4.1 Critical Density

Since Hubble's Law is $H_0 = \dot{R}_0/R_0 = \dot{a}_0/a_0$ at the current epoch, one can rewrite Eq. (23) for the present as **C & O, p. 1150**

$$\left(H_0^2 - \frac{8\pi}{3} G\rho \right) R_0^2 \; = \; \text{const.} \quad \Rightarrow \quad \frac{8\pi}{3} G(\rho_c - \rho)\, R_0^2 \; = \; \text{const.} \quad , \qquad (28)$$

where

$$\boxed{\rho_c \; = \; \frac{3H_0^2}{8\pi G}} \qquad (29)$$

is called the **critical density**. If we set $H_0 = h\,100$ km/sec/Mpc, then at the present epoch we have

$$\rho_c \; = \; 1.9 \times 10^{-29} h^2 \;\; \text{gm/cm}^3 \;\; \approx \;\; 9.5 \times 10^{-30} \;\; \text{gm/cm}^3 \quad , \qquad (30)$$

when using the WMAP value of $h = 0.71$. This is $\sim 10^{-5}$ protons per cc!

- $*$ If $\rho < \rho_c$, then const. > 0 is required and the universe is open.
- $*$ If $\rho > \rho_c$, then const. < 0 is required and the universe is closed.
- $*$ If $\rho = \rho_c$, then it is a flat universe with const. $= 0$.

Clearly, seeking to quantify the current mass density in the universe is of paramount importance to discerning its eventual fate!

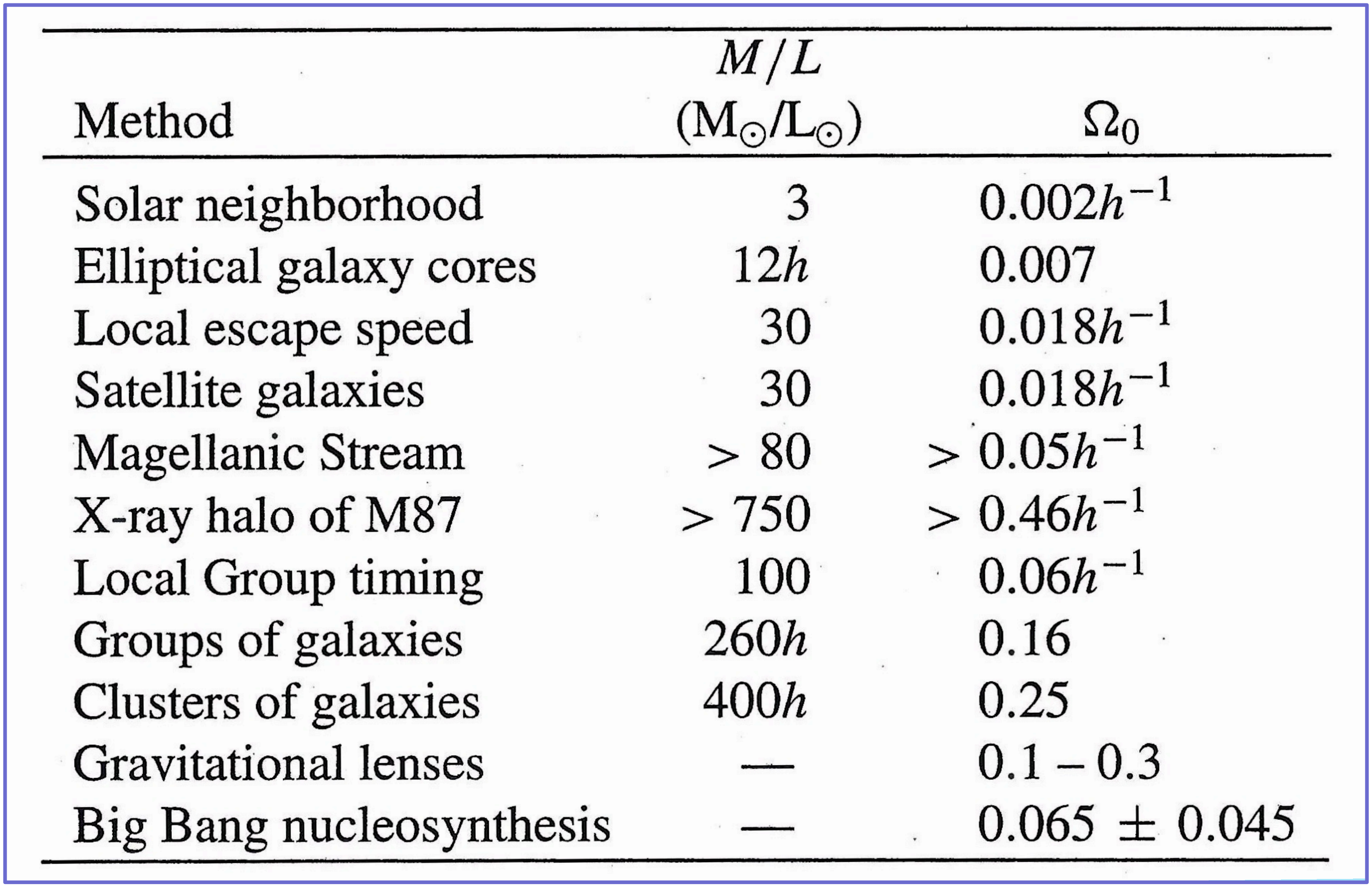
Plot: Mass-to-Light Ratios: Table for Various Astrophysical Systems

• The WMAP determinations of the cosmological parameters established that $\rho/\rho_c \approx 0.044$, i.e. that matter alone could not close the universe. This was consistent with earlier, **primordial nucleosynthetic** bounds that will be studied later.

Plot: Mass-to-Light Ratios verses Scale for Astrophysical Systems

• In general, the Hubble "constant" $H = H(t)$ evolves with time so that the critical density $\rho_c = \rho_c(t)$ evolves also.

Mass-to-Light Ratios



Method	M/L ($M_\odot/L_\odot$)	Ω_0
Solar neighborhood	3	$0.002h^{-1}$
Elliptical galaxy cores	$12h$	0.007
Local escape speed	30	$0.018h^{-1}$
Satellite galaxies	30	$0.018h^{-1}$
Magellanic Stream	> 80	$> 0.05h^{-1}$
X-ray halo of M87	> 750	$> 0.46h^{-1}$
Local Group timing	100	$0.06h^{-1}$
Groups of galaxies	$260h$	0.16
Clusters of galaxies	$400h$	0.25
Gravitational lenses	—	$0.1 - 0.3$
Big Bang nucleosynthesis	—	0.065 ± 0.045

Table 29.1 of Carroll & Ostlie *An Introduction to Modern Astrophysics*

Mass-to-Light Ratios for Different Scales

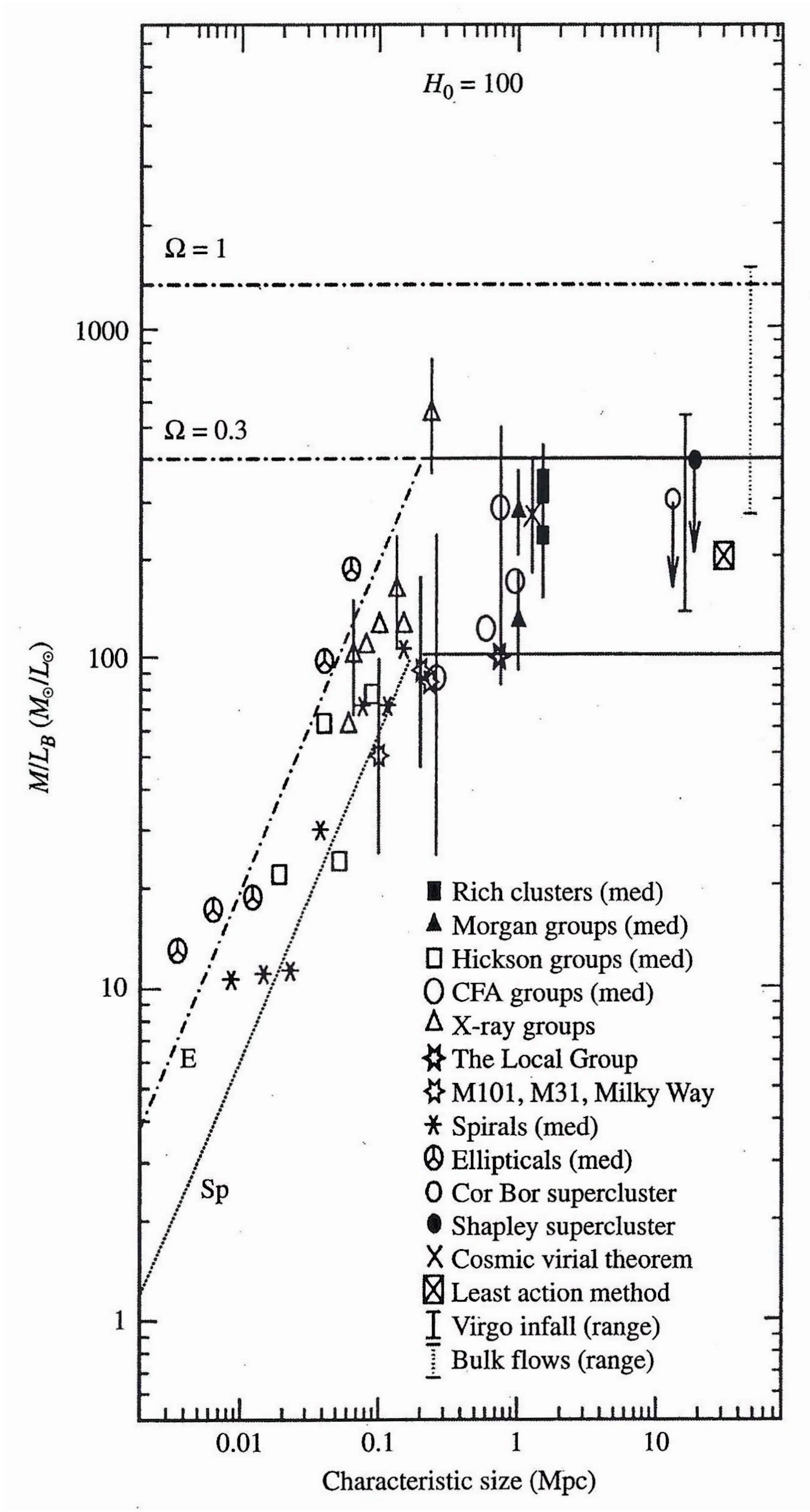


Fig. 29.4 of Carroll & Ostlie

An Introduction to Modern Astrophysics

From Dodelson *Modern Cosmology*

5 Matter-Dominated Universes

We now proceed to look at the qualitative features of solutions of Eq. (23), i.e. Friedmann's equation, first focusing on matter-dominated cosmologies. If we define the **density parameter**

C & O,
p. 1152-9

$$\Omega_m \;=\; \frac{\rho_0}{\rho_c} \tag{31}$$

as the *present* fractional matter density of the universe (often denoted Ω_0), then at any epoch, $\rho = \rho_c \Omega_m \left(R_0/R\right)^3$, taking into account dilution by volume expansion, *for a matter-dominated universe.* Then the Friedmann equation can be cast in the form

$$\left(\frac{dR}{dt}\right)^2 \;=\; H_0^2 \Omega_m \left(\frac{R_0}{R}\right)^3 R^2 - kc^2 \quad , \tag{32}$$

where we have used dimensional analysis to rewrite (with *a posteriori* motivations) the constant of integration as $-kc^2$. Using $a(t) = R(t)/R_0$, we can express this as

$$\left(\frac{da}{dt}\right)^2 \;=\; H_0^2 \left\{ \frac{\Omega_m}{a} - \frac{kc^2}{H_0^2 R_0^2} \right\} \quad . \tag{33}$$

Noting that when $a_0 = 1$ at present, $da/dt = H_0$ is the Hubble constant, establishing the identity

$$-\frac{kc^2}{H_0^2 R_0^2} \;=\; 1 - \Omega_m \quad . \tag{34}$$

Here there is a one-to-one correspondence: $k > 0 \Rightarrow \Omega_m > 1$ (closed universe) and $k < 0 \Rightarrow \Omega_m < 1$ (open universe).

$*$ The physical meaning of the constant k will be elucidated with the treatment of relativistic cosmology.

- Eq. (33) can then be readily integrated to yield the solution

$$\boxed{t \;=\; \frac{1}{H_0} \int_0^a \frac{d\Gamma}{\sqrt{\Omega_m/\Gamma + (1-\Omega_m)}} \quad .} \tag{35}$$

At early epochs when $a \ll 1$,

$$\frac{dt}{da} \propto a^{1/2} \quad \Rightarrow \quad a(t) \propto t^{2/3} \quad . \tag{36}$$

All matter-dominated universes generate this regardless of the value of k.

$\boxed{\textbf{Plot:}}$ Model universes: $a(t)$ versus time

- For such matter-dominating cosmologies, gravity pulls and $\dot{a}$ is higher in the past (i.e. higher value of the Hubble constant) since we have *deceleration*, i.e. $\ddot{a} < 0$. This forces a singularity at $t = 0$ at a finite past.

$$\boxed{\Rightarrow \quad \text{BIG BANG}} \tag{37}$$

As $t \to 0^+$, then $H(t) \to \infty$. Measuring H_0 does not determine the mass of the universe nor ρ_0, since $0 < H(t) < \infty$ is spanned in a closed universe during its evolution.

- Therefore, the signature of a **Big Bang**, or past, high-density epoch, becomes a *Holy grail* of cosmology. By thermodynamics, this must correspond to a high temperature epoch.

$*$ Because nuclear epochs are shrouded by subsequent radiative opacity of the universe, the atomic scale/epoch becomes the most relevant, spawning the **cosmic background radiation** is the keystone marker of a Big Bang.

- Gravity causes deceleration in all $\Lambda = 0$ models. This implies $1/H_0$ *overestimates* the age of the universe in a matter-dominated cosmology.

$\boxed{\textbf{Plot:}}$ Model universes pinned at current Hubble flow

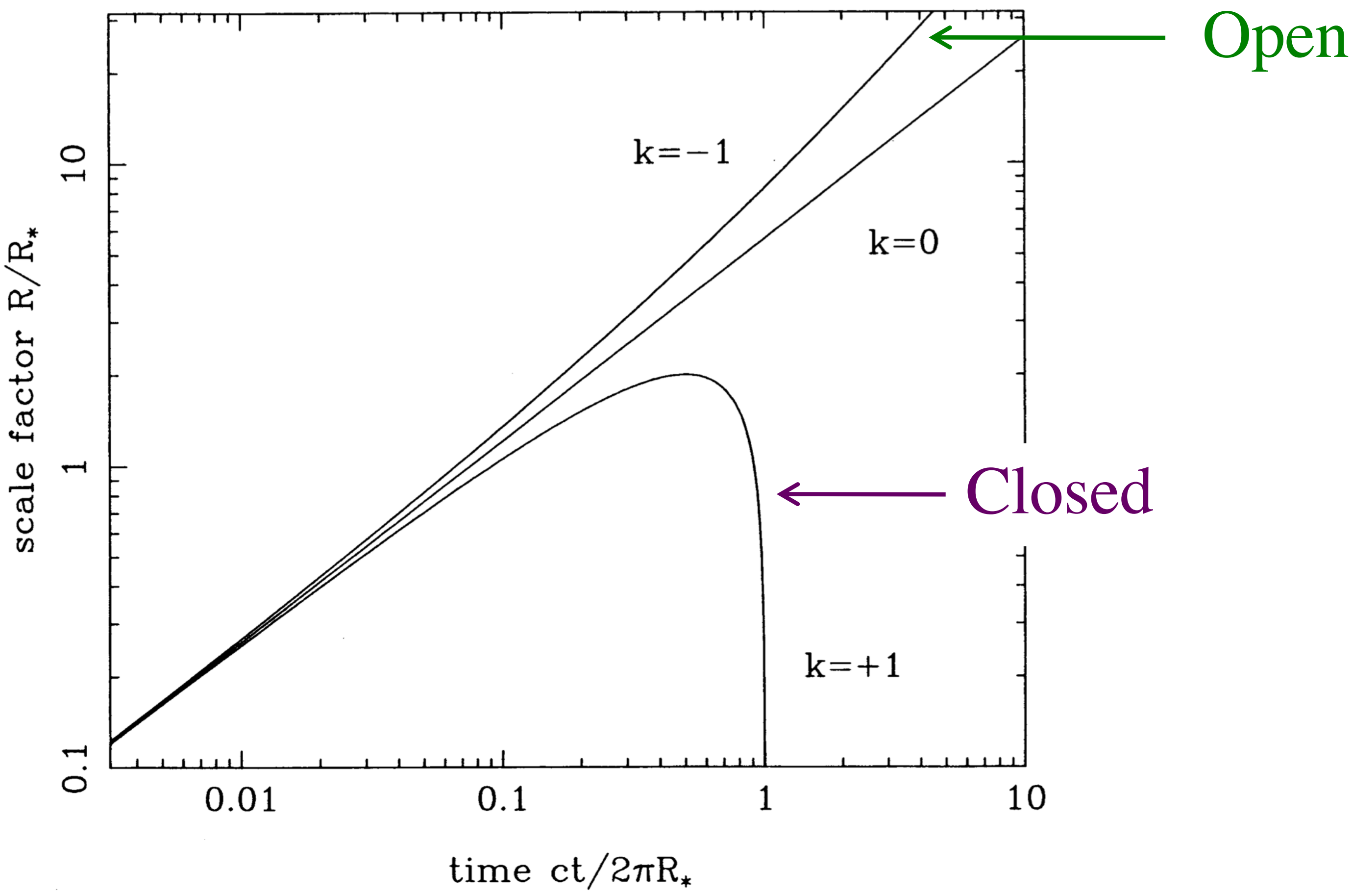
$*$ this impacts cosmochronological bounds on the age of the universe.

- From Eq. (35), it can be quickly deduced that for a closed universe, the expansion stops ($da/dt = 0$) when $a = \Omega_m/(\Omega_m - 1)$. It then follows, that the **lifetime** of a closed universe is

$$t_c = \frac{2}{H_0} \frac{\Omega_m}{(\Omega_m - 1)^{3/2}} \int_0^1 \frac{dx}{\sqrt{1/x - 1}} = \frac{\pi}{H_0} \frac{\Omega_m}{(\Omega_m - 1)^{3/2}} \quad , \tag{38}$$

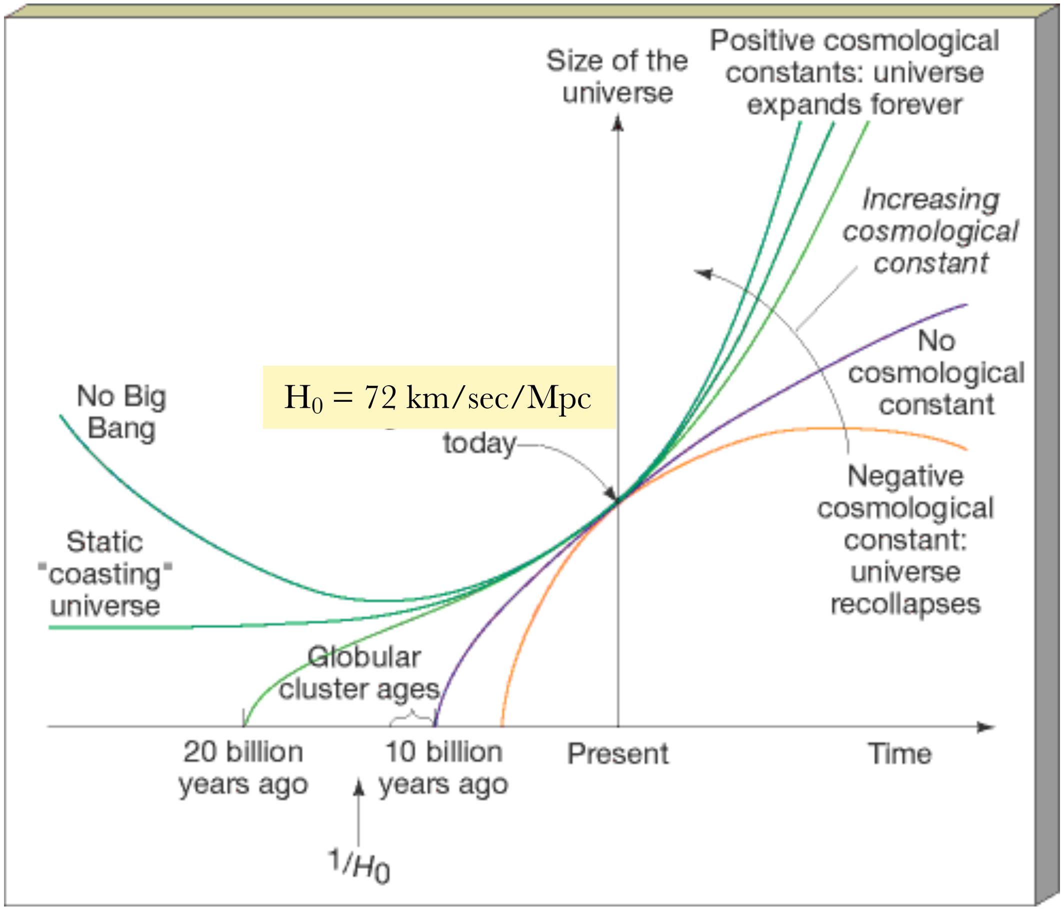
where the factor of 2 includes recollapse to a *big crunch*.

Friedmann Equation Solutions



- Friedmann-Robertson-Walker solutions for the scale factor a(t) in matter universes with positive (k=1, closed), zero (k=0, flat) and negative (k=-1, open) curvature.
- Observe the logarithmic scales for the axes. This is designed to illustrate the early time behaviour for matter-dominated cosmologies, $a(t)\ \alpha\ t^{2/3}$, independent of the sign of the curvature. [Fig. 3.4 of Peacock, *Cosmological Physics*]

Various Cosmologies



- Model universes appropriate for cosmologies with various Ω_m and Ω_Λ.
- The age of the universe is generally less than $1/H_0$.
- Bound universes have $\Omega_m > 1$, and are generally younger than unbound ones due to stronger gravitational deceleration of the expansion.
- Figure from Chaisson & McMillon *Astronomy Today* (Interlude 27-1, on page 643 therein).

5.1 Mathematical Solutions

- By using appropriate sinusoidal or hyperbolic substitutions for the cases of closed and open universes, the solution in Eq. (35) can be written for $k = 0, \pm 1$ in the parametric form

$$\begin{aligned} a &= \frac{\Omega_m}{2|\Omega_m - 1|} \left|1 - C_k(\Theta)\right| \quad , \\ H_0 t &= \frac{\Omega_m}{2|\Omega_m - 1|^{3/2}} \left|\Theta - S_k(\Theta)\right| \quad , \end{aligned} \tag{39}$$

where

$$S_k(\Theta) = \begin{cases} \sin\Theta, & k = 1, \\ \Theta, & k = 0, \\ \sinh\Theta, & k = -1, \end{cases} \tag{40}$$

and

$$C_k(\Theta) = \sqrt{1 - k S_k^2(\Theta)} \quad . \tag{41}$$

The parameter Θ is called the **development angle**, and assumes the form

$$\Theta = H_0 \sqrt{|\Omega_m - 1|} \int_0^t \frac{dt'}{a(t')} \quad , \tag{42}$$

- To eliminate the mystical origin of these parameterizations, we derive them for the special case of *closed universe* to illustrate the essentials. Start with the scale parameter form in Eq. (39):

$$a(t) = \frac{\Omega_m}{2[\Omega_m - 1]} \Big[1 - \cos\Theta\Big] = \frac{\Omega_m}{\Omega_m - 1} \sin^2\!\left(\frac{\Theta}{2}\right) \quad , \tag{43}$$

for, as yet, unspecified Θ. This motivates the change of variables $\Gamma = \Omega_m/(\Omega_m - 1)\ \sin^2[\chi/2]$ in the integral Eq. (35), which then quickly evaluates to

$$H_0 t = \frac{\Omega_m}{2[\Omega_m - 1]^{3/2}} \Big(\Theta - \sin\Theta\Big) \quad , \tag{44}$$

as the companion parametric form. Taking the time derivative of this and then dividing by Eq. (43) yields

$$\frac{H_0}{a(t)} = \frac{1}{\sqrt{\Omega_m - 1}} \frac{d\Theta}{dt} \quad \Rightarrow \quad \Theta = H_0 \sqrt{\Omega_m - 1} \int_0^t \frac{dt'}{a(t')} \quad , \tag{45}$$

as desired. Marginal and open universes offer similar derivations.

- Motivations for such a parametric formulation are really for more complicated cosmologies involving radiation and a cosmological constant, when tractable analytic integration is not possible.

In this case of a matter-only universe (case $\Omega_m > 1$ only here), the correspondence $a = 1/(1+z)$ motivates a change of variables $\Gamma \to 1/(1+q)$ in Eq. (35) to derive a complete analytic form for $t = t(z) \equiv t(a)$:

$$\begin{aligned} H_0 t &= \int_z^\infty \frac{dq}{(1+q)^2 \sqrt{1+\Omega_m q}} \\ &= \frac{\Omega_m}{2[\Omega_m - 1]^{3/2}} \arccos\left[1 - \frac{2(\Omega_m - 1)}{\Omega_m (1+z)}\right] - \frac{\sqrt{1+\Omega_m z}}{(\Omega_m - 1)(1+z)} \quad . \end{aligned} \tag{46}$$

Carroll & Ostlie derive this by inversion of Eq. (43) and substitution into Eq. (44). Setting $z = 0$ establishes the age of the universe in terms of Ω_m, and $z \to -1/\Omega_m$ derives the *lifetime* of a matter-only universe.

The $\Omega_m < 1$ case is similar, with inverse hyperbolic functions appearing instead. The $\Omega_m = 1$ case must be taken as the continuous limit of either of the other two. For example, taking $\Omega_m \to 1^+$ in Eq. (46) uses the Taylor series expansion $\arccos(1 - x^2/2) \approx x + x^3/24$ for $x \ll 1$. Then one obtains

$$H_0 t = \frac{2}{3\,(1+z)^{3/2}} \quad , \quad \Omega_m = 1 \quad . \tag{47}$$

using the `Series[` $H_0 t$ `, {omega, 1, 0}, Assumptions -> {omega > 1, z > 0}]` command in Mathematica.

- The **lookback time** to any redshift is then simply obtained as

$$t_{\rm L} = t(0) - t(z) \quad , \tag{48}$$

and in a flat, matter-dominated universe takes the form

$$t_{\rm L} = \frac{2}{3H_0}\left[1 - \frac{1}{(1+z)^{3/2}}\right] \quad . \tag{49}$$

5.2 Deceleration Parameter

A Taylor series expansion of the scale factor gives

C & O, p. 1162

$$a(t) \;=\; a_0 \left[1 - H_0(t_0 - t) - \frac{q_0}{2}\, H_0^2\, (t_0 - t)^2 \ldots \right] \quad , \tag{50}$$

where

$$q_0 \;=\; -\frac{\ddot{a}_0 a_0}{(\dot{a}_0)^2} \;=\; -\frac{\ddot{a}_0}{a_0}\,\frac{1}{H_0^2} \tag{51}$$

is the **deceleration parameter**, with $a_0 = 1$. It gives a measure of how the Hubble expansion was faster (or slower) in the past. Here $t_0 - t$ is the lookback time.

- Since we have $\ddot{a}_0 = -(4\pi/3)G\rho_0 a_0$ from the time derivative of the Friedmann equation,

$$q_0 \;=\; \frac{4\pi}{3}\,\frac{G\rho_0}{H_0^2} \;=\; \frac{\rho}{2\rho_c} \;=\; \frac{\Omega_m}{2} \quad . \tag{52}$$

Again, Ω_m is the fraction of the critical density, and measuring the acceleration of the expansion putatively probes the nature of the universe.

- Hence, we have the correspondence:

$$\begin{array}{llllll} q_0 > \frac{1}{2} & \Leftrightarrow & \Omega_m > 1 & \Leftrightarrow & k = 1 & \Leftrightarrow \;\; \textbf{closed} \\ q_0 = \frac{1}{2} & \Leftrightarrow & \Omega_m = 1 & \Leftrightarrow & k = 0 & \Leftrightarrow \;\; \textbf{flat} \\ q_0 < \frac{1}{2} & \Leftrightarrow & \Omega_m < 1 & \Leftrightarrow & k = -1 & \Leftrightarrow \;\; \textbf{open} \end{array} \tag{53}$$

Clearly, determining the value of q_0 can establish the global nature and eventual fate of the universe in this matter-only scenario.