1. **Answer: 6**

   This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at \((0,0), (4,0), (0,4)\), respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).

![Graph of circles](image)

2. **Answer: 6**

   For positive integers \(a, b\), we have
   \[
   a! | b! \iff a! \leq b! \iff a \leq b.
   \]
   
   Thus,
   \[
   (n! | (2004)! \iff (n!)! \leq 2004! \iff n! \leq 2004 \iff n \leq 6.
   \]

3. **Answer: 8**

   Let \(x = 2004\). Then the expression inside the floor brackets is
   \[
   \left(\frac{x+1}{x-1}\right)^3 - \left(\frac{x-1}{x+1}\right)^3 = \frac{(x+1)^3 - (x-1)^3}{x(x+1)} = \frac{8x^3 + 8x}{x^3 - x} = 8 + \frac{16x}{x^3 - x}.
   \]
   
   Since \(x\) is certainly large enough that \(0 < 16x/(x^3 - x) < 1\), the answer is 8.

4. **Answer: \(x = 10, y = 5, z = 4\)**

   Factoring, \((x-7)(y-3) = 6, (x-7)(z-2) = 6, (y-3)(z-2) = 4\). This implies that
   \[
   \begin{align*}
   x - 7 &= 3 \\
   y - 3 &= 2 \\
   z - 2 &= 2
   \end{align*}
   \]
   
   Thus \(x = 10, y = 5, z = 4\).

5. **Answer: \(-\frac{1+\sqrt{5}}{2}\)**

   Draw a right triangle with legs 1, x; then the angle \(\theta\) opposite x is \(\tan^{-1} x\), and we can compute
   \[
   \cos(\theta) = \frac{1}{\sqrt{x^2 + 1}}.
   \]
   
   Thus, we only need to solve \(x = \frac{1}{\sqrt{x^2 + 1}}\). This is equivalent to \(x\sqrt{x^2 + 1} = 1\). Square both sides to get \(x^4 + x^2 = 1 \Rightarrow x^4 + x^2 - 1 = 0\). Use the quadratic formula to get the solution \(x^2 = \frac{1+\sqrt{5}}{2}\) (unique since \(x^2\) must be positive).

6. **Answer: 2 hours**

   Adding the individual rates, we get \(\frac{1}{3} + \frac{1}{11} + \frac{1}{14} = \frac{1}{2}\) of the room is cleaned per hour so the whole room takes two hours.
7. **Answer:** $0 < x < 1$ or $2 < x < 3$ or $4 < x < 5$

The sign of one of the terms switches every time $x$ moves from the range $(I, I + 1)$ to $(I + 1, I + 2)$. When $x$ is less than zero, all terms are negative so the LHS is negative. Also note that $x$ is undefined at $1, 3, 5$.

8. **Answer:** $128$

For any integer $n \geq 0$, the given implies $x^{n+3} = -4x^{n+1} + 8x^n$, so we can rewrite any such power of $x$ in terms of lower powers. Carrying out this process iteratively gives

$$x^7 = -4x^5 + 8x^4$$

$$= 8x^4 + 16x^3 - 32x^2$$

$$= 16x^3 - 64x^2 + 64x$$

$$= -64x^2 + 128.$$

Thus, our answer is $128$.

9. **Answer:** $677$

If $d$ is the relevant greatest common divisor, then $a_{1000} = a_{500}^2 + 1 \equiv 1 \equiv a_0 \pmod{d}$, which implies (by induction) that the sequence is periodic modulo $d$, with period $1000$. In particular, $a_4 \equiv a_{2004} \equiv 0$. So $d$ must divide $a_4$. Conversely, we can see that $a_4 = a_4^2 + 1 \equiv 0 \pmod{a_4}$, so (again by induction) the sequence is periodic modulo $a_4$ with period $5$, and hence $a_{999}, a_{2004}$ are indeed both divisible by $a_4$. So the answer is $a_4$, which we can compute directly; it is $677$.

10. **Answer:** $-\frac{2010012}{2010013}$

Let $z_1, \ldots, z_5$ be the roots of $Q(z) = z^5 + 2004z - 1$. We can check these are distinct (by using the fact that there’s one in a small neighborhood of each root of $z^5 + 2004z$, or by noting that $Q(z)$ is relatively prime to its derivative). And certainly none of the roots of $Q$ is the negative of another, since $z^5 + 2004z = 1$ implies $(-z)^5 + 2004(-z) = -1$, so their squares are distinct as well. Then, $z_1^2, \ldots, z_5^2$ are the roots of $P$, so if we write $C$ for the leading coefficient of $P$, we have

$$P(1) = \frac{C(1-z_1^2) \cdots (1-z_5^2)}{C(-1-z_1^2) \cdots (-1-z_5^2)}$$

$$= \frac{[(1-z_1) \cdots (1-z_5)] [(1+2z_1) \cdots (1+2z_5)]}{[(1-z_1) \cdots (1-z_5)] [(1+z_1) \cdots (1+z_5)]}$$

$$= \frac{[(1-z_1) \cdots (1-z_5)] [(1+2z_1) \cdots (1+2z_5)]}{[(1+2z_1) \cdots (1+2z_5)] [(1-z_1) \cdots (1-z_5)]}$$

$$= \frac{(1^2 + 2004 - 1)(1^2 + 2004)(-1) - 1}{(1^2 + 2004 - 1)(-1) - 1}$$

$$= \frac{(-1)(-1 - 2005)}{2005^2 - 1}$$

$$= -\frac{2005^2 - 1}{2010013} = -\frac{2005^2 - 1}{2010013}.$$