



Geoffrey Baring pushes his bike through snow while making his commute to classes Tuesday, Jan. 14, 2025, as snow accumulates in Lafayette, Ind.

2.2 Planck Spectrum

The actual shape of the blackbody spectrum was explained by Planck, who found an empirical formula

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$$B_\lambda(T) = \frac{a}{\lambda^5 (e^{b/\lambda T} - 1)} \quad , \quad (10)$$

where $B_\lambda(T) d\lambda dA_\perp d\Omega$ is the radiation energy per unit area ($dA_\perp = dA \cos \theta$), per unit wavelength $d\lambda$, per unit solid angle $d\Omega$.

* The units of the **specific intensity** $B_\lambda(T)$ are $\text{erg cm}^{-3} \text{sec}^{-1} \text{sr}^{-1}$, or often, to avoid unit confusion, $\text{erg cm}^{-2} \text{\AA}^{-1} \text{sec}^{-1} \text{sr}^{-1}$.

Plot: B_λ specific intensity geometry

• Planck proposed that the waves could only have discrete, integer multiples of a minimum wave energy. The quantum of minimum energy was

$$h\nu = \frac{hc}{\lambda} \quad \Rightarrow \quad \varepsilon_\gamma = \frac{nhc}{\lambda} \quad . \quad (11)$$

Since the volume scales as λ^3 with $\lambda = c/\nu$, the isotropic phase space density scales as $d^3\nu \propto \nu^2 d\nu$, so that

$$dN = \frac{8\pi V}{c^3} \nu^2 d\nu \quad (12)$$

is the number of modes. This must be combined with thermal statistics of a Bose gas: occupation numbers are exponential Boltzmann factors, and are summed over all states:

$$\sum_{n=1}^{\infty} e^{-\varepsilon_\gamma/kT} \equiv \sum_{n=1}^{\infty} e^{-nhc/\lambda kT} \quad . \quad (13)$$

These ingredients led Planck to the semi-quantum formula:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5 (e^{hc/\lambda kT} - 1)} \quad . \quad (14)$$

This is the famous **Planck spectrum** of blackbody radiation. The value of **Planck's constant** h was determined to be

$$h = 6.626 \times 10^{-27} \text{erg sec} \quad , \quad (15)$$

and the quantum theory of light was established by Einstein's analysis of the photoelectric effect.

B_λ Specific Intensity Geometry

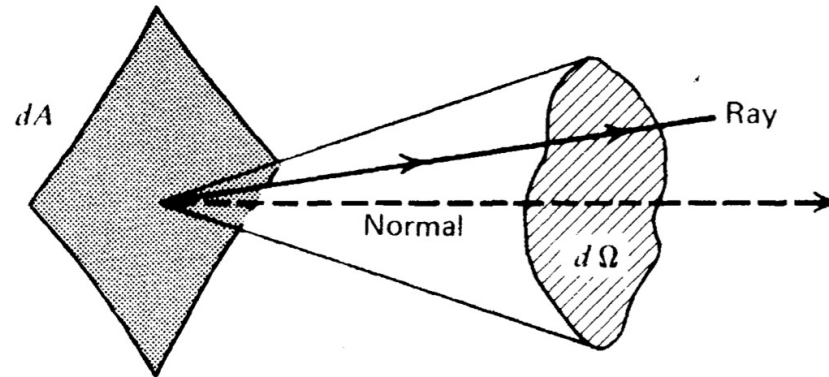


Figure 1.2 Geometry for normally incident rays.

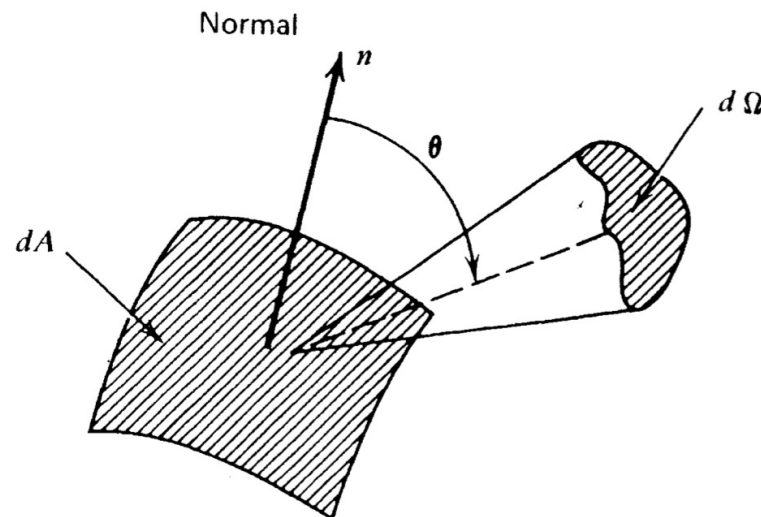


Figure 1.3 Geometry for obliquely incident rays.

In frequency space, the Planck spectrum is

$$B_\nu(T) = \frac{2h\nu^3}{c^2(e^{h\nu/kT} - 1)} \quad . \quad (16)$$

Plot: Blackbody spectrum in ν space

- One can quickly identify the Wien high energy form and the Raleigh-Jeans tail from the spectrum. What is the peak of $B_\lambda(T)$? This can be found by setting $\chi = hc/(\lambda kT)$, we have

$$\frac{d \log B_\lambda(T)}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d}{d\chi} \left\{ \frac{\chi^5}{e^\chi - 1} \right\} = 0 \quad \Rightarrow \quad \chi = 4.965 \quad . \quad (17)$$

It follows that

$$\lambda_{\max} = \frac{0.2898}{T(^{\circ}K)} \text{ cm} \quad . \quad (18)$$

The Planck spectrum integrates over frequencies to yield the total or **bolometric** flux:

$$\mathcal{F} = \pi \int_0^\infty B_\nu(T) d\nu = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} = \frac{2\pi(kT)^4}{c^2 h^3} \int_0^\infty \frac{\chi^3 d\chi}{e^\chi - 1} \quad . \quad (19)$$

Note that the factor of π represents the flux-weighting factor $\cos\theta$ integrated over a hemisphere, giving $2\pi \times 1/2$.

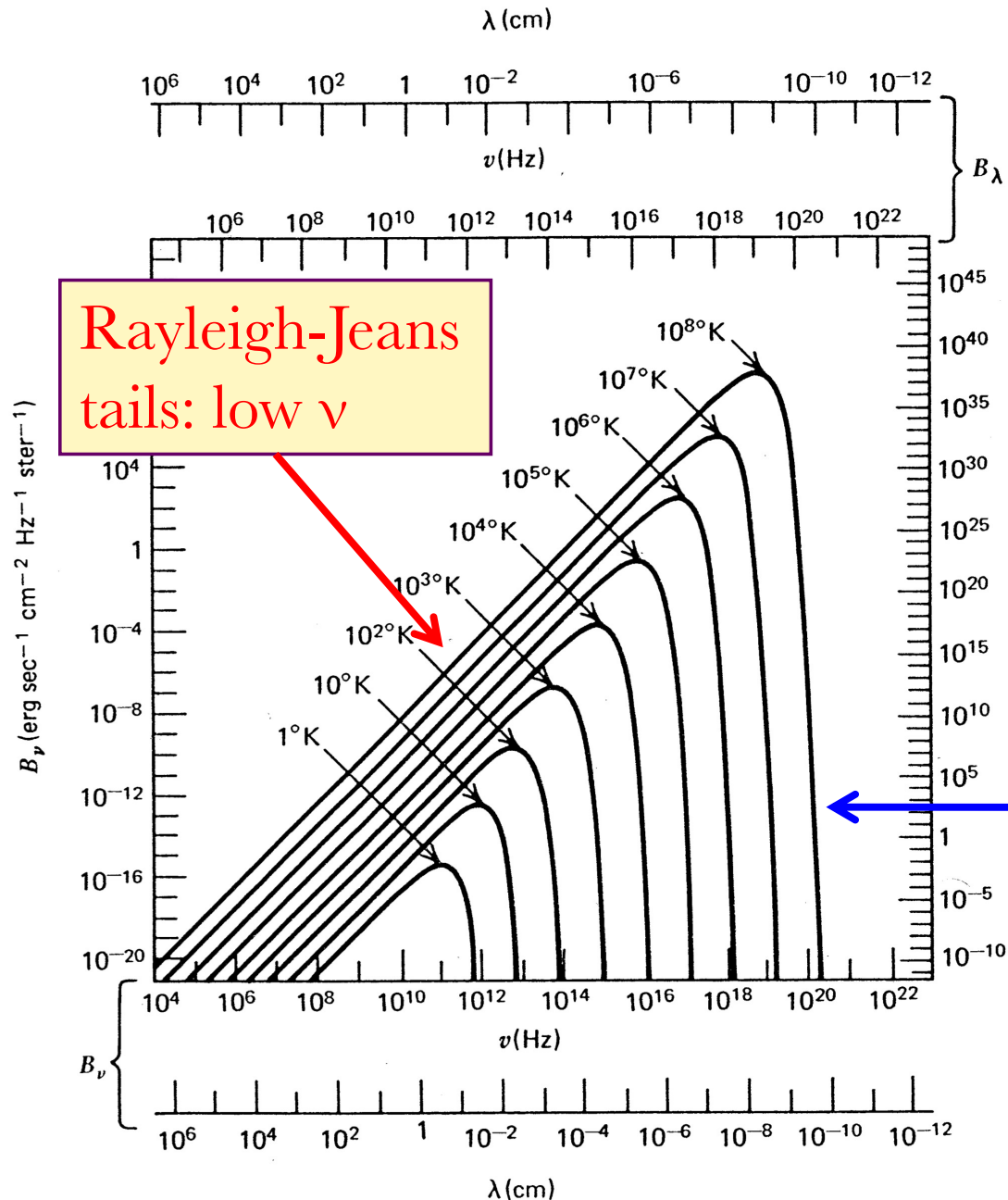
The integral evaluates to $\pi^4/15$, reproducing the Stefan-Boltzmann law:

$$\mathcal{F} = \sigma T^4 \quad , \quad \sigma = \frac{2\pi^5 k^4}{15c^2 h^3} = 5.67 \times 10^{-5} \text{ erg sec}^{-1} \text{ cm}^{-2} \text{ K}^{-4} \quad , \quad (20)$$

with the Stefan-Boltzmann constant expressed purely in terms of fundamental constants.

- The determination of stellar spectra leads to an evaluation of the effective temperature T_e of a star, and hence through the Stefan-Boltzmann law $L_* = 4\pi R_*^2 \sigma T_e^4$ to a determination of stellar radii.

Planck Spectra: ν Representation



- Spectra of **blackbody radiation** at various temperatures. Taken from the book by [Kraus, J. D. *Radio Astronomy*](#) (1966, McGraw-Hill Book Company)

Wien tails: high ν

3 Atomic Spectroscopy and the Bohr Model

C & O,
Sec. 5.3

Profound developments were made in atomic physics around the end of the 19th century and the beginning of the 20th century, including the discovery of the electron (Thomson, 1880s), the neutrality of atoms, the comparatively small size of the positive nucleus in atoms (Rutherford, 1911).

- In 1885, Balmer discovered an empirical relationship for atomic lines in hydrogen:

$$\boxed{\frac{1}{\lambda} = R_H \left(\frac{1}{4} - \frac{1}{n^2} \right)}, \quad n = 3, 4, 5, \dots \quad (21)$$

Here $R_H = 1.0968 \times 10^5 \text{ cm}^{-1}$ is the **Rydberg constant**.

- * $n = 3 \rightarrow H_\alpha$ Balmer line; $n = 4 \rightarrow H_\beta$ Balmer line, etc.

Balmer *predicted* a more general relationship for non-optical lines:

$$\boxed{\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right)}, \quad n = m + 1, m + 2, m + 3, \dots \quad (22)$$

- Today $m = 1$ denotes Lyman lines, typically in the UV, and $m = 3$ denotes the Paschen series that appear in the IR.

In classical electromagnetic theory, any orbital model of an $e + p$ atom is basically radiatively unstable on timescales of 10^{-12} seconds, and generating a continuous spectrum that conflicts with observed spectral lines.

- In 1911, Niels Bohr postulated a semi-classical model of the atom in which *angular momentum was quantized*:

$$\boxed{L \equiv \mu v r = \frac{nh}{2\pi} = n\hbar} \quad (23)$$

for $E = nh\nu$. Here $\mu = m_e m_p / (m_e + m_p) \approx m_e$ is the reduced mass (for hydrogen).

- Coupling Bohr's idea of angular momentum quantization to classical centripetal dynamics yields

$$r = r_n \equiv \frac{\hbar^2}{\mu e^2} n^2 = a_0 n^2 \quad , \quad (24)$$

where $a_0 = 5.29 \times 10^{-9} \text{ cm}$ is called the **Bohr radius**; it is the natural atomic scale, and is approximately equal to the electron Compton wavelength divided by $\alpha_f = e^2/(\hbar c)$.

* Bohr atomic radii are quantized!

- The *total energy is then also quantized*:

$$E_{\text{tot}} = -\frac{\mu e^4}{2\hbar^2} \frac{1}{n^2} \approx -\frac{13.6\text{eV}}{n^2} \quad , \quad (25)$$

and this gives the binding energy of hydrogen.

The scale $\mu e^4/(2\hbar^2)$ is often referred to as the **Rydberg energy**, the cutoff energy in the UV for the photoelectric effect for hydrogen.

* n is known as the **principal quantum number** in Bohr's theory.

- Transitions between levels $n > m$ lead to photon energies $h\nu = E_n - E_m$ or

$$\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \quad , \quad (26)$$

with

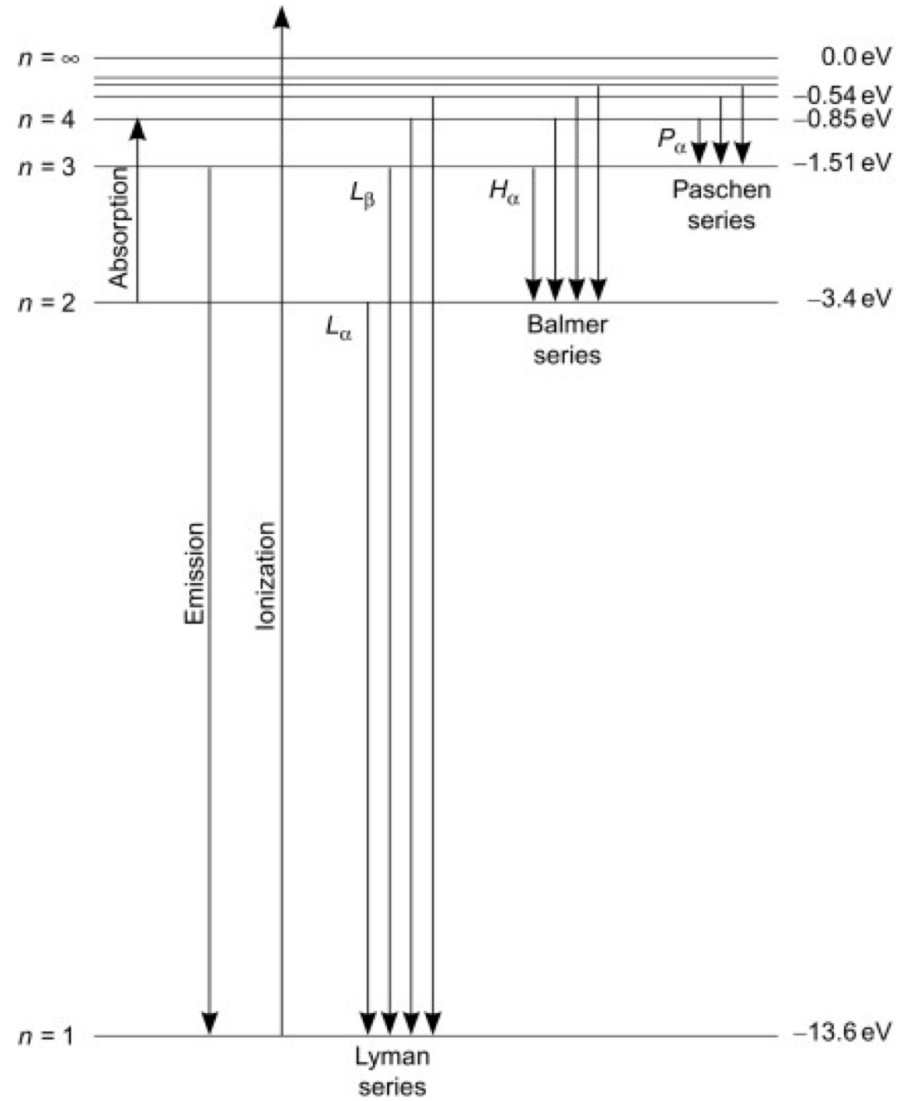
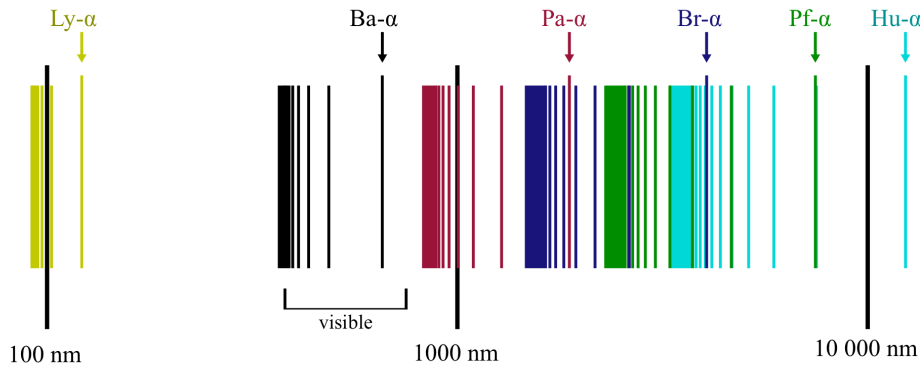
$$R_H = \frac{\mu e^4}{4\pi\hbar^3 c} = 1.0968 \times 10^5 \text{ cm}^{-1} \quad . \quad (27)$$

Hence, we have recovered an expression for the Rydberg constant in terms of fundamental constants.

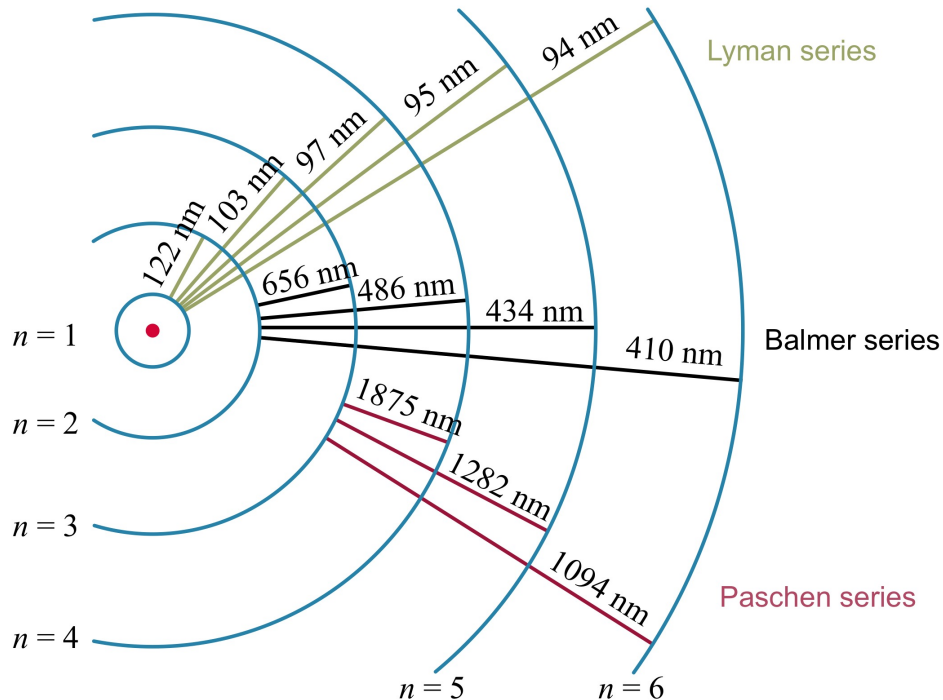
- The excellent agreement between Bohr's model and Balmer's observations were a strong vindication, despite Bohr's ideas not being truly quantum.

Plot: Bohr atomic transitions

Bohr Atom Transitions



Credits: Wikimedia commons + Science Direct



- Energy level and wavelength diagrams for the hydrogen atom.

4 Preliminaries: Kepler's Laws (1609)

Celestial mechanics within and outside the solar system are governed by Kepler's three key laws of orbital motion:

C & O,
Sec. 2.1

- I. Planets move about the sun in elliptical orbits with the sun at one focus.
- II. The radius vector of a planet sweeps out equal areas in equal times:

$$\frac{dA}{dt} = \frac{L}{2\mu} = \frac{\mathcal{J}}{2} = \text{const.} \quad , \quad L = |\mathbf{L}| \quad , \quad (28)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the planet and the sun. Herein, $\mathcal{J} = L/\mu$ is referred to as the **specific angular momentum**.

Plot: Equal area elliptical diagram

- III. If a is the semi-major axis, and P is the orbital period, then the general form of **Kepler's Third Law** for eccentricities $e \neq 0$ is

$$\boxed{\frac{a^3}{P^2} = \frac{GM}{4\pi^2}} \quad (29)$$

This law follows from his second law, i.e. by integrating Eq. (28) over an entire period, giving $A = LP/(2\mu)$. The angular momentum L is then eliminated using the gravitational force law either along the major or minor axes of the elliptical orbit.

* Here $M = m_1 + m_2$ so that the total mass of the system determines the period. This law is used ubiquitously in astrophysics as a means of mass determination.

Plot: Kepler III: observational evidence

Q. Why are these laws valid?

A. Conservation of energy and angular momentum in an inverse square (gravitational) 2 body force law.

* 3+ body dynamics destroys the symmetry underpinning Kepler's Laws.

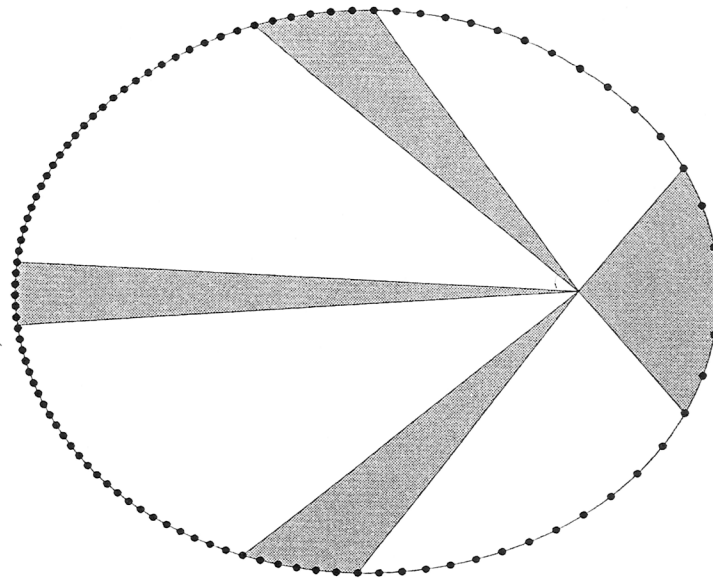


Figure 2.2 Kepler's second law states that the area swept out by a line between a planet and the focus of an ellipse is always the same for a given time interval, regardless of the planet's position in its orbit. The dots are evenly spaced in time.

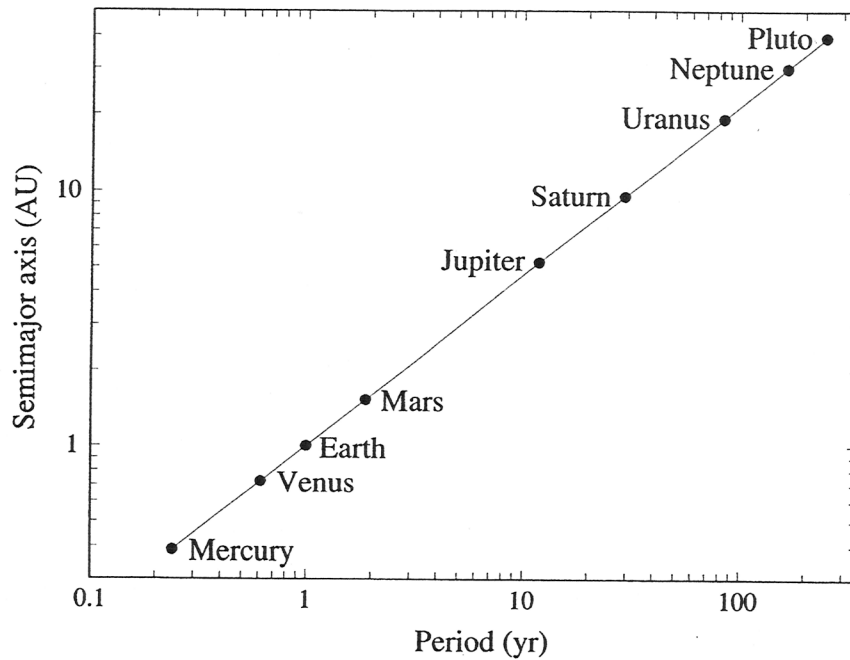


Figure 2.3 Kepler's third law for planets orbiting the Sun.

5 The Hertzsprung-Russell Diagram

The advances in spectroscopy afforded by gratings enabled more precise quantification and classification of stars, leading to the spectral categorizations O,B,A,F,V,G,K,M that correlated with blackbody temperatures.

Evidence that O stars were more massive than M stars, inferred from binary systems over a century ago, led to an incorrect evolutionary scenario:

**C & O,
Sec. 8.2**

- * that as stars age, they were thought to shed mass and exhaust their fuel before growing into old, red dim stars. This is now known as the early-type vs. late-type misnomer.

- Hertzsprung in 1905, and Russell in 1913, observed a correlation between absolute magnitude M and spectral type that supported such a contention.

- * 80% – 90% of stars reside in a diagonal band called the **main sequence**.

- * A handful of bright, redder stars with similar spectral types but much more negative M are called (red) giants. Their larger size is inferred from the Stefan-Boltzmann law:

$$R = \frac{1}{T_{\text{eff}}^2} \sqrt{\frac{L}{4\pi\sigma}} \quad . \quad (30)$$

- Russell’s first plot was the forerunner of the modern **Hertzsprung-Russell (H-R) Diagram**, which can take either an observational (M_V vs. B-V) or theoretical (L vs. T) form.

Plot: Hertzsprung-Russell Diagram in the Gaia Era

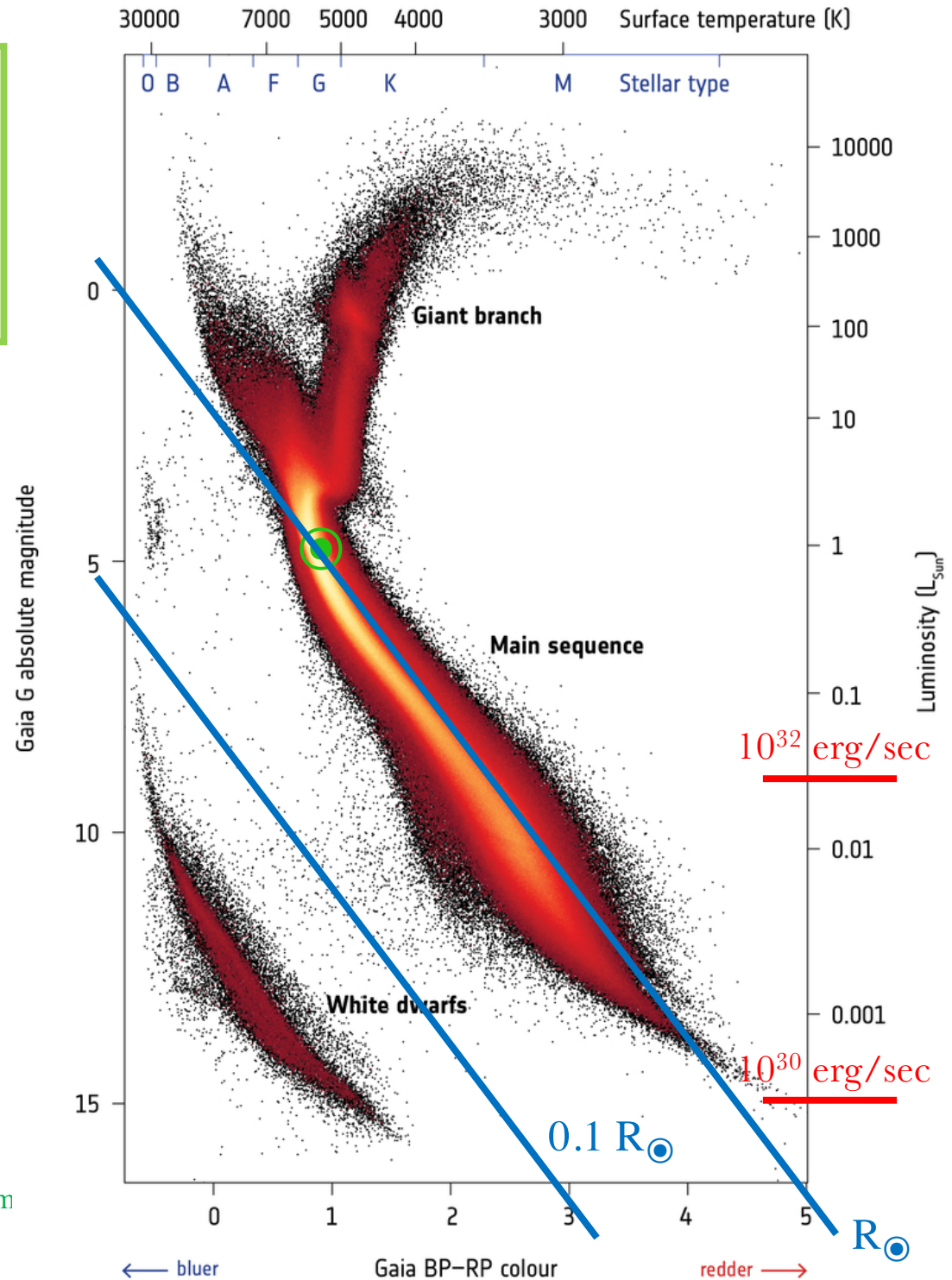
The radius of stars is easily inferred from the theorist’s plot via diagonal “iso-radius” lines.

- * It is easily inferred that supergiants such as Betelgeuse ($M \sim 10M_{\odot}$) are very tenuous, and white dwarfs such as 40 Eridani B are very dense.

Hertzsprung-Russell Diagram: Gaia Era

- The H-R diagram for Gaia's Data Release 2 (DR2) in 2018.
- From the ESA Science and Technology site.

<https://sci.esa.int/web/gaia/-/60198-gaia-hertzsprung-russell-diagram>



1.1 Units Choice for Electrodynamicists

There are two main conventions for units generally adopted in physics: SI units (aka *MKSA* units) and **Gaussian units** (aka *cgs* units). Throughout the course, *we will adopt Gaussian units because of their convenience*, particularly for relativistic considerations. This is the choice of Landau & Lifshitz, and also for large portions of Jackson's book.

- It is simple to convert between the two for electromagnetic problems, using the following prescriptions for Gaussian to SI units:

$$\begin{aligned} \text{Gaussian} &\rightarrow \text{SI} \\ \text{charge } q &\rightarrow \frac{1}{\sqrt{4\pi\epsilon_0}} q \\ \text{electric field } \vec{E} &\rightarrow \sqrt{4\pi\epsilon_0} \vec{E} \\ \text{magnetic field } \vec{B} &\rightarrow \sqrt{\frac{4\pi}{\mu_0}} \vec{B} \\ \text{speed of light } c &\rightarrow \frac{1}{\sqrt{\epsilon_0\mu_0}} \end{aligned}$$

With these correspondences, one can quickly determine the conversion of any of Maxwell's equations. Focusing on Gauss' equation for electrostatics, i.e. the Poisson equation for the electric potential, one has

$$\nabla^2\phi = \nabla \cdot \vec{E} = 4\pi\rho \quad \rightarrow \quad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad (2)$$

since $\rho \propto q$. Obviously, $\nabla \cdot \vec{B} = 0$ is invariant in form in translating between Gaussian and SI units. The Lorentz force equation obeys the correspondence

$$\frac{d\vec{p}}{dt} = q \left\{ \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right\} \quad \rightarrow \quad \frac{d\vec{p}}{dt} = q \left\{ \vec{E} + \vec{v} \times \vec{B} \right\} \quad . \quad (3)$$

Clearly in Gaussian units, E and B have identical dimensions, contrasting the SI case where they are scaled by a speed. This highlights an attractive convenience of the Gaussian system.

- *Anecdote:* Green eggs and ham (Seuss) – you will grow to like cgs!