1. Answer: $\frac{9}{49}$

$$P(\text{True}|\text{Droop}) = \frac{P(\text{True})P(\text{Droop}|\text{True})}{P(\text{Droop})}$$

$$= \frac{\frac{9}{25} \cdot \frac{1}{2} \cdot \frac{9}{2}}{\frac{9}{25} + \frac{1}{2} \cdot \frac{9}{2}} = \frac{9}{49}$$

2. Answer: 3125

$$(2000 + 5)^{2005} = 2000^k \cdot 5^m + 5^{2005}$$

Since $2000^k \cdot 5^m$ is divisible by 10,000, we want to find the last digits of $5^{2005}$. [The following are all mod 10,000.]

$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$
$5^5 = 3125$
$5^6 = 5625$
$5^7 = 8125$
$5^8 = 625 = 5^4$

...so these repeat every four starting with $5^3$.

$$(2005 - 3) \mod 4 = 2002 \mod 4 = 2$$

Therefore, $5^{2005} = 3125$.

3. Answer:

4. Answer: $\frac{7}{2} - \frac{\sqrt{3}}{2}$

The square projects out of the hexagon on top and bottom in isosceles right triangles. The area of the hexagon is $3s^2\sqrt{3}$ where $s = 1$. Each triangle is easiest dealt with as two triangles: $A = \frac{3\sqrt{3}}{2} + 2 \cdot 2 \cdot \frac{1}{2} (1 - \frac{\sqrt{3}}{2})^2 = \frac{3\sqrt{3}}{2} + 2 - 2\sqrt{3} + \frac{3}{2} = \frac{7}{2} - \frac{\sqrt{3}}{2}$

5. Answer: 500

$$\left\lfloor \frac{2005}{5} \right\rfloor = 401$$
$$\left\lfloor \frac{401}{5} \right\rfloor = 80$$
$$\left\lfloor \frac{80}{5} \right\rfloor = 16$$
$$\left\lfloor \frac{16}{5} \right\rfloor = 3$$
$$\left\lfloor \frac{3}{5} \right\rfloor = 0$$

$401 + 80 + 16 + 3 = 500$
6. Answer: $\frac{\sqrt{2005 \cdot 2009} - 2005}{2}$

Let the continued fraction be $x$.

$$x = \frac{2005}{2005 + x}$$

$$x^2 + 2005x - 2005 = 0$$

$$x = \frac{-2005 \pm \sqrt{2005^2 + 4 \cdot 2005}}{2}$$

$$x = \sqrt{2005 \cdot 2009} - 2005$$

Note: (-) is dropped since clearly positive.

7. Answer: $\frac{31}{30}$ muffins

Let $\phi(n) = \text{number of integers relatively prime to } n$.

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

where $p_1, \ldots, p_k$ are distinct primes dividing $n$.

We need $\phi(n) = 4$. So $5, 8, 10, 12$ are the only solutions to this equation.

Thus your expected winnings are

$$\frac{1}{20} - \frac{1}{5} \cdot \frac{3}{8} + \frac{3}{10} - \frac{1}{4} = \frac{72 - 6 + 15 + 36 - 18 + 30 - 20}{120} = \frac{124}{120} = \frac{31}{30}$$

8. Answer: $\frac{19}{20}$

If $P(x) = \sum_{i=0}^{n} c_i x^i$, then $P(x) - P(y) = \sum_{i=0}^{n} c_i (x^i - y^i)$. Note that $x^i - y^i$ is divisible by $x - y$. If $x - y \geq 2$, then $P(x) - P(y)$ will be composite. Since the degree is at least 2, $P(x) - P(y) > c_2 (x^2 - y^2) = c_2 (x+y)(x-y)$. Note that $x+y > 1$, so $\frac{P(x) - P(y)}{x-y}$ is an integer larger than 1. So we only need $x - y \geq 2$. There are 780 total pairs $(x, y)$. All will work except $(x, y) = (2, 1), (3, 2), \ldots, (40, 39)$. The answer is $\frac{780 - 39}{780} = \frac{19}{20} = \frac{19}{20}$.

The answer is $\sum_{n \in S_{mk}} n = \sum_{i=0}^{k-1} m + i$

$$2005 = \frac{(k - 1)(k) + km}{2}$$

$$4010 = (k - 1)(k) + 2km$$

$$4010 = k(k - 1 + 2m)$$

$k$ and $k - 1 + 2m$ must be factors of 4010

4010 = 2 \cdot 5 \cdot 401

$k = 1$ yields $m = 2005$

$k = 2$ yields $m = 1002$

$k = 5$ yields $m = 399$

$k = 10$ yields $m = 196$

For $k = 401$, we get $10 = 2m + 400$, which has no positive integers solutions for $m$.

Thus, $k = 1, 2, 5, 10$ are the only solutions.
10. **Answer:** \((2, 8), (2, 12), (4, 8), (4, 12), (6, 8), (6, 12), (8, 8), (8, 12)\)

\[
5^m + 3^n - 1 \equiv 0 \pmod{15}
\]

Taking mod 5:

\[
3^n - 1 \equiv 0 \pmod{5}
\]

\[
n \equiv 0 \pmod{4}
\]

Taking mod 3:

\[
5^m - 1 \equiv 0 \pmod{3}
\]

\[
m \equiv 0 \pmod{2}
\]

Indeed,

\[
5^{2p} + 3^{4q} - 1 \equiv 10^p + 6^q - 1 \equiv 10 + 6 - 1 \equiv 0 \pmod{15}.
\]

Hence, all solutions are of the form

\[
m = 2p, n = 4q, p > 0, q > 0.
\]