∗ The strong dependence of $T(M)$ is a consequence of the Maxwellian exponential folding into nuclear reaction rates in the stellar cores.

Plot: Gaia Hertzsprung-Russell diagram: Open Clusters

• Such L-T diagrams can calibrate the age t_{gc} of a globular cluster: **spectroscopic parallaxes** or **main sequence fitting**. Van den Bergh quotes recent estimates as **C & O, pp. 475-6**

$$t_{gc} \sim (1.5 \pm 0.2) \times 10^{10} \text{ years} \quad . \tag{8}$$

This was a critical piece of evidence conflicting with Hubble's original estimates of the speed of the flow.

3.4 White Dwarf Cooling

• White dwarfs are good chronometers because their equation of state is virtually independent of T for applicable ranges of temperature. This yields a simple cooling law for radiative loss through outer non-degenerate atmosphere (Mestel 1952): **C & O, pp. 574-8**

$$L \propto \frac{M}{t^{7/5}} \quad . \tag{9}$$

Hence, for the Chandrasekhar upper bound to M, there should be a sharp lower cutoff in the luminosity function in a system of particular age t_0

∗ The cutoff is sharp if the white dwarf mass distribution is not too skewed towards lower mass objects (i.e. $dn_{\rm wd}/dM \propto M^{\alpha}$ with $\alpha > -1$).

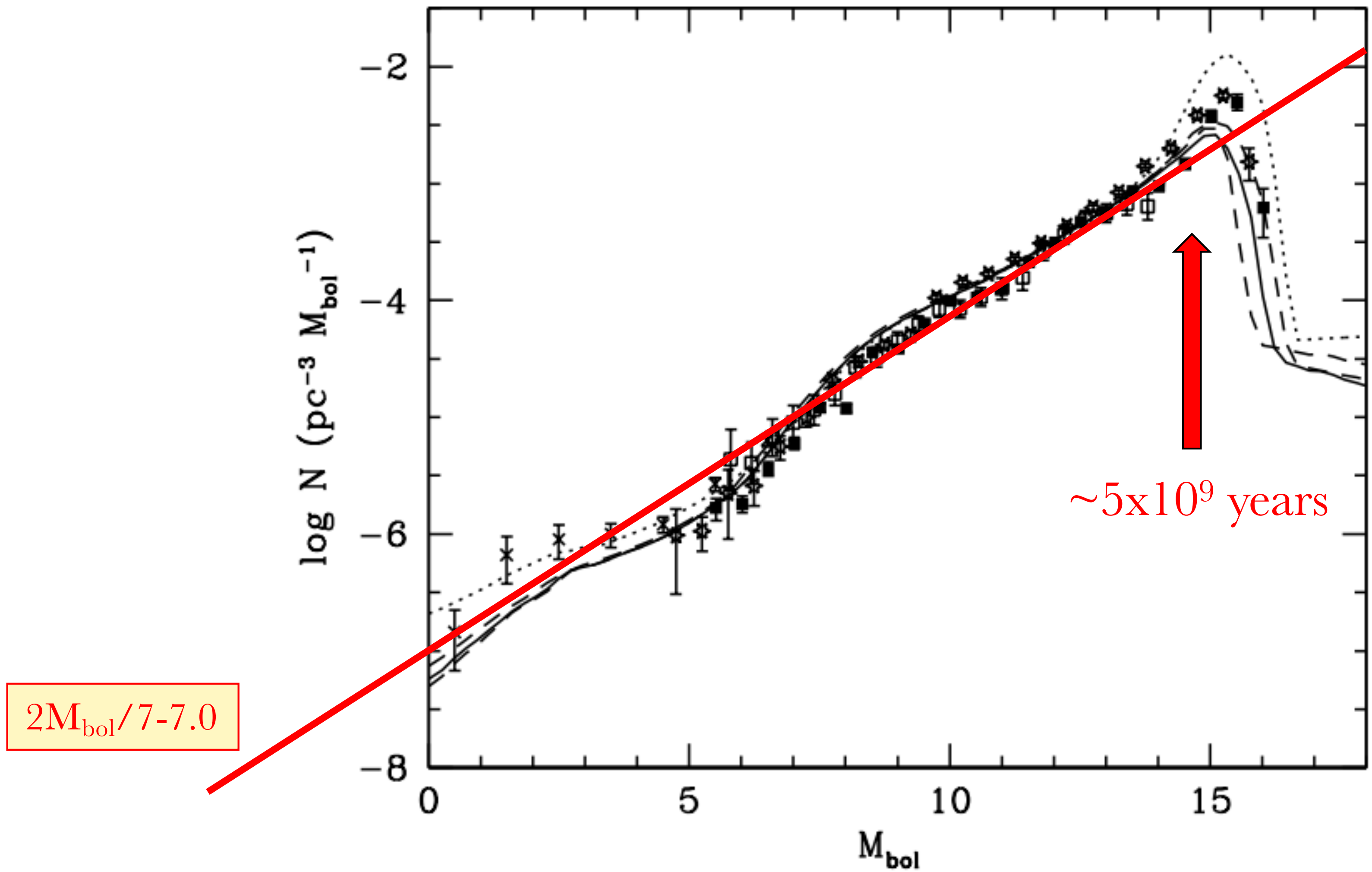
Plot: White Dwarf Cooling Curves and Luminosity Function

• Observation of such a cutoff in the Milky Way disk white dwarf population leads to an age estimate of

$$t_{disk} \sim (9.3 \pm 2.0) \times 10^{9} \text{ years} \quad . \tag{10}$$

White Dwarf Luminosity Function

(Isern, Artigas & García-Berro, EPJ 43, 05002, 2013)



L α t$^{-7/5}$

- Large survey data for the white dwarf luminosity function. Models for curves comprise different Galactic disk descriptions and star formation rates.

3.5 Radio-Isotope Dating

• Radio-isotope dating is common practice in archaeology and also in the study of galactic cosmic rays. The essential guide is to select elements with decay times on the order of the timescale of interest for a particular problem — in this case the Hubble expansion time. **Longair, pp. 214**

This analysis requires more than one decay chain because of the unknown initial conditions and chemical composition. **Peebles, pp. 104-6**

∗ **Uranium fits the bill!**

$$\begin{array}{cccc} (\text{mother} & \longrightarrow & \text{daughter}) & \\ {}^{238}U & \xrightarrow[\text{series}]{\text{radium}} & {}^{206}Pb & (\text{e-life} = t_{238}) \\ {}^{235}U & \xrightarrow[\text{series}]{\text{actinium}} & {}^{207}Pb & (\text{e-life} = t_{235}) \end{array} \tag{11}$$

Note that ${}^{208}Pb$ comes from ${}^{252}Th$. The measured e-lifes for the two series are $t_{235} = 1.015 \times 10^9$ years and $t_{238} = 6.45 \times 10^9$ years.

If $t = 0$ initially at the epoch of *chemical* mixing and we are at time t now, then

$$\begin{aligned} {}^{238}U(t) &= \exp\left(-\frac{t}{t_{238}}\right) {}^{238}U(0) \\ {}^{235}U(t) &= \exp\left(-\frac{t}{t_{235}}\right) {}^{235}U(0) \end{aligned} \tag{12}$$

leading to equations for the various isotopes of lead as end-products:

$$\begin{aligned} {}^{206}Pb(t) &= {}^{206}Pb(0) + \left\{\exp\left(+\frac{t}{t_{238}}\right) - 1\right\} {}^{238}U(t) \\ {}^{207}Pb(t) &= {}^{207}Pb(0) + \left\{\exp\left(+\frac{t}{t_{235}}\right) - 1\right\} {}^{235}U(t) \end{aligned} \tag{13}$$

• Comparing at least two different meteorites *and/or* Earth samples leads to the elimination of ${}^{206}Pb(0)/{}^{204}Pb(t)$ and ${}^{207}Pb(0)/{}^{204}Pb(t)$ (${}^{204}Pb(t)$ is the stable isotope reference which dominates the sample) when subtracting the abundances in the two meteorites (A + B).

The *key assumption* is that different samples start out with different U/Pb *chemical* abundances but identical *nuclear* isotope ratios. Hence,

$$\frac{{}^{207}Pb(t)\Big|_A - {}^{207}Pb(t)\Big|_B}{{}^{206}Pb(t)\Big|_A - {}^{206}Pb(t)\Big|_B} = \frac{{}^{235}U(t)}{{}^{238}U(t)}\,\frac{e^{t/t_{235}}-1}{e^{t/t_{238}}-1} \quad . \tag{14}$$

The age can then be determined, using the measured contemporary abundance ratio ${}^{235}U(t)/{}^{238}U(t) = 0.00725$:

$$t_U \;\sim\; (4.6 \pm 0.1) \times 10^9 \text{ years} \quad . \tag{15}$$

This result is consistent with checks using the products of ${}^{40}K \rightarrow {}^{40}Ar$ and ${}^{87}Rb \rightarrow {}^{87}Sr$ decays.

• **As model-dependent checks**, we can borrow from a stellar nucleosynthesis model, such as that due to Burbidge, Burbidge, Fowler & Hoyle (1957) of the r and s processes; the pure r process guarantees greater reliability.

∗ Then, one can use the decay formulae in Eq. (12) plus the present abundance ratio ${}^{235}U(t)/{}^{238}U(t) = 0.00725$ to deduce

$$t_{rs} \;\sim\; 6.5 \times 10^9 \text{ years} \quad . \tag{16}$$

Other checks include the ${}^{232}Th/{}^{238}U$ ratio; see Cowan et al. (*Ann. Rev. Astr. Astrophys.*, 1991 p. 447) for a summary.

• Such calculations can be made more realistic by porting in a time-dependent chemical evolution model that incorporates star formation and associated elemental depletions, and accretion and outflow from star formation environements. These result in the general **Tinsley equation**.

4 Newtonian Cosmology: Expansion/Collapse

For a baseline description of the universe, assume isotropy and homogeneity in an expanding (comoving) sphere of radius $R(t)$. The relevant mass is

C & O, Sec. 29.1

$$M_g = \frac{4\pi}{3} \left(\rho + \frac{3P}{c^2} \right) R^3 \tag{17}$$

for a sphere. Hereafter, we set $c = 1$ so that $3P/c^2 \to 3P$. Pressure from matter motions contributes in a special relativistic sense via $E = mc^2$. Newtonian gravity establishes

Longair, pp. 155–7

$$\frac{d^2R}{dt^2} = -\frac{GM_g}{R^2} = -\frac{4\pi}{3} G\,(\rho + 3P)\,R \quad , \tag{18}$$

which essentially comes from the field equation, i.e. Poisson's equation $\nabla^2 \phi = 4\pi G \rho$. In general, $R = R(t)$, $\rho = \rho(t)$ and $P = P(t)$.

$*$ N.B. Birkhoff's theorem (G.R.) implies that external matter exerts no gravitational force.

• Integration of this equation of motion is only possible by imposing mass and energy conservation: Assuming the ideal gas equation of state gives an energy content

$$U = \rho V \quad , \tag{19}$$

in units of $c = 1$, where the pressure contribution to the total internal energy has been assumed to be small (i.e. $T = T_{\rm NR}$). Energy conservation (1st law of thermodynamics) gives [using Eq. (19)]

$$dU = -PdV = \rho dV + V d\rho \quad . \tag{20}$$

This can then be rearranged to yield the time derivatives

$$\dot{\rho} = -(\rho + P)\,\frac{\dot{V}}{V} = -3(\rho + P)\,\frac{\dot{R}}{R} \quad . \tag{21}$$

Hence, eliminating P, Newtonian gravity plus the ideal gas equation and energy conservation yields

$$\frac{d^2R}{dt^2} = \frac{8\pi}{3} G\rho R + \frac{4\pi}{3} G\dot{\rho}\frac{R^2}{\dot{R}} \quad . \tag{22}$$

Multiplying through by $\dot{R}$ gives a perfect differential, so that integration yields the Newtonian version of Friedmann's equation: **Peacock, p. 73**

$$\boxed{\left(\dot{R}\right)^2 - \frac{8\pi}{3}\,G\rho R^2 \;=\; \text{const.}} \qquad (23)$$

for $\rho = \rho(t)$. This is the basic equation of expansion/contraction for a Newtonian cosmology, with the constant not determined.

∗ Note that we get the constant from general relativity: no coupling between the field and spacetime curvature is made in a Newtonian formalism. Note also that the value of the constant will establish whether $\dot{R} \to 0$ as $R \to$ const (i.e. we have a bound universe) or $R \to \infty$.

∗ N.B. Setting $\dot{R} = H(t)\,R$ yields an expansion rate that couples only to the mean density of the sphere at a given time, i.e. $H(t) = H(\rho[t])$. Thus, the Hubble constant is defined for any epoch, not just the present one.

• The physics of the Newtonian cosmology is identical to that of the structure formation problem, i.e. it can be applied to collapse of gas clouds and galaxies. Setting the constant to zero, and assuming constant density in time (appropriate for perturbative structure formation),

$$\frac{\dot{R}}{R} \;=\; \pm\sqrt{\frac{8\pi}{3}\,G\rho} \quad , \qquad (24)$$

which solves for collapse as

$$R \;=\; R_0 \,\exp\left\{-\frac{t}{t_g}\right\} \quad , \quad t_g \;=\; \sqrt{\frac{3}{8\pi G\rho}} \quad . \qquad (25)$$

This is exponential collapse on a pre-established Newtonian timescale. There is also an exponential growth solution, and so a real Universe may sample both these pieces of information.

∗ The constant density assumption most closely mimics the effect of vacuum energy, but does not apply to matter-dominated or radiation-dominated cosmologies: *the choice of cosmology influences the temporal dependence.*

For a more representative density scaling $\rho(t) \propto R^{-3}$ corresponding to domination of the dynamics by matter, the solution of the ODE is

$$\boxed{R \;=\; R_0 \left(\frac{3t}{2t_g}\right)^{2/3} \quad ,} \tag{26}$$

where t_g is given as in Eq. (25) but with $\rho \to \rho_{\rm now}$. This defines the temporal dependence of the universe at epochs after a relativistic equation of state ceases to be applicable (i.e. after equipartition with radiation).

∗ One can quickly identify that $H_0 \equiv \dot{R}/R = 1/t_g$ gives an expression for the Hubble constant, and moreover that $2/(3H_0)$ is the age of a universe subject to this choice of cosmology.

• The **sign of the constant** in Eq. (23) (statement of its value is deferred to the treatment of relativistic cosmology) is critical to defining the character of the universe. The constant is essentially the **total energy per unit mass**. **C & O, p. 1148**

∗ If the constant is *negative*, then the $G\rho R^2$ term must always exceed the $(\dot{R})^2$ term, and so ρ cannot drop to arbitrarily small values, i.e. R cannot tend unbounded to infinity. This universe is **closed**, i.e. eventually stops expanding. It is bounded and has a *negative total energy*.

∗ If the constant is *positive*, then the $(\dot{R})^2$ term must always exceed the $G\rho R^2$ term, and as $R \to \infty$ and $\rho \to 0$, the universe eventually coasts at constant $\dot{R} > 0$, i.e. it is **open**, and has a *positive total energy* (unbounded).

∗ The in-between, marginal case of zero constant, expands forever, but constantly decelerates at progressively slower rates.

• By the Cosmological Principle, the sphere of radius R can be any portion of the universe, so that physical spatial scales can be removed. Hence, $R(t)$ is often designated $a(t)$, and is called the **scale parameter**.

∗ Arbitrarily, at the present epoch, we choose $a \equiv a_0 = 1$, so that

$$a(t) \;=\; \frac{a_0}{1+z} \;\equiv\; \frac{1}{1+z} \tag{27}$$

defines the redshift dependence of scale.