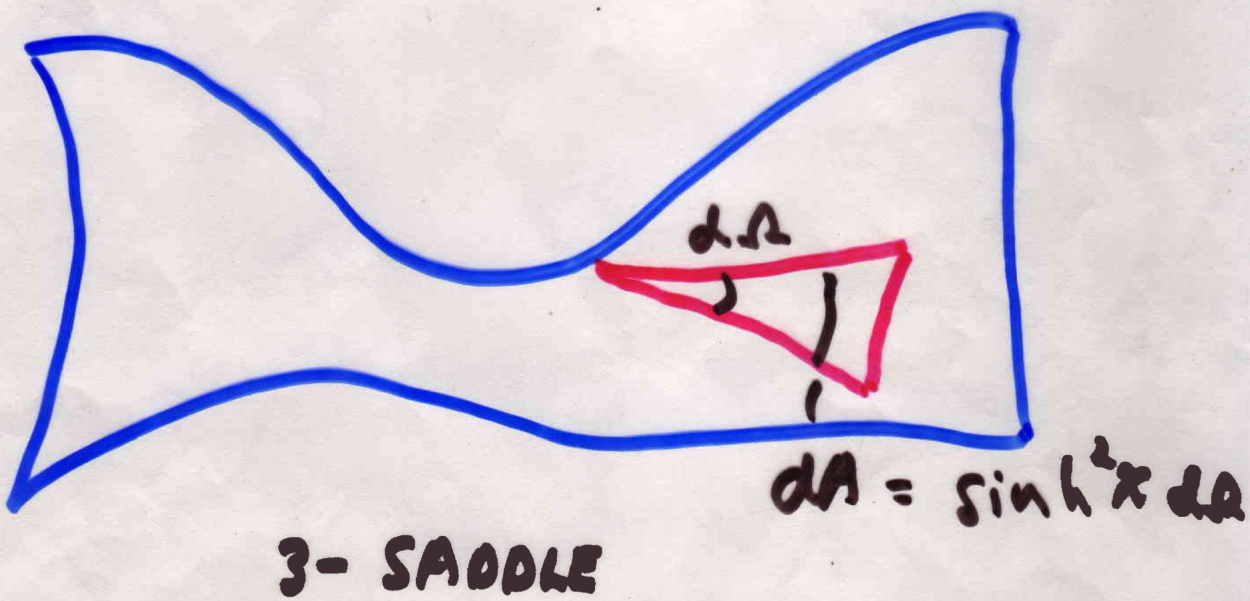
Clearly $k = 0$ is the flat spacetime Minkowski metric.

Plot: Hypersphere and Hypersaddle: Curvature and Solid Angles

• For $k = 1$, as $r \to 1$, we have $ds \to \infty$: the proper time diverges at finite radius, and we have a **closed universe**, just as with Newtonian cosmology. An observer starting off at (θ, ϕ) will return to this point when $\chi = \pi$ in a closed universe. In contrast, $k = -1$ yields no such finiteness, i.e. the universe is **open**.



Hypersphere and Hypersaddle



• Since all global lengthscales connected to the universe's gravitational potential as a whole are controlled by $a(t)$, photon wavelengths must "stretch" with $a(t)$, leading to a redshift as the universe expands. Hence

Longair, Sec. 5.5.1

$$\frac{a(t_{obs})}{a(t_{emit})} \;=\; \frac{\lambda_{obs}}{\lambda_{emit}} \;=\; 1+z \quad . \tag{5}$$

This relation governs the evolution of the cosmic background radiation that is presently in the microwave band. N.B. *It does not apply to radiation when it is subject to local gravitational perturbations such as in galaxies, etc.*

Plot: Coupling of wavelength to scale factor

2 Friedmann Equation

Thus far, we have looked only at generic forms for the metric of a universe that is consistent with the cosmological principle. Imparting the dynamical effects of mass/energy establishes a so-called *Friedmann universe/cosmology* using Einstein's field equations.

Peacock pp. 73–7

$$R_{\mu\nu} \;=\; \frac{8\pi G}{c^4}\, T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \quad . \tag{6}$$

Components of this can be integrated as an equation of motion, proceeding as we did in Newtonian cosmology, invoking mass/energy conservation to eliminate P via an equation of state. The result is identical to the Newtonian model:

C & O, pp. 1190

$$(\dot{R})^2 - \frac{8\pi}{3}\, G\rho\, R^2 \;=\; \text{const} \quad . \tag{7}$$

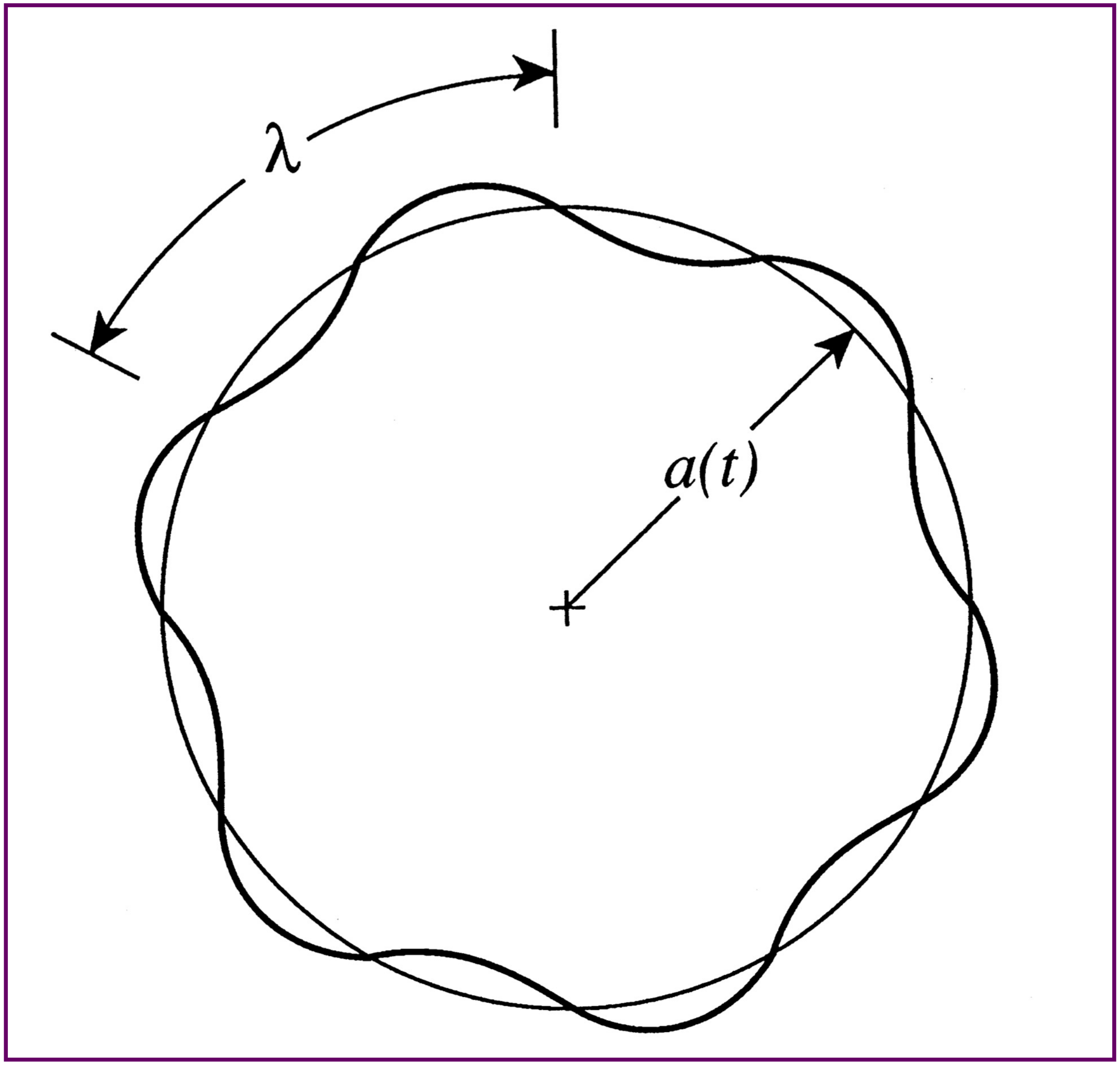
for $R/R_0 = a$. The derivation is beyond the scope of the course. However, Einstein's field equations provide more information, mostly superfluous, though the diagonal elements can be used to fix the constant of integration:

$$\boxed{(\dot{R})^2 - \frac{8\pi}{3}\, G\rho\, R^2 \;=\; -kc^2 \quad .} \tag{8}$$

This is the **Friedmann Equation**. We call universes that satisfy it, Friedmann-Robertson-Walker (FRW) models.

Coupling of Wavelength to Scale Factor

- Fig. 5.8 of Peebles.
- Model of redshift with universal expansion. The closed circle represents a universe of scale factor a(t). The oscillating line is the spatial part of a wave – this could be an acoustic mode connected to structure at CMB generation.
- The wavelength λ of this oscillation mode is proportional to a(t).



3 Radiation-Significant Cosmologies

So far, we have focused on baryon-dominated universes where the density fraction in matter is $\Omega_m = 8\pi G\rho/(3H_0^2)$. The Friedmann equation was developed for a non-relativistic equation of state (EOS), appropriate for dust. In general, the thermodynamic EOS can be written in the form

C & O, pp. 1174-6

$$P \;\propto\; \rho^{\gamma} \quad , \quad \gamma \;=\; \frac{\partial \log_e P}{\partial \log_e \rho}\Bigg|_{\text{S=Entropy}} \quad , \tag{10}$$

where γ is the familiar *adiabatic index*. Using the conservation of energy, $dU = -P\,dV \equiv -P\,d(1/\rho) = (P/\rho^2)\,d\rho$, the EOS can be written in the form

$$U \;=\; \frac{P}{(\gamma-1)\rho} \quad . \tag{11}$$

Here $\gamma = 5/3$ for non-relativistic matter, and $\gamma = 4/3$ for a relativistic gas such as radiation, or very hot matter in the pre-nucleosynthetic universe.

• For radiation, while the number density scales during expansion as $1/V \propto a^{-3}$, the wavelength changes according to $\langle\lambda\rangle \propto a$. Hence, we anticipate the evolution of the radiation energy density U_{rad} in the universe according to

$$U_{\text{rad}} \;\propto\; a^{-4} \quad . \tag{12}$$

Equivalently, quantum thermodynamics of the Planck spectrum can be considered with the dependence $kT = kT_0(1+z) \propto 1/a$ being invoked, and then applying the Stefan-Boltzmann law $U_{\text{rad}} \propto T^4$.

• Now we set $\rho \to U_{\text{rad}}/c^2$ to establish the correspondence, so that Friedmann's equation for a combination matter-radiation universe can be written in the form

$$\boxed{(\dot{R})^2 - \frac{8\pi}{3}\,G\left[\rho + \frac{U_{\text{rad}}}{c^2}\right] R^2 \;=\; -kc^2 \quad .} \tag{13}$$

The fractional density parameter for radiation can be written in the form

$$\Omega_{\text{rad}} \;=\; \frac{U_{\text{rad}}}{\rho_c c^2} \quad , \tag{14}$$

which is fixed by the temperature of the microwave background. Then Friedmann's equation can be cast in the form

$$\left(\frac{da}{dt}\right)^2 \;=\; H_0^2 \left\{ \frac{\Omega_m}{a} + \frac{\Omega_{\rm rad}}{a^2} - \overbrace{\frac{kc^2}{H_0^2 R_0^2}}^{\Omega-1} \right\} \quad . \tag{15}$$

The replacement $-kc^2/H_0^2/R_0^2 \to 1 - \Omega$ is deduced by setting $a \to 1$ now. From this it can be deduced that at early epochs, the universe must have been radiation-dominated regardless of the cosmology. The solution of Eq. (15) can be routinely determined:

$$\begin{aligned} H_0 t \;&=\; \int_0^a \frac{d\Gamma}{\sqrt{\Omega_m/\Gamma + \Omega_{\rm rad}/\Gamma^2 + (1-\Omega)}} \\ &=\; \frac{1}{\sqrt{|1-\Omega|}} \int_0^a \frac{\Gamma\, d\Gamma}{\sqrt{(a_+ - \Gamma)\,(\Gamma - a_-)}} \quad , \end{aligned} \tag{16}$$

where $\Omega = \Omega_m + \Omega_{\rm rad}$, and also

$$a_{\pm} \;=\; \frac{1}{2(\Omega - 1)} \left\{ \Omega_m \pm \sqrt{\Omega_m^2 + 4(\Omega - 1)\Omega_{\rm rad}} \right\} \quad . \tag{17}$$

At early epochs when $a \ll 1$ and radiation dominates,

$$\frac{dt}{da} \;\propto\; a \quad \Rightarrow \quad a(t) \;\propto\; t^{1/2} \quad . \tag{18}$$

All radiation-dominated universes generate this regardless of the value of k .

$*$ At present, WMAP gives $\Omega_{\rm rad}/\Omega_m \sim 3.1 \times 10^{-4} = 1/(1+z_{\rm eq})$ so that $z_{\rm eq} \sim 3200$ is the redshift at which radiation and matter are in equipartition.

The age t_c of a closed, combination matter + radiation universe can be obtained by noting that the expansion stops when $da/dt = 0$. Using Eq. (16), and introducing a factor of 2 to address recollapse to a *big crunch*, t_c can be expressed in terms of elementary functions of a_+ and a_- , leading to

$$t_c \;=\; \frac{1}{H_0} \left\{ \frac{2\sqrt{\Omega_{\rm rad}}}{\Omega - 1} + \frac{\Omega_m}{(\Omega-1)^{3/2}} \left[\frac{\pi}{2} + \arcsin \frac{\Omega_m}{\sqrt{\Omega_m^2 + 4\Omega_{\rm rad}(\Omega - 1)}} \right] \right\} \quad . \tag{19}$$

For a radiation-dominated universe ($\Omega_m = 0$), this simply reduces to an age $t_c = 2\sqrt{\Omega_{\rm rad}}/[\Omega_{\rm rad} - 1]/H_0$. The function $t(z)$ can also be obtained in closed form (left as an exercise).