Justify all answers, or give examples where appropriate. Partial credit will be given when appropriate.

**Interval Notation**

An interval from \(a\) to \(b\) of real numbers is denoted \([a, b]\) where this is the set \(x: a = x < b\). A "[" means that the endpoint is included in the set while a "]" means that the endpoint is not included. So the interval \([\frac{x}{2}, \pi]\) is the set of real numbers between \(\frac{x}{2}\) and \(\pi\) with \(\frac{x}{2}\) included but \(\pi\) not included.

**Partition:**

Define a partition of the interval \([a, b]\) as a finite subset of points \(x_0, x_1, ..., x_n\) such that \(a = x_0\), \(b = x_n\), and \(x_i < x_{i+1}\) for all \(i\) such that \(0 < i < n\). When we talk about a partition of a set into subsets, we mean the set of subsets of \([a, b]: [x_0, x_1], [x_1, x_2], ..., [x_n-1, x_n]\).

**Translation congruent:**

A set \(E\) is said to be \(2\pi\)-translation congruent to \(I_1 = [0, 2\pi]\) (denoted \(E \sim_{2\pi} I_1\)) if there is a partition of \(E\) into subsets such that adding a multiple of \(2\pi\) to each subset will produce disjoint sets whose union is \([0, 2\pi]\).

Example: \([-\pi, -\frac{x}{2}] \cup [\frac{3\pi}{2}, 3\pi] \sim_{2\pi} [0, 2\pi]\) because the set \(A = [-\pi, -\frac{x}{2}) + 2\pi \cup [\frac{3\pi}{2}, 2\pi) \cup [2\pi, 3\pi) - 2\pi = [\pi, \frac{3\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi) \cup [0, \pi) = [0, 2\pi]\).

1. Show that \(E = [-\frac{3\pi}{2}, -4\pi) \cup [-\pi, -\frac{4\pi}{3}) \cup [\frac{4\pi}{3}, \pi) \cup [4\pi, \frac{32\pi}{3}]\) is \(2\pi\)-translation congruent to \([0, 2\pi]\).

**Dilation congruent:**

A set \(E\) is said to be \(2\)-dilation congruent to \(I_2 = [-2\pi, -\pi) \cup [\pi, 2\pi)\) if there is a partition of \(E\) into subsets such that multiplying each subset by a power of \(2\) will produce disjoint sets whose union is \(I_2\).

Example: \([-\frac{\pi}{2}, -\frac{\pi}{4}) \cup [\frac{\pi}{4}, \frac{\pi}{2}) \cup [2\pi, \pi] \sim_{2} [-2\pi, -\pi) \cup [\frac{\pi}{4}, \frac{\pi}{2}) \cup [\pi, 2\pi)\) because the set \(A = [-\frac{\pi}{2}, -\frac{\pi}{4}) \cdot 2^2 \cup [\frac{\pi}{4}, \frac{\pi}{2}) \cdot 2^0 \cup [2\pi, \pi] \cdot 2^{-1} = [-2\pi, -\pi) \cup [\frac{\pi}{4}, \frac{\pi}{2}) \cup [\pi, 2\pi)\).

2. Show that \(E = [-\frac{3\pi}{2}, -4\pi) \cup [-\pi, -\frac{4\pi}{3}) \cup [\frac{4\pi}{3}, \pi) \cup [4\pi, \frac{32\pi}{3}]\) is \(2\)-dilation congruent to \(I_2\).

**Wavelet Sets**

Define a wavelet set to be any set \(E\) which is \(2\pi\)-translation congruent to \(I_1\) and \(2\)-dilation congruent to \(I_2\). So the set from our first two problems is a wavelet set.

3. Show that the two example sets are also wavelets sets. (i.e. Show \([-\pi, -\frac{\pi}{2}) \cup [\frac{3\pi}{2}, 3\pi) \cup [\pi, 2\pi)\) \(\cup [\frac{\pi}{2}, \pi) \cup [\frac{7\pi}{2}, \pi) \cup [2\pi, 0, 2\pi)\).)

Prove:

4. If \(E\) is a wavelet set, then zero is not an element of \(E\).

5. The interval \([a, b]\) is \(2\pi\)-translation congruent to \([0, 2\pi]\) if and only if \(b - a = 2\pi\).

6. For \(a > 0\), the interval \([a, b]\) is \(2\)-dilation congruent to \([\pi, 2\pi]\) if and only if \(b = 2a\).

**1-interval wavelet sets**
7. Prove that there can be no wavelet sets of just one interval (of the form \([a, b]\)).

**2-interval**

Now study wavelet sets which are the union of 2 intervals. Find any wavelet sets which are of the form \(E = [a, b] \cup [c, d]\) with \(b < c\).

8. Prove \(a = 2b\) and \(d = 2c\) if the set is a wavelet set.

9. Write \(c\) as a function of \(b\) and using one variable write out the form that all 2 interval wavelet sets must have.

**3-interval**

Now we try to find all wavelet sets which are a union of 3 intervals. \([u, v) \cup [x, y) \cup [w, z)\)

10. Reduce this case to finding 2 intervals \([x, y) \cup [w, z)\) which are \(2\pi\)-translation congruent and 2-dilation congruent to one interval from the 2 interval case.

11. If one of the three intervals is \([-2b, -b)\) where \(b > 0\), prove \(b < \pi\).

12. Show that we need our sets to be of the form \([-2b, -b) \cup [x, 2\pi - 2b) \cup [2n\pi - b, 2n\pi + x)\) for some \(n > 1\) and some \(x > 0\).

13. Choose some dilation \(2^j\) that makes the condition in (10) hold. Solve for \(x\) and \(b\) in terms of \(n\) and \(j\).

**4-interval**

Define a 4-interval wavelet set of the form \([a, b) \cup [c, d) \cup [e, f) \cup [g, h)\) to be symmetric if \(a = -h\), \(b = -g\), \(c = -f\), and \(d = -e\). Now examine symmetric 4 interval wavelet sets of the form: \([-y, -2^j\pi) \cup [-\pi, -x) \cup [x, \pi) \cup [2^j\pi, y)\).

14. Write \(y\) in terms of \(x\) in two separate ways to solve for \(x\) and \(y\) as a function of \(j\), and then write out the wavelet set in terms of only \(j\).