4 The Cosmological Constant

• Einstein's theory came out before we had knowledge of the Hubble expansion. The prevailing view in 1916 was of a steady-state universe, for which we would ascribe $H_0 = 0$ in our present description. **C & O, pp. 1190–1**

The field equations do not exhibit $H_0 \approx 0$ solutions for most of parameter space. This led Einstein to propose a simple way to effect such behavior in model steady-state universes.

• Logic: gravity attracts, so decelerative solutions need to be balanced by a repulsive force. Since $\nabla^2\phi = 4\pi G(\rho + 3P)$ results from an attractive force, a $T_{00} < 0$ contribution to the energy-momentum tensor $T_{\mu\nu}$ is desired. The simplest path to this is to generalize the field equations to the form

$$\boxed{G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \;=\; \frac{8\pi G}{c^4}\, T_{\mu\nu} - \Lambda g_{\mu\nu} \quad .} \tag{19}$$

Here Λ is the **cosmological constant**.

* $\Lambda > 0 \Rightarrow$ repulsion.
* $\Lambda =$ constant is consistent with the cosmological principle.

Friedmann's equation then generalizes to the form

$$(\dot{R})^2 - \frac{8\pi}{3}\, G \left[\rho + \frac{U_{\rm rad}}{c^2} \right] R^2 \;=\; -kc^2 + \frac{\Lambda}{3}\, R^2 \quad . \tag{20}$$

Thus, it is easy to discern that when the Λ term dominates, $\dot{a} \propto a$, i.e.

$$\frac{1}{a}\,\frac{da}{dt} \;=\; \pm\frac{1}{t_\Lambda} \;=\; \text{const.} \tag{21}$$

for $t_\Lambda = \sqrt{3/\Lambda}$, which leads to the exponential solution

$$a(t) \;\propto\; \exp\left\{ \pm\frac{t}{t_\Lambda} \right\} \quad . \tag{22}$$

The negative exponent corresponds to the growth of density perturbations in forming large scale structure.

For larger scale (cosmological) evolution, as the universe is expanding, the *positive exponent is pertinent*, resulting in the **de-Sitter solution**, which is realized when $\Lambda \gtrsim 8\pi G\rho$.

- The integration of Friedmann's equation parallels previous developments: **C & O, pp. 1192–3**

$$t \; = \; \frac{1}{H_0} \int_0^a \frac{d\Gamma}{\sqrt{\Omega_m/\Gamma + \Omega_{\rm rad}/\Gamma^2 + \Omega_\Lambda \Gamma^2 + (1-\Omega)}} \quad , \tag{23}$$

where $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$. Often $\Omega_\Lambda = \Lambda/(3H_0^2)$ is written Ω_v for the **vacuum contribution** to the universe's energy density.

Observe that the cosmological or vacuum term has a density that is independent of the scale parameter, i.e. constant in time, so that *eventually it wins out* over matter and radiation, whose densities decline with expansion.

∗ If $\Lambda < 0$, recollapse is inevitable, either forced by Λ in low density universes at large a, or by $\Omega_m > 1$.

∗ If $\Lambda > 0$ but $\Omega_m < 1$, then the model expands to infinity.

∗ If $\Lambda > 0$ but $\Omega_m > 1$, then collapse can be avoided provided $\Omega_\Lambda > f(\Omega_m)$, where f is a cubic function (we won't worry about the details).

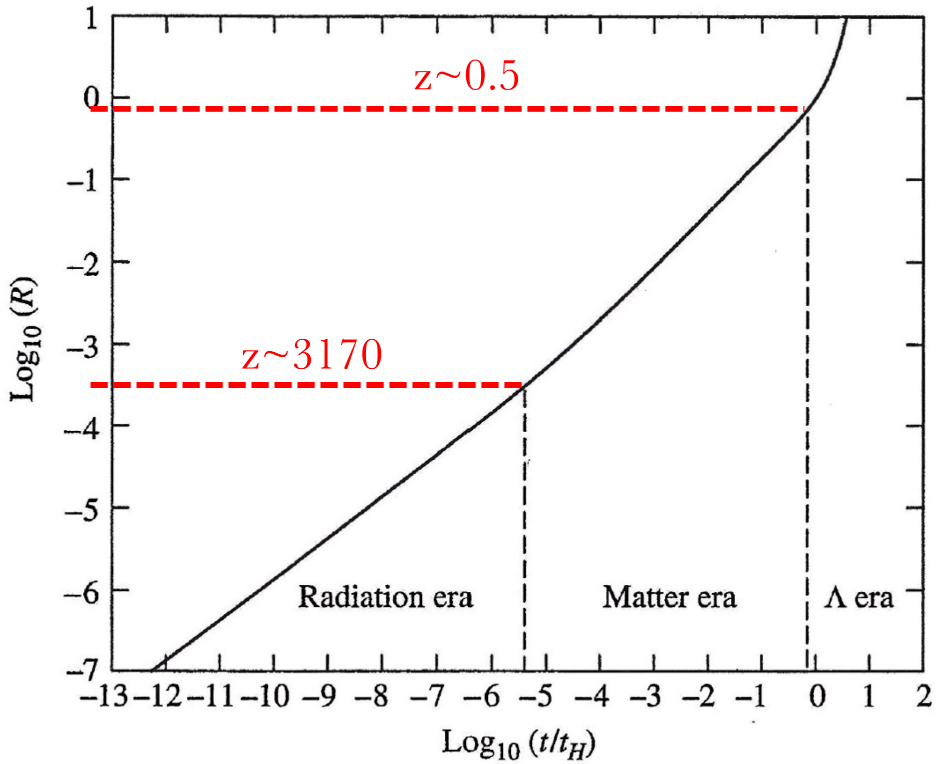
Plot: Expansion for a WMAP Universe

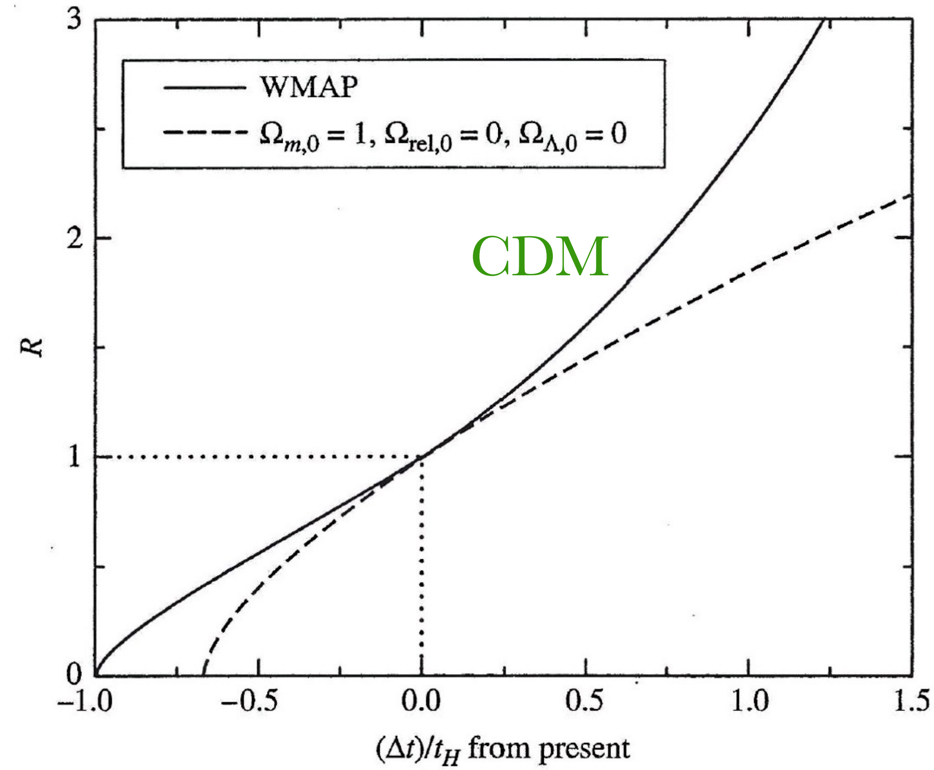
The flat, $k = 0$ case for $\Omega_{\rm rad} = 0$ is called the *Einstein-de Sitter* model.

- Back to Einstein's original premise, if we set $H_0 \to 0$ to mimic the perception at his time, then Friedmann's equation admits $\dot{a} = 0$ solutions of constant density and $k = {\rm sign}\Lambda$, constituting positive curvature for $\Lambda > 0$.

This generates a **static universe**, as desired by Einstein. Upon Hubble's discovery of expansion, Einstein labelled his cosmological constant hypothesis *his biggest blunder*. Perhaps not!

Expansion of a WMAP Universe





- *Left*: Scale factor R=a during a broad range of epochs. Recently ($z<0.5$), we have a vacuum-dominated epoch following a matter-dominated one.
- *Right*: Comparison of scale factor evolution for a WMAP CDM universe (solid curve) and a flat, matter-dominated universe (dotted curve).
- Figures 29.19 and 29.20 of Carroll & Ostlie *An Introduction to Modern Astrophysics*

5 Connecting to the Real World

It is expedient to express some of the key expansion equations in terms of salient observables. The key observable is the (spectroscopic) redshift z:

$$1 + z \;=\; \frac{a_0}{a(t)} \quad . \tag{24}$$

5.1 Deceleration Parameter

The full version of Friedmann's equation in Eq. (20) is equivalent to

$$\left(\frac{\dot{a}}{a}\right)^2 \;=\; H_0^2 \left\{ \Omega_m (1+z)^3 + \Omega_{\rm rad} (1+z)^4 + \Omega_\Lambda + (1-\Omega)\,(1+z)^2 \right\} \quad , \tag{25}$$

where $\Omega = \Omega_m + \Omega_{\rm rad} + \Omega_\Lambda$ as before. The acceleration/deceleration can be obtained by differentiating Eq. (25) with respect to time (remember, $a_0 = 1$):

$$\begin{aligned} 2\,\ddot{a}\,\dot{a} \;&=\; \dot{a}\,H_0^2 a_0^2 \,\frac{d}{da} \left\{ \Omega_m \left(\frac{a_0}{a}\right) + \Omega_{\rm rad} \left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda \left(\frac{a}{a_0}\right)^2 + \text{const} \right\} \\ \Rightarrow \quad \frac{\ddot{a}}{a} \;&=\; H_0^2 \left\{ \Omega_\Lambda - \frac{\Omega_m}{2} \left(\frac{a_0}{a}\right)^3 - \Omega_{\rm rad} \left(\frac{a_0}{a}\right)^4 \right\} \\ &=\; H_0^2 \left\{ \Omega_\Lambda - \frac{\Omega_m}{2} (1+z)^3 - \Omega_{\rm rad} (1+z)^4 \right\} \quad . \end{aligned} \tag{26}$$

Evaluating this at $z = 0$ and combining with Friedmann's equation yields

$$q_0 \;=\; -\frac{\ddot{a}_0\, a_0}{(\dot{a}_0)^2} \;=\; \frac{\Omega_m}{2} + \Omega_{\rm rad} - \Omega_\Lambda \tag{27}$$

as the general form of a quantity called the **deceleration parameter**. As a perturbation parameter about the local Hubble flow, before 1995, q_0 was highly sought after as an indicator of the level of closure of the universe. It essentially expresses the curvature in Hubble diagrams to leading order.

- Observe that if $\Omega_\Lambda > 0$ dominates, then $q_0 < 0$, *implying an accelerating universe,* a total surprise to the cosmology community in 1997.

5.2 Lookback Times

Since $1 + z = a_0/a$, the Taylor series for the lookback time $t_0 - t$ is

$$a(t) \;=\; a_0 \left[1 - H_0(t_0 - t) - \frac{q_0}{2}\, H_0^2\, (t_0 - t)^2 \ldots \right] \quad , \tag{28}$$

which can routinely be inverted to give $t_0 - t$ as a function of z. However, we already have this inversion as a solution integral for Friedmann's equation:

$$H_0 t(z) \;=\; H_0 \int_0^a \frac{da}{\dot{a}} \;\equiv\; \int_0^a \frac{da}{a\, E(z)} \;=\; \int_z^\infty \frac{dz'}{(1+z')\, E(z')} \quad , \tag{29}$$

where

$$E(z) \;=\; \sqrt{\Omega_m (1+z)^3 + \Omega_{\rm rad} (1+z)^4 + \Omega_\Lambda + (1-\Omega)\, (1+z)^2} \tag{30}$$

appears in the Eq. (25) version of Friedmann's equation. Note that as $z \to 0$, $E(z) \to 1$. Consequently,

$$t_0 - t(z) \;=\; \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z')\, E(z')} \quad , \tag{31}$$

Plot: Lookback time versus redshift

5.3 Angular Diameter Distance

Because of the distortion from Euclidean geometry introduced by curved spacetime, the coupling between apparent angular size θ and distance d for an object of of fixed size D deviates from $\theta = D/d \propto 1/z$. Considering

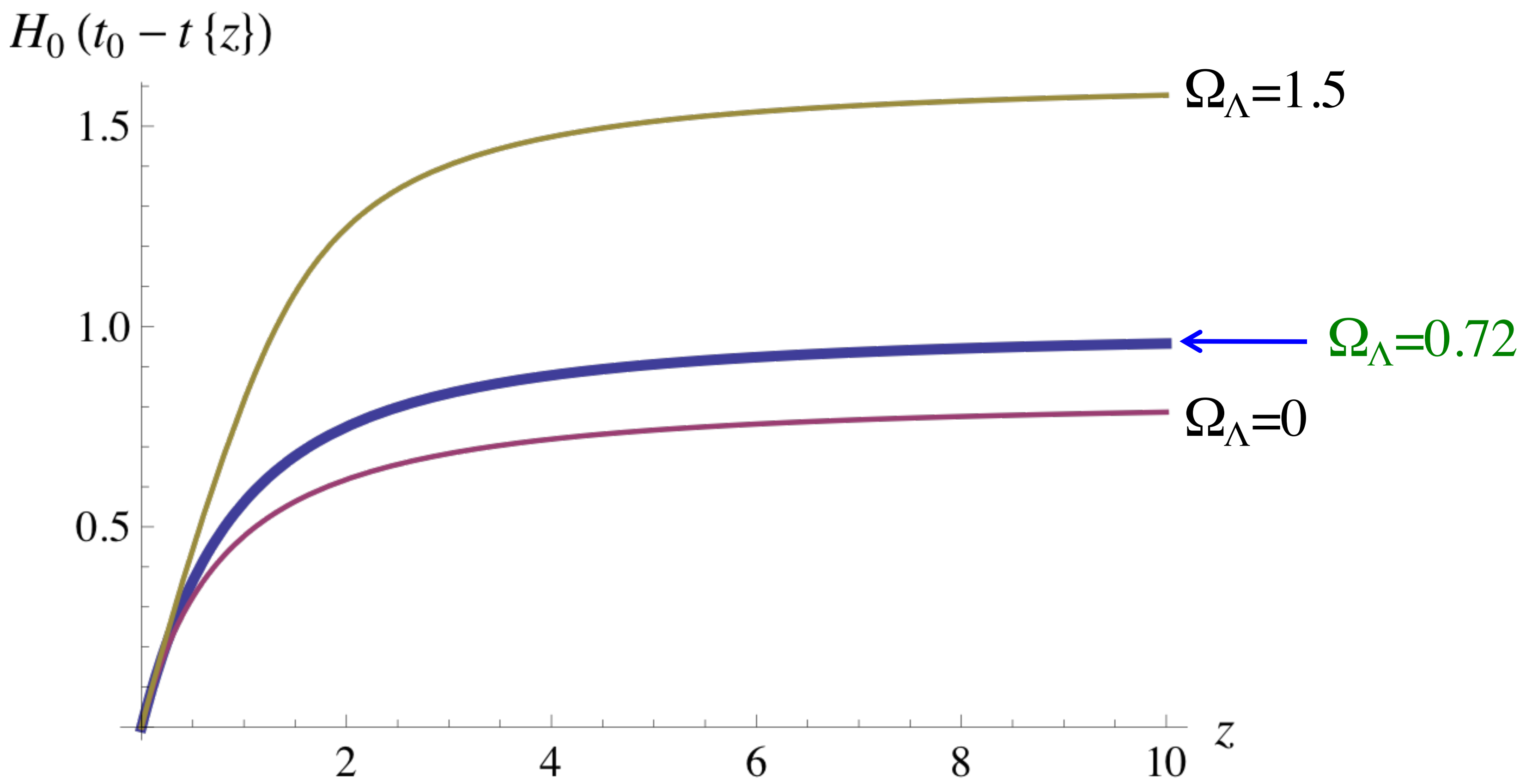
C & O, pp. 1215–8

$$ds^2 \;=\; dt^2 - a^2(t) \left\{ d\chi^2 + S_k^2(\chi)\; d\Omega \right\} \quad , \tag{32}$$

the RW line element, we now set $\chi \to \Theta$ to establish correspondence with the development angle Θ. The angular portion is captured via

$$S_k(\Theta) \;=\; \begin{cases} \sin\Theta, & k = 1, \\ \Theta, & k = 0, \\ \sinh\Theta, & k = -1 \;\; . \end{cases} \tag{33}$$

Lookback Times in Ω_m=0.28 Cosmologies



- Lookback time, scaled via $H_0(t-t_0[z])$, as a function of redshift back to around the re-ionization epoch. Radiation is insignificant (Ω_{rad}=0) and matter (including dark matter) is set at present density Ω_m=0.28.
- Different choices of the cosmological constant, labelled by Ω_Λ, are adopted, illustrating its influence on the total asymptotic age t_0 of the Universe as z becomes very large.

The physical area subtended by the source is then $a^2(t)\, S_k^2(\Theta)$, with $a(t)\, d\Theta$ being the radial distance element. Angular diameters therefore scale as $S_k(\Theta)/(1+z)$. For proper time elements $ds = 0$, then the radial scale (for $d\Omega = 0$) satisfies $d\Theta \propto dt/a \propto da/(\dot{a}\, a)$, so that

$$\Theta \;=\; H_0 \sqrt{|\Omega - 1|} \int_t^{t_0} \frac{a_0\, dt'}{a(t')} \;=\; \sqrt{|\Omega - 1|} \int_0^z \frac{dz'}{E(z')} \qquad (34)$$

expresses the development angle in terms of redshift z . Accordingly, we *define* the **angular diameter distance** via

$$\boxed{d_A \;\equiv\; \frac{c}{H_0 \sqrt{|\Omega - 1|}} \, \frac{S_k[\Theta(z)]}{1+z} \;=\; \frac{D}{\theta} \quad .} \qquad (35)$$

This gives the apparent behavior of angular scales of structures within the universe with redshift: angular size scales as $1/d_A$.

• As $z \to 0$, $\Theta \to z\sqrt{|\Omega - 1|}$, so that $d_A \to cz/H_0$. This then yields angular scales $\theta \propto H_0/z$ in the local (Euclidean) universe, as is expected.

Plot: Angular diameter distance versus redshift (from Peacock)

N.B. Since Θ remains finite as $z \to \infty$ [by inspection of Eq. (30)], then $d_A \propto (1+z)^{-1}$ i.e. angular scale $\theta \propto (1+z)$. Hence, placing a fixed object back at high redshift generates a large apparent angular size in the smaller universe, i.e. the object was looming over us when it emitted its light!

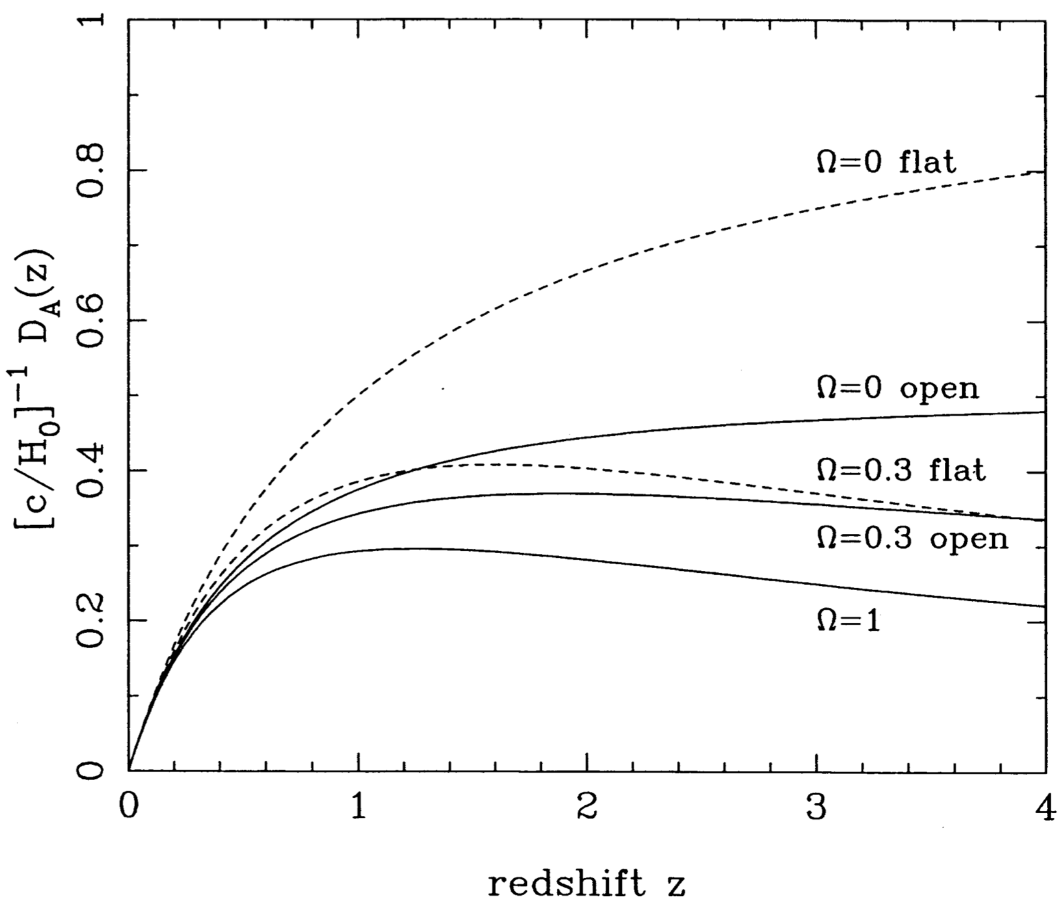
• For matter-dominated cosmologies, $\Omega_{\rm rad} = 0 = \Omega_\Lambda$, the parametric form for $a(t)$ can be use to obtain an analytic form for $S_k(\Theta)$, and it can be shown that

$$d_A \;=\; \frac{c}{H_0} \, \frac{1}{q_0^2 (1+z)^2} \left\{ q_0 z + (q_0 - 1) \left[\sqrt{1 + 2 q_0 z} - 1 \right] \right\} \quad . \qquad (36)$$

Hence, in principle, observing variations in angular scales of known "standard size" objects leads to a determination of q_0 . In practice, this is hard to do, as size variations hamper q_0 diagnostics.

Plot: Angular diameter – redshift diagram for galaxy clusters

Angular Diameter Distance versus Redshift



- Scaled angular-diameter distance versus redshift for various cosmologies (Fig. 3.7 from Peacock *Cosmological Physics*). The solid curves show models with zero vacuum energy; the broken curves show flat models with $\Omega_m+\Omega_\Lambda=1$. In both cases, curves for $\Omega_m=0$, 0.3, 1 are depicted. Higher density results in lower diameter distances at high z due to greater gravitational focusing of light rays.