Incentive compatible technology pooling: Improving upon autarky.*

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Abstract

When $n$ agents each own a different technology to produce a common private good, what is the optimal way to pool their production possibilities? The technologies are common knowledge, but information concerning the preferences of the agents is private. There exists a unique technologically efficient, incentive compatible (in the strongest sense) pooling method guaranteeing voluntary participation. This method is an asymmetric version of serial surplus sharing along a certain fixed path. In the two-person case, we provide a related characterization of fixed path methods based on their incentive properties.

Keywords: Joint production, serial rule, technology pooling, incentive compatibility.

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1 Introduction

We consider a situation where $n$ agents each own a private technology and decide to pool their production possibilities. The technologies (production sets) are common knowledge, but information concerning the preferences of the agents is private. What kind of incentive compatible surplus sharing methods gives all agents an incentive to engage in the process of technology pooling, where incentive compatibility is given its most demanding interpretation?

To fix ideas, think of farmers each owning a parcel of land or, alternatively, producers owning machines which convert a certain (divisible) input into a certain (also divisible) output. When the $n$ technologies do not exhibit constant returns to scale, or when they are not identical, inputs can efficiently be reallocated across the various technologies. For instance, if technology 1 is always

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more productive than the others, efficiency requires the \((n - 1)\) other agents to provide input toward technology \(1\) instead of their own. Thus, the autarkic use of the production possibilities, where each agent \(i\) can only supply input toward technology \(i\), can be Pareto-improved. But a first best efficient pooling method must rely on accurate knowledge of individual preferences, and hence will typically not be incentive compatible (see Leroux[9]).

We show that there exists a unique technologically efficient pooling method which satisfies a strong form of incentive compatibility and voluntary participation.

Our results are closely related to the more familiar problem of sharing a single technology between a finite number of agents. It is known (see below) that sharing rules allocating incremental surplus (or cost) along a prespecified path, or “fixed path methods”, have extremely strong incentive properties (even stronger than group strategy-proofness, see Theorem 1). We show (Theorem 3) that in the two-person case these properties actually characterize the class of fixed path methods. We conjecture that the result holds for an arbitrary number of agents.

Finally, we uncover a relationship between the issue of sharing a single technology and that of technology pooling. Each fixed path method can be interpreted as the optimal distribution of property rights to the shared technology, such that all agents would in fact be willing to pool the resulting private production possibilities (see Theorem 2). This is a novel interpretation for fixed path methods.

**Relation to the literature** The present paper is a contribution to the general question of trade-offs between efficiency and incentive compatibility. In pure exchange economies, these two concepts are incompatible with any kind of fairness requirement (see Hurwicz[7], Serizawa and Weymark[18]). The class of incentive compatible mechanism satisfying voluntary participation has been characterized in Barberà and Jackson[3].

In public good economies, much work has been done to characterize the class of strategy-proof social choice functions (e.g. Barberà and Jackson[2]) and that of strategy-proof and individually rational social choice functions (e.g. Serizawa[16][17]). Our voluntary participation criterion is reminiscent of the “autarkic individual rationality” axiom in Saijo[13]; the latter has been shown to be incompatible with strategy-proofness and non-bossiness (the requirement that no agent can influence the social outcome without affecting her own bundle). When the public good is excludable, Moulin[10] shows that serial cost-sharing improves upon more conservative cost allocation rules.

In economies with production of a private good, the question of aggregating individual production possibilities is not new (see, e.g., Weitzman[22]). The management of a common property technology has also received much attention (see, among many others, Israelsen[8], Moulin and Shenker[12], Sen[15], Shenker[19], Sprumont[21]). Fixed path methods were introduced recently (Friedman[5]) as non-anonymous extensions of the serial sharing rule inheriting its very strong
incentive properties. They are analogous to the incremental paths studied in Moulin[11] in a discrete version of the sharing problem.

2 Overview of the paper

Denote by \( f_i \) the (increasing and strictly concave) production function corresponding to Agent \( i \)'s private technology. When the \( n \) technologies are efficiently pooled, the resulting aggregate technology, \( F \), is:

\[
F(t) = \max_{(x_1, \ldots, x_n) \in \mathbb{R}_+^n} \sum_{i=1}^n f_i(x_i) \quad \text{for any } t \geq 0. \tag{1}
\]

Thus, the problem of technology pooling amounts to one of sharing a single production function under voluntary participation and incentive compatibility constraints. By voluntary participation we mean that every agent gets at least her stand-alone utility, \( u_i(x_i) = \max\{u_i(x, y) | y \leq f_i(x)\} \). By incentive compatibility, we mean group strategy-proofness: coordinated deviations from truthful reporting is harmful for at least one member of the deviating coalition.\(^1\)

We now describe the unique pooling method characterized by these two properties (see Theorem 2 and Theorem 3). This “optimal” pooling method is best illustrated on an example with two agents because one can represent graphically the supply game where each agent supplies a positive quantity of input, \( x_i \), and receives \( y_i \) units of output such that \( y_1 + y_2 = F(x_1 + x_2) \). See Figure 2. The marginal product, \( F' \), of the aggregate technology is represented on the graph by ACM: it is the horizontal sum of the individual marginal product lines AK and BJ. For simplicity, let the utility function of Agent \( i \) be of the form:

\[
u_i(x_i; y_i) = y_i - \frac{1}{2} x_i^2.
\]

In autarky, agents are constrained to only using their own technology. Given the preferences of the agents, the preferred point of Agent 1 (resp. Agent 2) is E (resp. H). Let \( x_{1a} \) and \( x_{2a} \) denote respectively the autarky supply levels of Agent 1 and Agent 2.

The “optimal” pooling method works as follows. Assign the agent with the largest value of \( f'_i(x^*_i) \) her autarky bundle. In Figure 2, \( f'_1(x^*_1) \geq f'_2(x^*_2) \), therefore Agent 1 receives her autarky bundle \( (x^*_1, y^*_1) \) with \( x^*_1 = x^a_1 \) and \( y^*_1 = f_1(x^*_1) = \text{Area}(AEx^*_1O) \). Now that Agent 1 has been served, Agent 2 can actually benefit from the pooling of \( f_1 \) and \( f_2 \). Her opportunity set is BDL, where DL is obtained by shifting the line segment GM to the left by the amount \( x^*_1 \). Agent 2’s preferred point on BDL is I; she therefore receives the bundle \( (x^*_2, y^*_2) \) with \( x^*_2 = x^I_2 \) and \( y^*_2 = \text{Area}(BDIx^*_2O) \).

Here, Agent 1 receives her stand-alone utility. On the other hand, Agent 2’s utility \( \text{Area}(BDIO) \) is greater than her stand-alone utility \( \text{Area}(BHO) \); her utility gain is represented in Figure 2 by a shaded triangle.

\(^1\)We refer the reader to Barberà[1] for an introduction to strategy-proof social choice functions.
This “optimal” pooling method therefore improves upon autarky. This feature still holds if one technology does not dominate the other, as can be seen in Figure 2. But because technology 1 is less productive than in Figure 2, Agent 2’s utility gain is smaller: the benefit obtained from pooling the two technologies is less.

Interestingly, this method does not compensate agents for “lending” their technology: an agent supplying little input will barely receive her stand-alone utility even if her technology dominates all the others (as is the case in Figure 2). In this sense, one could say that this “optimal” pooling method rewards input contributions more than technological ones.

Our results are now reviewed. Section 3 sets up the framework. In section 4, we define a class of methods for sharing a single technology: “fixed path methods”, and discuss their strategic properties. We also present the issue of technology pooling and use fixed path methods to establish a correspondence between the two questions (Theorem 2). Section 5 is devoted to a characterization of fixed path methods. Section 6 concludes.

Our “optimal” pooling method is a fixed path method corresponding to the path $\phi$, where $\phi(t) = (\phi_1(t), ..., \phi_n(t))$ is the (unique) solution of (1) for any $t \geq 0$; i.e. $\phi(t)$ is the optimal allocation of $t$ units of input between the various technologies. More generally, fixed path methods are an asymmetric version of serial surplus sharing along a certain fixed path. Theorem 1 states that fixed path methods satisfy an incentive compatibility requirement even stronger than group strategy-proofness. This result has been shown in Friedman[5] in a more
general context; the proof we provide is tailored to our setting, and gives a construction of the unique Nash equilibrium of a game induced by a fixed path method.

Our Theorem 2 establishes a bijection between the issue of technology pooling and that of sharing a single technology with a fixed path method. On the one hand, to each technology profile \((f_1, \ldots, f_n)\) corresponds a unique fixed path method that shares the aggregate technology, \(F\), while guaranteeing each agent their stand-alone utility. Conversely, given a technology \(F\), to any fixed path method \(\zeta\) corresponds a unique technology profile \((f_1, \ldots, f_n)\) such that \(\zeta\) is the only fixed path method sharing \(F\) while improving upon autarky, relative to \((f_1, \ldots, f_n)\). In other words, \(\zeta\) disaggregates \(F\) into (virtual) private technologies, reflecting the property rights of the members of the cooperative. Thus, Theorem 2 constitutes a novel interpretation of fixed path methods.

Theorem 3 generalizes the characterization of the serial rule in Moulin and Shenker ([12], Theorem 2) to sharing methods which are not anonymous in the two-agent case. It states that a sharing rule which is monotonic (my output share is strictly increasing in my input contribution), smooth (i.e. continuously differentiable) and which satisfies the requirement that an agent providing no input receives nothing, is a fixed path method if and only if the induced sharing game admits a unique Nash equilibrium when the technology \(F\) is concave.

Our characterization is akin to that of the serial rule in Moulin and Shenker[12]: we replaced their anonymity axiom with our “zero-input-zero-output” requirement. The justification for this axiom is twofold: it ensure that an agent must
provide some input in order to be rewarded, and it also places a mild lower bound on the utility level of each agent. Our requirement being much weaker than anonymity, it comes to no surprise that we obtain a whole class of sharing methods. Just like Theorem 2 in [12], our characterization makes use of the acyclicity of strategy-proof sharing rules (see Satterthwaite and Sonnenschein[14]).

3 The framework

Let $N = \{1, \ldots, n\}$ be the set of agents. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a production function which is strictly increasing, strictly concave such that $F(0) = 0$. We denote by $\mathcal{F}$ the class of such functions. If in addition $F$ is continuously differentiable, we write $F \in \mathcal{F}^c$. Each agent $i$ provides a non-negative amount $x_i$ of input to the common technology, and receive a non-negative quantity $y_i$ of output such that $\sum_i y_i = F(\sum_i x_i)$. We write $x = (x_1, \ldots, x_n)$ and for any $i \in N$, $(x'_i, x_{-i})$ is the vector of inputs where the $i$th entry of $x$ has been replaced by $x'_i \in \mathbb{R}_+$. Let $y_i \in \mathbb{R}$ denote the output share of agent $i$. A bundle is an element $z_i = (x_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$; we define an allocation $z$ to be a list of $n$ bundles, one for each agent. We denote by $Z_F = \{ z \in (\mathbb{R}_+ \times \mathbb{R})^N \mid \sum_i y_i = F(\sum_i x_i) \}$ the set of feasible allocations under $F$.

Each agent $i$ has a bounded endowment of input $M_i$ (with possibly $M_i = +\infty$). Her preferences over bundles are defined on $[0, M_i] \times \mathbb{R}$; they are continuous, convex, strictly increasing in $y_i$, strictly decreasing in $x_i$ and representable by a utility function $u_i$. While all our results are purely ordinal in nature, we will use utility representations for the preferences rather than the more cumbersome binary relation notation. We adopt the convention that $u_i(x_i, y_i) = -\infty$ if $x_i > M_i$. We denote by $\mathcal{U}$ the class of preferences. A preference profile (or utility profile) is a list of $n$ preferences, $u = (u_1, \ldots, u_n) \in \mathcal{U}^N$, one per agent. For any $j \subseteq N$, we will sometimes abuse notations and write $u = (u_j, u_{-j})$.

An $F$-sharing method (or $F$-sharing rule) is a mapping

$$
\xi : \mathbb{R}_+^N \to \mathbb{R}_+^N
$$

$$
x \mapsto (\xi_1(x), \ldots, \xi_2(x)) \quad \text{s.t.} \quad \sum_{i \in N} \xi_i(x) = F(\sum_{i \in N} x_i)
$$

We denote by $\mathcal{S}_F$ the class of $F$-sharing rules satisfying the following two properties:

$\rightarrow$ Monotonicity: $\frac{\partial \xi_i}{\partial x_i} > 0$,

$\rightarrow$ Zero output for zero input (ZOZI): $\forall x_{-i} \in \mathbb{R}_+^{N \setminus \{i\}} \quad \xi_i(0, x_{-i}) = 0$.

If in addition $F \in \mathcal{F}^c$ and $\xi$ satisfies the following smoothness property, we write $\xi \in \mathcal{S}^c_F$.

$\rightarrow$ Smoothness: $\xi$ is continuously differentiable on $\mathbb{R}_+^n$. 

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A few comments are in order regarding these three properties. Monotonicity is a normatively appealing requirement. It states that an agent should receive strictly more output as her input contribution increases. Thus, Monotonicity gives agents an incentive to supply input. Also, from the point of view of fairness, it implies that every agent will receive at least some part of the output resulting from her input contribution.

The normative aspect of the ZOZI property is twofold. The more obvious one is that an agent contributing no input to the economy \((x_i = 0)\) should not be rewarded \((u_i(x_i, x_i) = 0)\). The other consequence is that an agent can always guarantee her utility level to be no less than \(u_i(0, 0)\), by choosing to supply nothing to the system \((x_i = 0)\).

Finally, we may demand that sharing methods be smooth. This requirement is a technical one. One of our proofs (Theorem 3) relies heavily on this assumption and, while we were not able to prove our results without imposing smoothness, we do not know whether it is a necessary condition. The same remark applies to results in Moulin and Shenker[12] and in Shenker[19].

For any production function \(F \in \mathcal{F}\), any preference profile \(u \in \mathcal{U}^N\) and any \(F\)-sharing method \(\xi \in \mathcal{S}_F\), we denote by \(G(F, \xi; u)\) the game in which each agent’s strategy space is \(\mathbb{R}_+\) and agent \(i\)’s payoff is \(u_i(x, \xi_i(x))\) when \(x\) is the strategy played by agent \(j \in N\). The game form \(G(F, \xi)\) is a mapping from \(\mathcal{U}^N\) to the set of games such that \(G(F, \xi)(u) = G(F, \xi; u)\) for any preference profile \(u \in \mathcal{U}^N\). We say that the game form \(G(F, \xi)\) is induced by the \(F\)-sharing method \(\xi\).

4 Sharing a production function along a path

Fix a production function \(F \in \mathcal{F}\). We are interested in sharing methods which induce game forms with strong strategic properties. Namely, the induced game form should admit a unique Nash equilibrium at every profile; moreover, this equilibrium should also be strong (i.e. immune to coalitional deviations). In this section, we restrict our attention to fixed path methods. These methods were introduced and analyzed in Friedman[5], and consist in allocating to the agents the integral of the marginal product of \(F\) along some prespecified continuous increasing path of \(\mathbb{R}_+^n\).

A path is a mapping
\[
\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N
\]
\[
t \mapsto (\phi_1(t), ..., \phi_n(t))
\]
such that for all \(i \in N\) the following two properties hold:

(a) \(\phi_i\) is non-decreasing and differentiable on \(\mathbb{R}_+\),

(b) \(\sum_j \phi_j(t) = t\) for any \(t \in \left[0, \sum_j M_j\right]\) and \(\phi_i(t) = M_i\) for any \(t \geq \sum_j M_j\).

We denote by \(\mathcal{P}\) the class of paths. If a path \(\phi\) also satisfies the following condition (c) for every \(i\), we write \(\phi \in \mathcal{P}^c\):
Finally, by inspecting formula (3) at the points \( x \) for which \( \phi(t) = 0 \), the sharing rule \( \delta_i \) jumps wherever \( \phi_i \) is flat. Hence \( \delta_i \) is continuous on \([0, M_i]\) if and only if \( \phi \in \mathcal{P}^c \).

Given a production function \( F \in \mathcal{F} \) and a path \( \phi \in \mathcal{P} \), we define the output sharing rule \( \xi^{F, \phi} \) as follows. Let \( x \in \times_i[0, M_i] \), and \( t \geq 0 \) such that \( \phi(t) \leq x \), i.e. such that the supply \( x \) of Agent \( i \) has not yet been met for any \( i \in N \). The fixed path method \( \xi^{F, \phi}(x) \) precisely recommends that the marginal product \( F'(t) \) be split between the agents according to the vector of proportions \( (\phi'_1(t), \ldots, \phi'_n(t)) \) (recall that \( \sum_i \phi'_i(t) = 1 \)). Once the amount of input supplied by Agent \( i \) has been met along the path \( \phi(t) \geq x_i \), we freeze her output share and continue the sharing procedure with the remaining “active” agents.

Formally, fix any \( x \in \times_i[0, M_i] \) and compare the \( \delta_i(x_i) \)'s for all \( i \in N \); without loss, we will assume \( \delta_1(x_1) \leq \delta_2(x_2) \leq \ldots \leq \delta_n(x_n) \). Thus,

\[
\begin{align*}
\xi_1^{F, \phi}(x) &= \int_0^{\delta_1(x_1)} F'(t) d\phi_1(t) \\
\xi_2^{F, \phi}(x) &= \int_0^{\delta_1(x_1)} F'(t) d\phi_2(t) + \int_{\delta_1(x_1)}^{\delta_2(x_2)} F'(t + \sum_{i=2}^{\infty} \phi_i(t)) d\phi_2(t) \\
&\vdots \\
\xi_n^{F, \phi}(x) &= \int_0^{\delta_1(x_1)} F'(t) d\phi_n(t) + \int_{\delta_1(x_1)}^{\delta_2(x_2)} F'(t + \sum_{i=2}^{\infty} \phi_i(t)) d\phi_n(t) + \int_{\delta_2(x_2)}^{\delta_3(x_3)} F'(t + \sum_{i=2}^{\infty} \phi_i(t)) d\phi_n(t) + \ldots + \int_{\delta_n(x_n)}^{\delta_{n+1}(x_{n+1})} F'(t + \sum_{i=1}^{\infty} \phi_i(t)) d\phi_n(t) \\
\end{align*}
\]

A more compact notation is used in Friedman ([5] and [6]): for any \( i \in N \),

\[
\xi_i^{F, \phi}(x) = \int_0^\infty F'(t) \left( |\phi(t) \land x_i| \right) d(\phi_i(t) \land x_i)
\]

where \( | \cdot | \) returns the sum of the coordinates of a vector and \( \land \) is the componentwise minimum of two vectors\(^2\).

It follows easily from the monotonicity of \( F \) and the \( \delta_j \)'s that \( \xi_i^{F, \phi} \) is monotonic \( (\frac{\partial \xi_i^{F, \phi}}{\partial x_i} > 0 \) for all \( i \)). Moreover, one can check (or see Friedman[5], Lemma 1) that \( \xi_i \) is strictly concave in \( x_i \). Because each function \( \delta_j \) takes on the value zero at zero, the sharing rule \( \xi^{F, \phi} \) satisfies the ZOZI condition. Hence, \( \xi^{F, \phi} \in \mathcal{S}_F \).

Finally, by inspecting formula (3) at the points \( x \) such that \( \delta_i(x_i) = \delta_j(x_j) \) and those where \( x_i = 0 \), one can check that \( \xi_i^{F, \phi} \in \mathcal{S}_F \) if \( F \in \mathcal{F}_c \) and \( \phi \in \mathcal{P}_c \).

We next illustrate the definition of fixed path methods with a few examples for \( n = 2 \).

**Example 1a.** Assume \( M_1 < +\infty \). Consider the path

\[
\phi^f : t \mapsto \begin{cases} (t, 0) & \text{if } t \leq M_1 \\
(M_1, t-M_1) & \text{if } M_1 \leq t \leq M_1 + M_2 \end{cases}
\]

\(^2\)For any \( p, q \in \mathbb{R}^n_+ \), \( |p| = \sum_j p_j \) and \( (p \land q)_i = \min\{p_i, q_i\} \) for every \( i \in N \).
i.e., $\phi^I$ is a parametrization of the horizontal axis up to $x_1 = M_1$. Let $x = (x_1, x_2) \in [0, M_1] \times [0, M_2]$, then if we write $\xi^I$ instead of $\xi^{F, \phi^I}$:

$$
\begin{align*}
\xi^I_1(x) &= F(x_1) \\
\xi^I_2(x) &= F(x_1 + x_2) - F(x_1)
\end{align*}
$$

The sharing rule $\xi^I$ gives priority to Agent 1, who ends up receiving her stand-alone output level, and then takes care of Agent 2, who then receives her stand-alone output level via the “truncated” production function $F_2(\cdot) = F(x_1 + \cdot)$. Given the concavity of $F$, Agent 1 is clearly at an advantage relative to Agent 2 as the first units of input are more “productive”. $\xi^I$ is the incremental cost sharing method starting with Agent 1.

**Example 1b.** $M_1 < +\infty$ and let $0 < Q_1 < M_1$. Then the fixed path method $\xi^I$ associated with the path

$$
\phi^I : t \mapsto \begin{cases} (t,0) & \text{if } t \leq Q_1 \\
(Q_1, t - Q_1) & \text{if } Q_1 \leq t \leq Q_1 + M_2 \\
(Q_1 + t - M_2, M_2) & \text{if } Q_1 + M_2 \leq t \leq M_1 + M_2
\end{cases}
$$

is not smooth because $\phi^I \notin \mathcal{P}$. Figure 1. Indeed, for any $x_2 \in [0, M_2]$, $\xi^I_1(x) = F(x_1)$, $\xi^I_2(x) = F(x_1) + F(x_1 + x_2) - F(Q_1 + x_2)$ if $x_1 < Q_1$, and $\xi^I_2(x) = F(x_1 + x_2)$ if $x_1 > Q_1$.

Therefore $\frac{\partial}{\partial x_2}(\cdot, x_2)$ is not continuous at $x_1 = Q_1$.

**Example 2.** Next, assume $M_1 = M_2 = +\infty$. Let $\alpha_1, \ldots, \alpha_n > 0$ such that $\sum_j \alpha_j = 1$ and consider the path $\phi^S : t \mapsto (\alpha_1 t, \ldots, \alpha_n t)$. Let $x \in \mathbb{R}_+^N$ and assume without loss that $\frac{\alpha_2}{\alpha_1} \leq \frac{\alpha_3}{\alpha_2} \leq \ldots \leq \frac{\alpha_n}{\alpha_{n-1}}$. Then, writing $\xi^S$ instead of $\xi^{F, \phi^S}$, expression (3) yields:

$$
\xi^S_i(x) = \frac{\alpha_i}{\alpha^i} F(x^i) - \sum_{k=1}^{i-1} \frac{\alpha_i \alpha_k}{\alpha^i \alpha^{k+1}} F(x^k)
$$

for all $i = 1, \ldots, n$, where $\alpha^k = \sum_{j=k}^n \alpha_j$, and $x^k = x_1 + \ldots + x_{k-1} + \frac{\alpha_k}{\alpha^k} x_k$. $\xi^S$ is a generalization of the serial rule attributing a fixed weight $\alpha_i$ to Agent $i$. In the 2-person case, the graph of $\phi^S$ is the positive ray from the origin with slope $\frac{\alpha_2}{\alpha_1}$. And:

$$
\begin{align*}
\xi^S_1(x) &= \alpha_1 F(x_1) \\
\xi^S_2(x) &= \alpha_2 F(x_2) + F(x_1 + x_2) - F(\frac{x_1}{\alpha_1}).
\end{align*}
$$

When $\alpha_i = \frac{1}{n}$ for all $i$, $\xi^S$ is precisely the surplus sharing counterpart to the serial cost sharing rule analyzed in Moulin and Shenker[12]. This method splits the marginal product $F'(t)$ equally among agents who are still “active”; i.e. agents whose supply has not yet been met along the diagonal ($\frac{t}{n} < x_1$).
Example 3. As a last example, consider the following path:

\[ \phi^{BL}(t) = \begin{cases} 
(at, (1-a)t) & \forall t \leq 1 \\
(a-b+bt, b-a+(1-b)t) & \forall t \geq 1 
\end{cases} \]

where \(a, b \in ]0, 1[\). \(\phi^{BL}\) parametrizes a piecewise linear curve connecting the origin to the point \((a, 1-a)\), then increasing with constant slope \(\frac{1-b}{b}\). See Figure 2.

One checks that for any \(x \in \mathbb{R}^2\) the sharing rule \(\xi^{BL} \equiv \xi^F, \phi^{BL}\) can be additively decomposed into two parts:

\[ \xi^{BL}(x) = \begin{cases} 
\xi^a(x) & \text{if } x_1 \leq a \text{ or } x_2 \leq 1-a, \\
\xi^a(a, 1-a) + \xi^{H,b}(x_1-a, x_2-(1-a)) & \text{otherwise.} 
\end{cases} \]

where we denote by \(\xi^a\) the rule sharing \(F\) according to the path \(\phi^a : t \mapsto (at, (1-a)t)\) and \(\xi^{H,b}\) the method sharing the production function \(H(\cdot) = F(\cdot+1)\) according to the path \(\phi^b : t \mapsto (bt, (1-b)t)\).

### 4.1 Incentive properties of fixed path methods

We now turn to the incentive properties of fixed path methods. For notational simplicity, given a production function \(F \in \mathcal{F}\), a path \(\phi \in \mathcal{P}\) and any \(u \in \mathcal{U}^N\), we write \(G(F, \phi; u)\) instead of \(G(F, \xi^F, \phi; u)\) when no confusion is possible.
Figure 2: When computing $\zeta^{BL}(A)$, only the linear part of $\phi^{BL}$ from the origin matters.

**Theorem 1** For any $F \in \mathcal{F}$, any $\phi \in \mathcal{P}$ and any $u \in \mathcal{U}^N$, the game $G(F, \phi; u)$ is dominance solvable: the successive elimination of dominated strategies converges to the unique Nash equilibrium of the game.

**Proof.** It is shown in Friedman [5] that for any production function $F \in \mathcal{F}$, any path $\phi \in \mathcal{P}$ and any preference profile $u \in \mathcal{U}^N$, the game induced by $\zeta^{F, \phi}$ actually satisfies a more demanding equilibrium property called $O$-solvability. For the sake of completeness, we will nonetheless give a detailed proof of Theorem 1. Our proof (in Appendix) is constructive and shows how the unique Nash equilibrium is obtained, it is inspired from the proof of Theorem 1 in Moulin and Shenker [12].

4.2 An interpretation of fixed path methods

We now turn to the situation where $n$ agents each own a private technology and decide to pool their production possibilities.

Let $f_1, \ldots, f_n \in \mathcal{F}$, and define $F^*$ to be the *aggregated production function* resulting from an efficient usage of the combined $n$ individual technologies:

$$\forall t \in \mathbb{R}_+ \quad F^*(t) = \max_{(x_1, \ldots, x_n) \in \mathbb{R}_+^N} \sum_{i=1}^n f_i(x_i), \quad (x_1, \ldots, x_n) \in \mathbb{R}_+^N, \sum_i x_i = t. \quad (4)$$
Notice that because the \( f_i \)'s belong to \( \mathcal{F} \), \( F^* \) must also belong to \( \mathcal{F} \). Also, if all the \( f_i \)'s belong to \( \mathcal{F^c} \), so does \( F^* \). However, the converse is not true (see Example 1b' below). We call \( f = (f_1, ..., f_n) \in \mathcal{F}^N \) the technology profile of the agents. An \( f \)-pooling method is an \( F^* \)-sharing rule \( \xi^f \in \mathcal{S}_{F^*} \). We denote by \( \mathcal{S}_f \) the class of \( f \)-pooling methods. If moreover \( \xi^f \) is smooth (\( \xi^f \in \mathcal{S}_{F^*}^c \)), we write \( \xi^f \in \mathcal{S}_f^c \). For any path \( \phi \in \mathcal{P} \), \( \xi^{f,\phi} \) denotes the \( F^* \)-sharing rule \( \xi^f \).

For any preference profile \( u \in \mathcal{U}^N \), we define the game \( G(f,\xi^f;u) \equiv G(F^*,\xi^f;u) \) as in Section 4. We are interested in sharing rules \( \xi^f \in \mathcal{S}_f \) such that the induced game satisfies voluntary participation in the following sense:

**Stand-Alone Test (SAT)** Let \( f \in \mathcal{F}^N \). An \( f \)-pooling method \( \xi^f \) satisfies SAT with respect to \( f \) if for any preference profile \( u \in \mathcal{U}^N \) and any Nash equilibrium \( x^* \) of \( G(f,\xi^f;u) \) the following holds:

\[
\quad u_i \left( x_i^*, \xi^f_i \left( x^* \right) \right) \geq sa_i(u_i) \equiv \max \{ u_i(x_i, y_i) | y_i \leq f_i(x_i) \} \quad \forall i \in N.
\]

SAT requires that no agent would rather use her own technology than participate in \( G(f,\xi^f;u) \) and play any Nash equilibrium strategy.

The following theorem suggests a novel interpretation of fixed path methods.

**Theorem 2** For any \( f \in \mathcal{F}^N \), there exists a unique path \( \phi^* \in \mathcal{P} \) such that \( \xi^{f,\phi^*} \) satisfies SAT with respect to \( f \). It is the unique solution of

\[
\max \quad \sum_{i=1}^{n} f_i(x_i)
\]

\[
\text{subject to} \quad \sum_i x_i = t
\]

for any \( t \geq 0 \).

Conversely, for any \( F^* \in \mathcal{F} \) and any \( \phi^* \in \mathcal{P} \), there exists a unique technology profile \( f \) decomposing \( F^* \) in the sense of (4) such that \( \xi^{f,\phi^*} \) satisfies SAT with respect to \( f \). For any \( i \in N \), \( f_i \) is given by

\[
f_i(x_i) = \int_{0}^{x_i} F^{**}(\delta^*_i(t))dt
\]

for all \( 0 \leq x_i \leq M_i \), where \( \delta^*_i \) is defined relative to \( \phi^*_i \) as in expression (2).

The path \( \phi^* \) is constructed as follows: for any \( t \geq 0 \), \( (\phi^*_1(t),...,\phi^*_n(t)) \) is the optimal distribution of \( t \) units of input between the \( n \) available technologies. In other words, it is the distribution of total input which equalizes the marginal product across the individual technologies in use. Formally, for any \( t \geq 0 \) and any \( i \in N \) such that \( \phi^*_i(t) > 0 \) (technology \( i \) is then in use), the following holds:

\[
F^{**}(t) = f_i^*(\phi^*_i(t)).
\]

The converse part of Theorem 2 says that, given a production function \( F^* \), the \( f_i \)'s can be viewed as the (virtual) individual contributions of the agents to
the common technology $F^*$. The fixed path method $\xi^{F^*,\phi^*}$ allocates surplus as if the common technology originated from the pooling of the individual $f_i$’s. In other words, the choice of a path is tantamount to that of a decomposition of the common technology, $F^*$, into individual production functions, rewarding the agents so that they would in fact be willing to pool these virtual individual production possibilities. Hence, Theorem 2 shows a one-to-one correspondence between the class of fixed path methods to manage $F^*$ and the set of possible distributions of property rights on $F^*$.

We now explain intuitively why $\xi^{F,\phi}$ not only satisfies SAT but also improves upon autarky. As long as all agents are active ($t \leq \min_j \delta_j^*(x_j)$), $\xi^{F,\phi}$ shares the marginal product $F''(t)$ according to the vector of ratios $(\phi_1^*(t), \ldots, \phi_n^*(t))$. Hence, assuming for clarity that $\delta_1^*(x_1)$ is the smallest of the $\delta_j^*(x_j)$’s, then

$$\xi^{F,\phi^*}(x) = \int_0^{\delta_1^*(x_1)} F''(t) \phi^*(t) dt = f_1(x_1)$$

and Agent 1 receives her stand-alone level of output. Now, for $\delta_1^*(x_1) \leq t \leq \min_{j \neq 1} \delta_j^*(x_j)$, $\xi^{F,\phi}$ shares the marginal output $F''(t)$ between agents 2,...,$n$ according to the ratios $(\phi_2^*(t), \ldots, \phi_n^*(t)) \times \frac{1}{\sum_{j > 1} \phi_j^*(t)}$. Clearly, for any $i \neq 1$, $\frac{\phi_i^*(t)}{\sum_{j > 1} \phi_j^*(t)} \geq \phi_i^*(t)$ and Agent $i$ receives no less (typically more) than her stand-alone share of output. And so on. Improvement upon autarky obtains by integration.

Proof. Before proving Theorem 2, we present a lemma establishing that under any fixed path method, $\xi^{F,\phi}$, any positive level of output, $x_i$, can be guaranteed at equilibrium by some preference $u^*_i$ for agent $i$. Its proof can be found in Appendix.

Lemma 1 Let $F \in \mathcal{F}$, $\phi \in \mathcal{P}$, $i \in N$. For any $x_i > 0$, there exists a preference $u^*_i \in \mathcal{U}$ such that the following holds:

$$\forall u_{-i} \in \mathcal{U}^{N\setminus i} \quad x^*_i = x_i;$$

where $x^*$ denotes the unique Nash equilibrium of $G(F, \phi; u^*_i, u_{-i})$.

We now prove the direct statement of Theorem 2. Let $f_1, \ldots, f_n \in \mathcal{F}$, and let $F^* \in \mathcal{F}$ be the aggregated production function. Let $\phi \in \mathcal{P}$ such that $\xi^{F,\phi}$ satisfies SAT with respect to $f$. For the rest of the proof we will write $\xi$ instead of $\xi^{F,\phi}$ and $F$ instead of $F^*$ as no confusion is possible.

Fix $j \in N$ and an arbitrary $x_j \in [0, M_j]$, and let $x \in \times_i [0, M_i]$ such that $\delta^*_i(x_i) = \delta^*_j(x_j)$ for all $i \in N$; i.e. $x$ is a point on the graph of $\phi^*$. We know from Lemma 1 that there exists a preference profile $u \in \mathcal{U}^N$ such that $x$ is the unique Nash equilibrium of $G(f, \xi; u)$. It follows that $\xi$ satisfies SAT with respect to $f$ only if for any $i \in N$ and any $x_i > 0$ the following holds:

$$\int_0^{\delta_i(x_i)} F'(t) d\phi_i(t) \geq \int_0^{x_i} f_i'(t) dt$$
By (5) and the definitions of $\delta_i$ and $\delta^*_i$, this transforms into

$$\int_0^{x_i} F'(\delta_i(t))dt \geq \int_0^{x_i} F'(\delta^*_i(t))dt \tag{6}$$

for all $i \in N$ and all $x_i > 0$. Let $i \in N$ and define $H_i(x_i) = \int_0^{x_i} F'(\delta_i(t))dt$ for any $x_i \geq 0$; $H_i$ is strictly increasing and strictly concave. Hence,

$$H_i(x_i) \leq H_i(\phi_i \circ \delta^*_i(x_i)) + H_i'(\phi_i \circ \delta^*_i(x_i)) \cdot (x_i - \phi_i \circ \delta^*_i(x_i))$$

i.e.

$$H_i(x_i) \leq H_i(\phi_i \circ \delta^*_i(x_i)) + F'(\delta^*_i(x_i)) \cdot (x_i - \phi_i \circ \delta^*_i(x_i)) \tag{7}$$

with equality if and only if $x_i = \phi_i \circ \delta^*_i(x_i)$. It follows from equations (6) and (7) that

$$\int_0^{x_i} F'(\delta^*_i(t))dt \leq \int_0^{x_i} F'(\delta_i(t))dt + \int_0^{x_i} F'(\delta_i(t))dt \cdot (x_i - \phi_i \circ \delta^*_i(x_i))$$

and

$$\int_0^{x_i} F'(\delta^*_i(t))dt \leq \int_0^{x_i} \phi_i(t)F''(t)dt + \int_0^{x_i} F'(\delta^*_i(x_i)) \cdot x_i$$

the last expression is obtained by integrating by parts. Rearranging yields:

$$\int_0^{x_i} \phi_i(t)F''(t)dt \leq F'(\delta^*_i(x_i)) \cdot x_i - \int_0^{x_i} F'(\delta^*_i(t))dt.$$ 

Recall that $\delta^*_i(x_i) = \delta^*_j(x_j)$ for all $i \in N$; and write $z = \delta^*_i(x_i)$ for any $i$. Summing up over all $i \in N$ and using the fact that $\sum_i \phi_i(t) = t$ for any $t \geq 0$ and $\sum_i x_i = \sum_i \phi^*_i(z) = z$, we get:

$$\prod_{i=1}^{n} \phi_i(t)F''(t)dt \leq F'(z) \cdot z - \sum_{i=1}^{n} \int_0^{\delta^*_i(z)} F'(\delta^*_i(t))dt$$

From $\sum_i \phi_i(t) = t$ and integrating by parts again, this yields an equality. Therefore, equation (6) must be an equality for all $i \in N$. The choice of $j$ and $x_j$ being arbitrary, it follows that $\delta_i(x_i) = \delta^*_i(x_i)$ for all $x_i \in [0, M_i]$ and for all $i \in N$. That is to say that $\phi_i \equiv \phi^*_i$ for all $i \in N$, proving the first statement of the theorem.

We now prove the converse statement. Let $F^* \in \mathcal{F}$, $\phi^* \in \mathcal{P}$ and $f \in \mathcal{F}^N$ decomposing $F^*$ in the sense of (4) such that $\xi^{f, \phi^*}$ satisfies SAT with respect to $f$. Hence, for any $i \in N$, expression (5) holds almost everywhere. I.e.,

$$f_i^*(t) = F^*(\delta^*_i(t)) \text{ almost everywhere.}$$

The result follows from integrating between 0 and $x_i$ (recall $f_i(0) = 0$).

**Remark:** In the statement of Theorem 2, we could replace SAT with a weaker requirement, demanding solely that the output each agent receives at the Nash equilibrium $x^*$ of $G(f, \xi; u)$ be no less than what she could have produced on her own by supplying $x^*_i$ units of input.
**weak SAT (wSAT)** A sharing rule $\xi \in \mathcal{S}$ satisfies *weak SAT* with respect to $f \in \mathcal{F}^N$ if for any profile $u \in \mathcal{U}^N$ and any Nash equilibrium $x^*$ of $G(f, \xi; u)$ the following holds:

$$
\xi_i(x^*) \geq f_i(x_i^*) \quad \forall i \in N.
$$

From the strict monotonicity of preferences, it is clear that SAT implies wSAT. ■

To illustrate Theorem 2, we provide the virtual production profiles corresponding to examples 1a, 1b, 2 and 3 of the previous section. In all of the following examples, Agent 1 receives her autarky bundle while Agent 2 strictly benefits from the pooling of the virtual technologies; her virtual utility gain is represented by a shaded area.

*Example 1a’.* $\xi^I$ gives priority to Agent 1. It is equivalent to pooling the production profile where Agent 2’s technology is useless ($f_2 \equiv 0$ and $f_1 \equiv F^*$), thus Agent 2 must wait for Agent 1 to be served before using the technology $F^*$. Clearly, $\xi^I$ improves upon autarky with respect to $(0, F^*)$ because Agent 2 can benefit from whatever is left over by Agent 1. See Figure 4.2.

*Example 1b’.* $\xi^J$ gives full priority to Agent 1 and Agent 2 alternatively. $\phi^J$ decomposes $F^*$ as in Figure 3. Observe that although $f_1$ and $f_2$ do not belong to $\mathcal{F}^c$, the aggregate technology $F^*$ does. Yet $\xi^J \notin S_j^c$.

*Example 2’.* $f_i(t) = \alpha_i F^*(\frac{x}{\alpha_i})$. Figure 4.2 is drawn for $\alpha_2 = 2\alpha_1 = \frac{2}{3}$.

*Example 3’.* The $f_i$’s can be recovered by using formula (5). See Figure 4.2.
Figure 3: The graph of $f_1'$ is ABIM, where IM is obtained by shifting CE $M_2$ units to the left. The graph of $f_2'$ is GHM, where GH is the arc BC shifted left $Q_1$ units.

### 4.3 Comment on the size of efficiency gains

While every agent is at least as well off under our “optimal” pooling method as in autarky, some are typically better off. The relative utility gain of an agent decreases when the relative performance of that agent’s technology improves. Therefore, in order to properly assess the magnitude of the gains obtained from our pooling method, one must consider comparable individual technologies.

We consider a economy where the $n$ private technologies are identical: $f_i \equiv f_1$ for all $i$. In this setting, our pooling method coincides with the serial surplus sharing of the aggregated technology, $F^*(t) = n f_1(t/n)$ for any $t > 0$. Preferences are assumed to be of the form:

$$u_1(x_1, y_1) = y_1 - \frac{\lambda_1}{2} x_1^2$$

and

$$u_j(x_j, y_j) = y_j - \frac{\lambda_2}{2} x_j^2$$

for any $j \neq 1$,

with $\lambda_1, \lambda_2 > 0$. Assume $M_1 = +\infty$. The reader can easily check by examining Figure 4.3 that the relative utility gain of Agent 1 goes to $n$ as $\lambda_1$ goes to 0 and
\( \lambda_2 \) increases to infinity:

\[
\lim_{\lambda_1 \to 0} \frac{u_1^{pooling}}{u_1^{autarky}} = n.
\]

In the limit, Agent 1 ends up operating the \( n \) individual technologies by herself. Hence, the potential relative utility gain from pooling is \((n - 1) \times 100\%\).

5 A characterization of fixed path methods

Let \( F \in \mathcal{F}^c \). In this section we show for the two-person case that fixed path methods are the only \( F \)-sharing rules exhibiting the strong incentive properties mentioned in Section 4.

**Theorem 3** Assume \( n = 2 \). For any \( F \in \mathcal{F}^c \) and any \( \xi \in \mathcal{S}^c_F \), the following statements are equivalent:

i) \( \xi \) is a fixed path method: \( \xi = \xi^{F,\phi} \) with \( \phi \in \mathcal{P}^c \),

ii) The game \( G(F, \xi; u) \) has at most one Nash equilibrium,

iii) In the game \( G(F, \xi; u) \), every Nash equilibrium is a strong equilibrium.

This result is analogous to Theorem 2 in Moulin and Shenker[12]. Our characterization of fixed path methods is obtained by replacing anonymity with
ZOZI in their characterization of the serial mechanism. Clearly, ZOZI is a weaker requirement than anonymity, hence the resulting larger class of $F$-sharing methods. We were only able to prove the result for $n = 2$, but suspect it holds for an arbitrary number of agents.

Theorems 2 and 3 together lead to an interesting result upon recalling that for any $f_1, \ldots, f_n \in \mathcal{F}^c$, the aggregate technology $F^*$ also belongs to $\mathcal{F}^c$. In the statement of the Corollary 1, incentive compatibility is meant in the sense of provisos ii) or iii) in the statement of Theorem 3; $\xi^{f, \phi^*}$ is the same as in the statement of Theorem 2.

**Corollary 1** Assume $n = 2$. For any $f \in (\mathcal{F}^c)^N$, $\xi^{f, \phi^*}$ is the unique smooth and incentive compatible $f$-pooling method satisfying SAT with respect to $f$.

Now to the proof of Theorem 3.

**Proof.** *Notation:* We say that a matrix $[\alpha_{i,j}]$ has a cycle $(i_1, i_2, i_3, \ldots, i_L)$ if the $i_k$’s form a non-repeating sequence with $\alpha_{i_L,i_1} \neq 0$ and $\alpha_{i_k,i_{k+1}} \neq 0$ for all $1 \leq k \leq L - 1$. A matrix which has no cycles of length greater than 1 is called acyclic. A square acyclic matrix must have an element $j$ such that $\alpha_{i,j} = 0$ for all $i \neq j$; we call such an element a tail element. Fix $F \in \mathcal{F}^c$, we say that a mechanism $\xi \in \mathcal{S}_F^c$ is acyclic at a point $x \in \mathbb{R}^N_+$ if the Jacobian matrix of $\xi$, $\frac{\partial \xi}{\partial x_j}$, is acyclic at that point. Notice that if element $j$ is a tail element of the Jacobian matrix of $\xi$ at a point $x$, then

$$\frac{\partial \xi_j}{\partial x_j}(x) = F'(|x|).$$

We are given an $F$-sharing rule $\xi \in \mathcal{S}_F^c$ satisfying one of the properties (ii) or (iii) in the statement of Theorem 3. We will show that $\xi$ must be a fixed path method.

We start the proof by restating two lemmas from the proof of Theorem 2 in [12]. Their proofs still hold in our setting, and we will only state these lemmas.

**Lemma 2** (Lemma 5 in [12]) $n \in \mathbb{N}$. Consider $\xi \in \mathcal{S}_F^c$. If every Nash equilibrium is a strong equilibrium, then $\xi$ is acyclic at all $x \in \mathbb{R}^N_+$.

**Lemma 3** (Lemma 6 in [12]) $n \in \mathbb{N}$. Consider $\xi \in \mathcal{S}_F^c$. If there is at most one Nash equilibrium of the game $G(F, \xi; u)$ for every profile $u \in \mathcal{U}^N$, then $\xi$ is acyclic at all $x \in \mathbb{R}^N_+$.

The heart of the proof consists in establishing the following lemma.

**Lemma 4** $n = 2$. Consider an $F$-sharing rule $\xi \in \mathcal{S}_F^c$. Such a rule is a fixed path method if and only if the matrix $\frac{\partial \xi}{\partial x_j}$ is acyclic for all $x \in \mathbb{R}^N_+$.

The “only if” part follows directly from the defining formula of fixed path methods, where it is clear that $\frac{\partial \xi}{\partial x_j}(x) = 0$ if and only if $\delta_j(x_j) \geq \delta_i(x_i)$ and $i \neq j$. 

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At this point, we are only able to prove the converse for $n = 2$. For notational simplicity, we define $SW$ (resp. $NE$) to be the subset of $\mathbb{R}^2_+$ where element 1 (resp. element 2) is a tail element of Jacobian matrix of $\xi$: $\frac{\partial \xi_2}{\partial x_1} = 0$ (resp. $\frac{\partial \xi_1}{\partial x_2} = 0$). We write $D = SW \cap NE$, $D$ is the subset of $\mathbb{R}^2_+$ on which the matrix $\frac{\partial \xi_i}{\partial x_j}$ is diagonal. We define also $SW^* = SW \setminus D$ and $NE^* = NE \setminus D$; by continuity of the partial derivatives of $\xi$ and acyclicity, $SW^*$ and $NE^*$ are open in $\mathbb{R}^2_+$ while $SW$ and $NE$ are closed.

The rest of the proof is divided into seven steps. We show that the set $D$ is the image of a path $\phi \in \mathcal{P}$ and deduce that $\xi$ must be the fixed path method generated by $\phi$. The statements of most steps will consist of two symmetrical statements (one per agent), we shall only establish one of them as the other follows by symmetry.

**Step 1** $D$ has empty interior.

By contradiction, assume there exists some $x \in D$ and some $\varepsilon > 0$ such that the closed ball $B(x, \varepsilon)$ is included in $D$. Choose four points $K, L, M, N \in B(x, \varepsilon)$ s.t.

$$
\begin{align*}
  x^K_1 &= x^M_1 < x^K_2 = x^N_2 \\
  x^K_2 &= x^L_2 < x^M_2 = x^N_2
\end{align*}
$$

See Figure 5. Notice that the line segments KM, KL, MN and LN are included in $D$.

Then,

$$
\xi_1(N) = \xi_1(K) + \int_{x^K_2}^{x^M_2} \frac{\partial \xi_1}{\partial x_2} (x^K_1, x_2) dx_2 + \int_{x^K_1}^{x^M_1} \frac{\partial \xi_1}{\partial x_1} (x_1, x^N_2) dx_1.
$$

Because $\frac{\partial \xi_1}{\partial x_2} = 0$ along KM and MN, it follows from (8) that

$$
\begin{align*}
  \xi_1(N) &= \xi_1(K) + \int_{x^K_1}^{x^M_1} F'(x_1 + x^N_2) dx_1 \\
  &\text{i.e. } \xi_1(N) = \xi_1(K) + F(|N|) - F(|M|).
\end{align*}
$$
Similarly,

\[ \xi_1(N) = \xi_1(K) + \int_{x^K}^{x^N} \frac{\partial \xi_1}{\partial x_1}(x_1, x_2^K) dx_1 + \int_{x_*^N} x_2 \frac{\partial \xi_1}{\partial x_2}(x_1^N, x_2) dx_2, \]

i.e., \( \xi_1(N) = \xi_1(K) + F(|L|) - F(|K|). \)

Hence, \( F(|N|) - F(|M|) = F(|L|) - F(|K|) \) in contradiction with the strict concavity of \( F \).

**Step 2**

(i) \( \mathbb{R}_+ \times \{0\} \subseteq SW \) and \( \{0\} \times \mathbb{R}_+ \subseteq NE. \)

(ii) Let \( a = (a_1, a_2) \in \mathbb{R}^2_+, \) then

\[ a \in SW^* \implies (a_1 + \lambda, a_2) \in SW^* \text{ for any } \lambda \geq 0, \text{ and} \]

\[ a \in NE^* \implies (a_1, a_2 + \lambda) \in NE^* \text{ for any } \lambda \geq 0. \]

The first statement follows from ZOZI: \( \xi_2(x_1, 0) = 0 \) for any \( x_1 \geq 0, \) therefore \( \frac{\partial \xi_2}{\partial x_1}(x_1, 0) = 0 \) for any \( x_1 \geq 0. \)

Let \( a = (a_1, a_2) \in SW^* \) and assume there exists \( \bar{a}_1 > a_1 \) such that \( (\bar{a}_1, a_2) \in NE. \) Because \( SW^* \) is open, let \( \bar{a}_1, a_1^+ \) be the largest interval containing \( a_1 \) on which \( (x_1, a_2) \in SW^*; \) note that it is non-empty. Because \( (0, a_2) \in NE \) (from (i)) and \( (\bar{a}_1, a_2) \in NE, \) it follows that \( 0 \leq a_1^+ < a_2^+ \leq \bar{a}_1. \) Also, by continuity of the partials of \( \xi, (a_1^-, a_2) \in D \) and \( (a_1^+, a_2) \in D. \)

\( SW^* \) being open, there exists a neighborhood of \( \{(x_1, a_2)|x_1 \in [a_1^-, a_1^+]\} \) which is included in \( SW^*. \) On this neighborhood, \( \frac{\partial \xi_2}{\partial x_1} = 0; \) i.e., \( \xi_2 \) is independent of \( x_1. \) Hence, the expression \( \frac{\partial^2 \xi_2}{\partial x_2}(x_1, x_2^+) - \xi_2(x_1, x_2) \) is also independent of \( x_1 \) on this neighborhood; therefore \( \frac{\partial \xi_2}{\partial x_2} \) is independent of \( x_1 \) on \( \{(x_1, a_2)|x_1 \in [a_1^-, a_1^+]\}. \)

Hence \( \frac{\partial \xi_2}{\partial x_2}(a_1^-, a_2) = \frac{\partial \xi_2}{\partial x_2}(a_1^+, a_2). \) Because \( j = 2 \) is a tail element of the Jacobian matrix of \( \xi \) at \( (a_1, a_2) \) and \( (a_1^+, a_2), \) it follows from (8) that \( \frac{\partial \xi_2}{\partial x_2}(a_1^-, a_2) = F'(a_1^- + a_2) \) and \( \frac{\partial \xi_2}{\partial x_2}(a_1^+, a_2) = F'(a_1^+ + a_2). \) Hence, \( F'(a_1^- + a_2) = F'(a_1^+ + a_2) \) contradicting the strict concavity of \( F. \)

We introduce some terminology. We say that a subset \( A \subseteq \mathbb{R}^2 \) is NW-comprehensive (resp. SE-comprehensive) if \( A + \mathbb{R}_- \times \mathbb{R}_+ \subseteq A \) (resp. \( A + \mathbb{R}_+ \times \mathbb{R}_- \subseteq A \)).

**Step 3** \( SW^* \) and \( SW \) are SE-comprehensive; \( NE^* \) and \( NE \) are NW-comprehensive.

Let \( a = (a_1, a_2) \in SW^* \) and \( x = (x_1, x_2) \) such that \( x_1 \geq a_1 \) and \( x_2 \leq a_2. \) If \( (x_1, x_2) \in NE^*, \) then we would have \( (x_1, a_2) \in SW^* \cap NE^* \) from the previous step, which is clearly impossible. Hence \( x \in SW. \) Therefore \( \xi_2 \) is independent of \( x_1 \) in the region south-east to \( a. \) It follows again (see Step 2) that \( \frac{\partial \xi_2}{\partial x_2} \) is also independent of \( x_1 \) on that domain.

Assume there exists \( b = (b_1, b_2) \in NE. \) The case \( b_2 = a_2 \) has been covered in the previous step, so we will assume \( b_2 < a_2 \) from now on. Consider the case where \( b_1 > a_1. \) From the preceding paragraph, \( b \in SW, \) therefore \( b \in D. \)
However, note that it follows from Step 2 that \((x_1, b_2) \in NE\) for any \(x_1 \in [a_1, b_1]\); hence, \((x_1, b_2) \in D\) for any \(x_1 \in [a_1, b_1]\). In particular, \((a_1, b_2) \in D\). Therefore, from the previous paragraph: \(\frac{\partial F}{\partial x_2}(a_1, b_2) = \frac{\partial F}{\partial x_2}(b_1, b_2)\). By (8), \(F'(a_1 + b_2) = F'(b_1 + b_2)\), contradicting the strict concavity of \(F\). If \(b_1 = a_1\), the result follows from the openness of \(SW^*\): there exists \(\varepsilon > 0\) such that \((x_1, a_2) \in SW^*\) for any \(x_1 \in [a_1 - \varepsilon, a_1]\). We repeat the previous argument.

We proved that \(SW^*\) is SE-comprehensive, a direct consequence is the NW-comprehensiveness of \(NE\). The rest of the claim can be proved symmetrically.

**Step 4** For any \((a_1, a_2) \in D\),

\[
\begin{align*}
\text{if } a_1 = 0 & \implies (0, x_2) \in D \quad \forall x_2 \in [0, a_2] \\
\text{if } a_2 = 0 & \implies (x_1, 0) \in D \quad \forall x_1 \in [0, a_1] & (5.\text{a}) \\
\text{if } a_2 > 0 & \implies (x_1, a_2) \in SW^* \quad \forall x_1 > a_1 \\
\text{if } a_1 > 0 & \implies (a_1, x_2) \in NE^* \quad \forall x_2 > a_2 & (5.\text{b}) \\
\text{if } a_1 > 0 & \implies (a_1, x_2) \notin D \quad \forall x_2 \neq a_2 \\
\text{if } a_2 > 0 & \implies (x_1, a_2) \notin D \quad \forall x_1 \neq a_1 & (5.\text{c})
\end{align*}
\]

Let \(a_2 \in \mathbb{R}_+\) such that \((0, a_2) \in D\). We know from Step 2 that \((0, x_2) \in NE\) for any \(x_2 \leq a_2\). It also follows from Step 2 that \((0, x_2) \notin NE^*\) for any \(x_2 \leq a_2\), thus proving (5.a).

We now establish (5.b). Let \((a_1, a_2) \in D\) such that \(a_2 > 0\). Assume there exists \(b_1 > a_1\) such that \((b_1, a_2) \in NE\); then by Step 3, \((b_1, a_2) \in D\). Because \(a_2 > 0\), it follows from Step 3 that for all \(x_2 < a_2\) and all \(x_1 \in [a_1, b_1]\), \((x_1, x_2) \in SW\). On that domain, \(\xi_3\) is independent of \(x_1\), therefore \(\frac{\partial F}{\partial x_2}\) is independent of \(x_1\) also. It follows that \(\frac{\partial F}{\partial x_2}(a_1, a_2) = \frac{\partial F}{\partial x_2}(b_1, a_2)\), which implies \(F'(a_1 + a_2) = F'(b_1 + a_2)\), in contradiction with the strict concavity of \(F\).

Let \((a_1, a_2) \in D\) such that \(a_1 > 0\). We only need to check \((a_1, x_2) \notin D\) for any \(x_2 < a_2\) as the case \(x_2 > a_2\) was covered in (5.b). For the sake of contradiction, assume there exists \(b_2 \in [0, a_2]\) such that \((a_1, b_2) \in D\). Because \(a_1 > 0\), it follows from (5.b) that \((a_1, a_2) \in NE^*\), a contradiction. Therefore (5.c) holds.

**Step 5** \(\exists i \in \{1, 2\} \quad \forall x_i \geq 0 \exists x_j \geq 0\) such that \((x_1, x_2) \in D\).

Assume the statement is not true. Then, there exists \(x_1 \geq 0\) such that for any \(\lambda \geq 0\), \((x_1, \lambda) \notin D\). From Step 1, \((x_1, 0) \in SW\), therefore by smoothness and acyclicity of \(\xi\), it must be that \((x_1, \lambda) \in SW\) for all \(\lambda \geq 0\). Similarly, there exists \(x_2 \geq 0\) such that \((\lambda, x_2) \in NE\) for any \(\lambda \geq 0\). Hence, \((x_1, x_2) \in SW \cap NE = D\), contradicting our assumption.

Without loss of generality, we will assume for the rest of the proof that for any \(x_1 \geq 0\) there exists \(x_2 \geq 0\) such that \((x_1, x_2) \in D\).

**Step 6** \(D\) is the graph of a continuous increasing path of \(\mathbb{R}_+^2\).
Define $P(x_1) = \max \{ x_2 \in \mathbb{R}_+ \mid (x_1, x_2) \in D \}$ for all $x_1 \geq 0$. It follows from Step 3 that $P$ is non-decreasing and strictly increasing on $P^{-1}(\mathbb{R}_+)$. Also, the graph of the restriction of $P$ to $\mathbb{R}_+$ is $D \cap [0, +\infty] \times \mathbb{R}_+$. Because the latter set is closed in $[0, +\infty] \times \mathbb{R}_+$ (as the intersection of $SW$ and $NE$), $P$ is continuous on $\mathbb{R}_+$.

Now let $a_1 = 0$, and define $l_2 = \lim_{x_1 \to 0} P(x_1)$; we claim that $l_2 = P(0)$. By closedness of $D$, $(0, l_2) \in D$. It follows that $l_2 \leq P(0)$ otherwise the very definition of $P$ would be contradicted. Now, if $P(0) > 0$, we show that $l_2 \geq P(0)$. If $l_2 < P(0)$, by continuity of $P$ on $\mathbb{R}_+$ there exists $x_1 > 0$ such that $P(x_1) < P(0)$, contradicting the fact that $P$ is non-decreasing. Hence $l_2 = P(0)$ and $P$ is continuous at zero.

Therefore $P$ is continuous on $\mathbb{R}_+$. It follows that

$$D = \{(0, x_2) \mid x_2 \in [0, P(0)]\} \cup \{(x_1, P(x_1)) \mid x_1 \geq 0\}.$$  

Define the function

$$\gamma : \quad t \mapsto \begin{cases} (0, t) & \text{if } t \leq P(0), \\ (\lambda, P(\lambda)) \text{ s.t. } \lambda + P(\lambda) = t & \text{otherwise.} \end{cases}$$

and write $\phi(t) = \gamma(t) \wedge (M_1, M_2)$ for all $t \geq 0$. $\phi$ is well defined by continuity and strict monotonicity of $P$.

**Step 7** $\xi$ is the $F$-fixed path method generated by $\phi$.

At any point $x = (x_1, x_2)$ on $D$, it follows from the definition of $D$ that $
abla_{x_1}^2 \xi_1(x) = \nabla_{x_2} \xi_2(x) = F'(|x|)$. Thus, for any point $x = (x_1, x_2) \in D$, taking the integral along $\phi$ from the origin to $x$ yields:

$$\xi_i(x) = \int_0^{\delta_i(x_1)} \frac{\partial \xi_1}{\partial x_1}(\phi(t))\phi_i(t) = \int_0^{\delta_i(x_1)} F'(t)\phi_i(t).$$

Now that $\xi$ is defined on $D$, it can easily be extended to all of $[0, M_1] \times [0, M_2]$ upon noticing that one agent receives all of the surplus after leaving $D$: it is the unique tail element of the Jacobian matrix of $\xi$ at $x$. I.e., for any $x = (x_1, x_2) \in [0, M_1] \times [0, M_2]$ (and, without loss, we assume $\delta_1(x_1) \leq \delta_2(x_2)$)

$$\xi_1(x) = \int_0^{\delta_1(x_1)} F'(t)\phi_1(t) \quad \text{and} \quad \xi_2(x) = \int_0^{\delta_2(x_2)} F'(t)\phi_2(t) + F(x_1 + x_2) - F(\delta_1(x_1)),$$

completing the proof of Theorem 3. 

**6 Concluding comments**

**A characterization of fixed path mechanisms** Let $f \in \mathcal{F}$. Consider the sharing rule $\xi^{f, \phi}$ as defined in Theorem 2, and define $\mu^* : \mathcal{U} \to \mathbb{Z}_{F^*}$ as the mechanism whose outcome is the unique Nash equilibrium outcome of the game.
\( G(f, \phi^*; u) \) for any \( u \in \mathcal{U}^N \). A standard result in implementation theory yields that \( \mu^* \) is group strategy-proof as it is a single-valued social choice function Nash implementable on a preference domain satisfying a "richness" condition, which is satisfied by \( \mathcal{U} \) (see, e.g. Dasgupta, Hammond and Maskin[4], Theorem 7.2.3).

We conjecture that \( \mu^* \) is the only mechanism satisfying simultaneously group strategy-proofness (GSP), voluntary participation with respect to \( f \) (VP(\( f \))), consumer sovereignty (CS: every level of input is attainable for all agents) and ZOZI.

**Conjecture 1** Let \( f \in \mathcal{F}^N \). A mechanism \( \mu : \mathcal{U}^N \to \mathcal{Z}_F \) satisfies GSP, VP(\( f \)), CS and ZOZI if and only if \( \mu = \mu^* \).

**Proof.** As argued above, the group strategy-proofness of \( \mu^* \) follows from Theorem 7.2.3 in Dasgupta, Hammond and Maskin[4]. ZOZI is obvious. VP(\( f \)) follows from Theorem 2 and CS from Lemma 1. The "if" part of the conjecture has been established.

We were not able to establish the "only if" part of the conjecture. We here provide a sketch of how we tried to proceed as well as a justification of our failure.

Lemma 1 in Moulin[11] can be readily adapted to the context of surplus sharing where the set of admissible supply levels is \( \mathbb{R}_+ \) for each agent (as opposed to \( \mathbb{N} \) in [11]). It follows that if \( \mu \) satisfies GSP, CS and ZOZI, there exists a function \( \xi \) from \( \mathbb{R}_{+}^N \) to itself which satisfies budget balance \((\sum_{i \in N} \xi_i(x) = F^*(|x|))\), ZOZI \((x_i = 0 \implies \xi_i(x) = 0)\) and strict supply monotonicity \((x'_i > x_i \implies \xi_i(x'_i, x_{-i}) > \xi_i(x))\) such that \( \mu(u) = (x_i, \xi_i(x))_{i \in N} \) for all \( u \in \mathcal{U}^N \). Moreover, for any \( u \) and \( x \) such that \( \mu(u) = (x_i, \xi_i(x))_{i \in N} \), \( x \) is a strong equilibrium of the game \( G(F^*, \xi; u) \). However, we do not know whether \( \xi \) is smooth, which prevents us from using our characterization theorem for fixed path methods (Theorem 3). If we could show that \( \xi \) is indeed a fixed path method, then VP would yield \( \xi = \xi^{\phi^*} \) with \( \phi^* \) defined as in Theorem 2. It would follow that \( \mu = \mu^* \). ■

**Technology pooling in the cost sharing context** For the past decade, cooperative production has been mostly modelled in the cost sharing context (see Moulin and Shenker[12]) where the variable controlled by an agent is her demand for output, and the sharing rule must divide the resulting total cost among the users. Formally, our problem of technology pooling amounts to having a profile of cost functions \( c = (c_1, \ldots, c_n) \), where each \( c_i \) is continuous, strictly convex and such that \( c_i(0) = 0 \). We denote by \( \mathcal{C} \) the class of cost functions, and by \( \mathcal{C}_c \) the subset of continuously differentiable cost functions. The aggregated cost function results from the cost-minimizing way to produce \( q \) units of output:
\[ C^*(q) = \min_{(y_1, \ldots, y_n) \in \mathbb{R}_+^n} \sum_{i=1}^n c_i(y_i) \]

Notice that if \( c_i \) and \( f_i \) are inverse of each other \((y_i = f_i(x_i) \iff x_i = c_i(y_i) \) for all \( x_i, y_i \in \mathbb{R}_+ \)) for all \( i \), they are dual representations of the same individual technologies. Therefore \( C^* \) and \( F^* \) are also inverse of each other as dual representation of the same aggregate technology.

A \( c \)-cost-pooling method \( \zeta_c^c \) is a \( C^* \)-cost-sharing rule: it associates to each vector of demands \( y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n \) a vector of cost shares \((x_1, \ldots, x_n) \in \mathbb{R}_+^n \) such that

\[ x_i = \zeta_i^c(y) \quad \text{for all} \quad i \in N, \quad \text{and} \quad \sum_i x_i = C^*(\sum_i y_i). \]

For any path \( \phi \in \mathcal{P} \), \( \zeta_c^{C,\phi} \) is a fixed path cost sharing method defined as in (3) by replacing \( F \) by \( C \). Given a cost function \( C \in \mathcal{C} \), a preference (utility) profile \( u \in \mathcal{U}_N \) and a \( C \)-sharing rule \( \zeta \), we denote by \( H(C, \zeta; u) \) the normal form game induced by \( \zeta \).

All of our results readily translate to the cost sharing framework.

**Theorem 4** For all \( C \in \mathcal{C} \), all \( \phi \in \mathcal{P} \) and all \( u \in \mathcal{U}_N \), the game \( H(C, \zeta^{C,\phi}; u) \) is dominance solvable. If \( F \in \mathcal{F} \) is the inverse of \( C \in \mathcal{C} \), the equilibrium outcomes of the games \( H(C, \zeta^{C,\phi}; u) \) and \( G(F, \zeta^{F,\phi}; u) \) coincide.

In the following statement, SAT is defined as before: no agent prefers the autarkic outcome.

**Theorem 5** 1)For any \( c \in \mathcal{C}^N \), there exists a unique path \( \psi^* \in \mathcal{P} \) such that \( \zeta^{c,\psi^*} \) satisfies SAT with respect to \( c \). It is the unique solution of

\[ \min_{(y_1, \ldots, y_n) \in \mathbb{R}_+^n} \sum_{i=1}^n c_i(y_i) \]

for any \( q \geq 0 \).

2)Conversely, for any \( C^* \in \mathcal{F} \) and any \( \psi^* \in \mathcal{P} \), there exists a unique technology profile \( c \) decomposing \( C^* \) such that \( \zeta^{c,\psi^*} \) satisfies SAT with respect to \( c \). For any \( i \in N \), \( c_i \) is given by

\[ c_i(y_i) = \int_0^{y_i} C^*(\varepsilon_i^*(q))dq \]

for all \( 0 \leq y_i \), where \( \varepsilon_i^* \) is defined relative to \( \psi_i^* \) as in (2).

3)Moreover, if \( f_i \) and \( c_i \) are inverse of each other for all \( i \), then for any \( q \geq 0 \) and for all \( i \in N \),

\[ \psi_i^*(q) = f_i(\phi_i^*(t)) \]

with \( t = C^*(q) \). I.e., \( \psi_i^*(q) \) is the efficient share of output produced with technology \( i \) when \( q \) total units are produced.
Finally, the statement of Theorem 3 is barely affected by the change of context.

**Theorem 6** Assume \( n = 2 \). For any \( C \in C^c \) and any \( C \)-cost sharing rule \( \zeta \), the following statements are equivalent:

i) \( \zeta \) is a fixed path method: \( \zeta = \zeta^{C,\phi} \) with \( \phi \in \mathcal{P}^c \).

ii) The game \( H(C, \zeta; u) \) has at most one Nash equilibrium.

iii) In the game \( H(C, \zeta; u) \), every Nash equilibrium is a strong equilibrium.

**Conclusion** The present paper describes the unique way to pool private technologies in a manner which is technologically efficient, incentive compatible and autarkically individually rational. It is a fixed path method uniquely defined by the technology profile of the agents. Uniqueness could only be proved rigorously for the two-agent case (Corollary 1). However, we conjecture that the result holds for an arbitrary number of agents: if Lemma 4 could be established for any \( n \in \mathbb{N} \), the generalization of our result would immediately follow.

Our model assumed the private technologies to be common knowledge while the possibility of strategic manipulations stemmed from private information about the agents’ preferences. It would be interesting to examine a situation where agents could also misrepresent their own production possibilities.\(^3\) This setting would give agents more possibilities of misrepresenting their characteristics while not giving the mechanism designer more freedom in devising a pooling method. Hence, the class of incentive compatible pooling methods satisfying voluntary participation would not expand relative to the present setting. Yet, we suspect our “optimal” pooling method to still belong to this class, and hence be its unique member.

**A Appendix**

**A.1 Proof of Theorem 1**

**Proof.** Notations and preliminary lemma. We fix a production function \( F \in \mathcal{F} \), a path \( \phi \in \mathcal{P} \) and a preference profile \( u \in \mathcal{U}^N \). As no confusion may arise, we shall write \( \xi(x) \) instead of \( \xi^{F,\phi}(x) \) for all \( x \in \times_{i \in N}[0, M_i] \). Also, we write:

(i) \( \delta(x_1, \ldots, x_n) = (\delta_1(x_1), \delta_2(x_2), \ldots, \delta_n(x_n)) \) for any \( x \in \times_{i \in N}[0, M_i] \),

(ii) \( (t_1, t_2, \ldots, t_{i-1}, t_i, (n-i)) \) is the vector of \( \mathbb{R}^N_+ \) with the last \((n-i)\) coordinates equal to \( t_i \),

(iii) for any \( (t_1, \ldots, t_n) \in \mathbb{R}^N_+ \), \( \phi(t_1, \ldots, t_n) = (\phi_1(t_1), \phi_2(t_2), \ldots, \phi_n(t_n)) \) with a slight abuse of notation.

We restate (but do not prove) the following useful lemma, which is the analog of Lemma 2 in Moulin and Shenker\(^1\) for the surplus sharing context.

\(^3\)In a different setting, Shin and Suh\(^2\) allowed for misrepresentations of technologies.
Lemma 5 Let $h_1$ and $h_2$ be two increasing and strictly concave functions of $\mathbb{R}_+$ to itself that coincide up to $\lambda_0$:

$$h_1(\lambda) = h_2(\lambda) \quad \text{for all } \lambda \text{ s.t. } 0 \leq \lambda \leq \lambda_0.$$ 

Then for every $u_i \in \mathcal{U}$, the (unique) maximizers of $u_i(\lambda, h_j(\lambda))$ on $\mathbb{R}_+$, denoted $\lambda_j$, $j = 1, 2$, are on the same side of $\lambda_0$:

$$\lambda_1 \geq \lambda_0 \iff \lambda_2 \geq \lambda_0, \quad \lambda_1 = \lambda_0 \iff \lambda_2 = \lambda_0.$$ 

The proof of Theorem 1 requires two steps. In Step 1, we define an outcome $(x_i^*, y_i^*)$, $i \in N$, and show that $x^*$ is a Nash equilibrium of the game $G(F, \phi; u)$. In Step 2 we show that the outcome left after the successive elimination of dominated strategies is unique.

**Step 1.** For any $i \in N$, let $x_i$ be the unique solution of

$$\max_{x_i} u_i(x_i, \xi_i \circ \phi(\delta_i(x_i) \cdot n)) \quad (9)$$

To be consistent with the terminology of Moulin and Shenker[12], $x_i$ could be called the “unanimity supply” of agent $i$ relative to the path $\phi$. Note that it follows from the definition of $\xi$ that the amount of output received by agent $i$ is non-increasing in $x_j$, $j \neq i$, and that $\xi_j(x_i, x'_{-i}) = \xi_i \circ \phi(\delta_i(x_i) \cdot n)$ for any $x'_{-i} \in \times_{j \in N \setminus \{i\}}[0, M_j]$ such that $x'_j \geq \phi_j(\delta_i(x_i))$, $j \neq i$. Therefore, for any level of supply $x_i \geq 0$, $\xi_i \circ \phi(\delta_i(x_i) \cdot n)$ is the smallest amount of output that agent $i$ can receive. Consequently, $x_i$ is the strategy for which agent $i$’s guaranteed utility level is the highest.

Pick an agent $i$ with the lowest $\delta_i(x_i)$; we assume, without loss, that $i = 1$. Set $x_1^* = x_1$, $t_1^* = \delta_1(x_1^*)$ and $y_1^* = \xi_1 \circ \phi(t_1^* \cdot n)$. Next, solve for all agents $i \geq 2$ the following program:

$$\max_{x_i} u_i(x_i, \xi_i \circ \phi(t_1^*, \delta_i(x_i) \cdot (n - 1))) \quad (10)$$

Note that the unique solution, $x_i$, cannot be smaller than $\phi_i(t_1^*)$. This follows from the fact that the solution of (9) is not smaller than $\phi_i(t_1^*)$ and, because $\xi_i \circ \phi(\delta_i(x_i) \cdot n)$ and $\xi_i \circ \phi(t_1^*, \delta_i(x_i) \cdot (n - 1))$ are two increasing and strictly concave functions coinciding up to $x_i = \phi_i(t_1^*)$, we can apply Lemma 5.

Pick an agent $i$ whose solution to program (10) is lowest and assume, without loss, that $i = 2$. Set $x_2^*$ to be the corresponding solution, $t_2^* = \delta_2(x_2^*)$ and $y_2^* = \xi_2 \circ \phi(t_1^* \cdot t_2^* \cdot (n - 1))$. We have just shown $t_1^* \leq t_2^*$.

To complete the inductive definition, assume we have constructed a sequence $(t_1^*, x_1^*, y_1^*)$ up to $i = k$, non-decreasing in the first argument. Then, compute for all $i \geq k + 1$ the solution of the program

$$\max_{x_i} u_i(x_i, \xi_i \circ \phi(t_1^*, \ldots, t_k^*, \delta_i(x_i) \cdot (n - k))) \quad (11)$$

and assume, without loss, that $x_{k+1} = x_{k+1}^*$ is one of the lowest such supplies. Write $t_{k+1}^* = \delta_{k+1}(x_{k+1}^*)$ and observe that $t_{k+1}^* \geq t_k^*$. This follows from
Lemma 5 again: the two increasing and strictly concave functions

\[ \xi_{k+1} \circ \phi(t^*_1, \ldots, t^*_{k-1}, \delta_{k+1}(x_{k+1}) \cdot (n - k + 1)) \quad \text{and} \quad \xi_{k+1} \circ \phi(t^*_1, \ldots, t^*_{k-1}, t^*_k, \delta_{k+1}(x_{k+1}) \cdot (n - k)) \]

coincide up to \( \phi_{k+1}(t^*_k) \) and the maximum of \( u_{k+1} \) on the graph of the former is not smaller than \( \phi_{k+1}(t^*_k) \). We set \( y^*_{k+1} = \xi_{k+1} \circ \phi(t^*_1, \ldots, t^*_{k}, t^*_k, \delta_{k+1}(x_{k+1}) \cdot (n - k)). \)

We now check that \( x^* \) is a Nash equilibrium of the game \( G(F(\phi; u)) \). Pick a player \( i \) and consider the two functions

\[ h_1(\lambda) = \xi_i \circ \phi(t^*_1, \ldots, t^*_{i-1}, \delta_i(\lambda) \cdot (n - i + 1)) \quad \text{and} \quad h_2(\lambda) = \xi_i \circ \phi(\delta_i(\lambda), t^*_{i-1}). \]

From the definition of \( \xi \) and the fact that the sequence \( t^*_j \) is non-decreasing in \( j \), we deduce that these two functions coincide on \( [0, \phi_i(t^*_i)] = [0, x^*_i] \). Moreover, \( x^*_i \) maximizes \( u_i(\lambda, h_1(\lambda)) \) on \( \mathbb{R}_+ \). Thus, Lemma 5 implies that \( x^*_i \) also maximizes \( u_i(\lambda, h_2(\lambda)) \) on \( \mathbb{R}_+ \), which is the desired Nash equilibrium property.

**Step 2.** This step consists in showing that the outcome \( (x^*_i, y^*_i), i = 1, \ldots, n \), constructed in Step 1 is the only outcome left after the successive elimination of strictly dominated strategies. The proof provided in Moulin and Shenker [12] can be easily adapted to our setting in the same way that we adapted Step 1.

### A.2 Proof of Lemma 1

**Proof.** Let \( F \in \mathcal{F}, \phi \in \mathcal{P}, i \in N \) and \( x_i > 0 \). Again, we write \( \xi \) instead of \( \xi^{F, \phi} \). Consider a preference (utility) \( u^*_i \) which is quasi-linear with respect to \( y_i \) such that its indifference curves are piecewise linear with a single kink at \( (x_i, y_i) \) for any \( y_i \in \mathbb{R} \). Set the slope of these indifference curves to be no greater than \( F'_{x_i}(\delta_i(x_i)) \) before \( x_i \) and no smaller than \( F'_{x_i}(x_i) \) after \( x_i \); where “before \( x_i \)” (resp. “after \( x_i \)” ) stands for “at any point of \( \mathbb{R}_+ \times \mathbb{R} \) with first coordinate smaller (resp. greater) than \( x_i \).”

We show below that the former quantity is the smallest variation in output that agent \( i \) can obtain via \( \xi \) by deviating infinitesimally from \( x_i \); it corresponds to the case where agent \( i \) is the first one served along the path (i.e., the agent with smallest \( \delta_j(x_j) \)). On the other hand, \( F'(x_i) \) is the largest variation in output obtainable via \( \xi \) at \( x_i \) by deviating marginally from \( x_i \). It corresponds to the case where agent \( i \) receives all the output up to \( F'(x_i) \) (\( \delta_j(x_j) = 0 \) for all \( j \neq i \)).

Indeed, let \( x_{-i} \in \mathbb{R}^{|N\setminus i|}_+ \); then, from the definition of \( \xi \), and keeping the notations from the proof of Theorem 1,

\[ \frac{\partial}{\partial x_i} \xi_i(\lambda, x_{-i}) = F'_{-i}([(\lambda, x_{-i}) \wedge \phi(\delta_i(\lambda) \cdot n)]) \quad \text{and} \quad \frac{\partial}{\partial x_i} \xi_i(\lambda, x_{-i}) = F'_{+i}([(\lambda, x_{-i}) \wedge \phi(\delta_i(\lambda) \cdot n)]) \]

As the \( i \)th component of both vectors \( x \) and \( \phi(\delta_i(x_i) \cdot n) \) is equal to \( x_i \), the concavity of \( F \) yields \( F'_+(|x \wedge \phi(\delta_i(x_i) \cdot n)|) \leq F'_{+i}(x_i) \). Moreover, the concavity of \( F \) also yields \( F'_-(|x \wedge \phi(\delta_i(x_i) \cdot n)|) \geq F'_{-i}(|\phi(\delta_i(x_i) \cdot n)|) \); notice that this
last term equals $F'_-(\delta_i(x_i))$. It follows from these two facts that:

$$\frac{\partial^-}{\partial \lambda} \xi_i(\lambda, x_{-i}) \bigg|_{\lambda=x_i} \geq F'_-(\delta_i(x_i)) \quad \text{and} \quad \frac{\partial^+}{\partial \lambda} \xi_i(\lambda, x_{-i}) \bigg|_{\lambda=x_i} \leq F'_+(x_i) .$$

Hence, for any $x_{-i} \in \mathbb{R}^{N\setminus i}$, the slope of $\xi_i(\lambda, x_{-i})$ at $\lambda = x_i$ lies between $F'_-(\delta_i(x_i))$ and $F'_+(x_i)$. It follows from the strict concavity of $\xi_i(\cdot, x_{-i})$ that $x_i$ maximizes $u_i^*(\lambda, \xi_i(\lambda, x_{-i}))$ on $\mathbb{R}^{N\setminus i}$ for any $x_{-i} \in \mathbb{R}^{N\setminus i}$, completing the proof of the lemma. ■

References


