TEAM TEST
2006 RICE MATH TOURNAMENT
FEBRUARY 25, 2006

1. Given \( \triangle ABC \), where \( A \) is at \((0,0)\), \( B \) is at \((20,0)\), and \( C \) is on the positive \( y \)-axis. Cone \( M \) is formed when \( \triangle ABC \) is rotated about the \( x \)-axis, and cone \( N \) is formed when \( \triangle ABC \) is rotated about the \( y \)-axis. If the volume of cone \( M \) minus the volume of cone \( N \) is \( 140\pi \), find the length of \( BC \).

2. In a given sequence \( \{S_1, S_2, \ldots, S_k\} \), for terms \( n \geq 3, S_n = \sum_{i=1}^{n-1} i \cdot S_{n-i} \). For example, if the first two elements are 2 and 3, respectively, the third entry would be \( 1 \cdot 3 + 2 \cdot 2 = 7 \), and the fourth would be \( 1 \cdot 7 + 2 \cdot 3 + 3 \cdot 2 = 19 \), and so on. Given that a sequence of integers having this form starts with 2, and the 7th element is 68, what is the second element?

3. A triangle has altitudes of lengths 5 and 7. What is the maximum possible integer length of the third altitude? (We restricted the third altitude to integer lengths after the contest)

4. Let \( x + y = a \) and \( xy = b \). The expression \( x^6 + y^6 \) can be written as a polynomial in terms of \( a \) and \( b \). What is this polynomial?

5. There exist two positive numbers \( x \) such that \( \sin(\arccos(\tan(\arcsin x))) = x \). Find the product of the two possible \( x \).

6. The expression \( 16^n + 4^n + 1 \) is equivalent to the expression \( (2^p(n) - 1)/(2^q(n) - 1) \) for all positive integers \( n > 1 \) where \( p(n) \) and \( q(n) \) are functions and \( \frac{p(n)}{q(n)} \) is constant. Find \( p(2006) - q(2006) \).

7. Let \( S \) be the set of all 3-tuples \((a,b,c)\) that satisfy \( a + b + c = 3000 \) and \( a, b, c > 0 \). If one of these 3-tuples is chosen at random, what’s the probability that \( a, b, \) or \( c \) is greater than or equal to 2,500?

8. Evaluate: \( \lim_{n \to \infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} \).

9. \( \triangle ABC \) has \( AB = AC \). Points \( M \) and \( N \) are midpoints of \( \overline{AB} \) and \( \overline{AC} \), respectively. The medians \( \overline{MC} \) and \( \overline{NB} \) intersect at a right angle. Find \( \left( \frac{AB}{MC} \right)^2 \).

10. Find the smallest integer \( m > 8 \) for which there are at least eleven even and eleven odd positive integers \( n \) so that \( n^3 + m \) is an integer. (We restricted the solution to \( m > 8 \) after the contest since \( m = 8 \) is a trivial solution, with \( n^3 + 8 \) divisible by \( n + 2 \)).

11. Polynomial \( P(x) = c_{2006}x^{2006} + c_{2005}x^{2005} + \ldots + c_1x + c_0 \) has roots \( r_1, r_2, \ldots, r_{2006} \). The coefficients satisfy \( 2i \cdot \frac{c_i}{c_{2006-i}} \) for all pairs of integers \( 0 \leq i, j \leq 2006 \). Given that \( \sum_{i=1}^{2006} \frac{c_i}{r_j} = 42 \), determine \( \sum_{i=1}^{2006} (r_1 + r_2 + \ldots + r_{2006}) \).

12. Find the total number of \( k \)-tuples \((n_1, n_2, \ldots, n_k)\) of positive integers so that \( n_{i+1} \geq n_i \) for each \( i \), and \( k \) regular polygons with numbers of sides \( n_1, n_2, \ldots, n_k \) respectively will fit into a tesselation at a point. That is, the sum of one interior angle from each of the polygons is \( 360^\circ \).

13. A ray is drawn from the origin tangent to the graph of the upper part of the hyperbola \( y^2 = x^2 - x + 1 \) in the first quadrant. This ray makes an angle of \( \theta \) with the positive \( x \)-axis. Compute \( \cos \theta \).

14. Find the smallest nonnegative integer \( n \) for which \( \binom{2006}{n} \) is divisible by \( 7^3 \).
15. Let \( c_i \) denote the \( i \)th composite integer so that \( \{c_i\} = 4, 6, 8, 9, \ldots \). Compute

\[
\prod_{i=1}^{\infty} \frac{c_i^2}{c_i^2 - 1}.
\]

(Hint: \( \sum_{i=1}^{n} \frac{1}{n^2} = \frac{\pi^2}{6} \)).