Risk, Uncertainty and Asset Prices

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Abstract:

We identify the relative importance of changes in the conditional variance of fundamentals (which
we call "uncertainty") and changes in risk aversion in the determination of the term structure, equity
prices and risk premiums. Theoretically, we introduce persistent time-varying uncertainty about
the fundamentals in an external habit model. The model matches the dynamics of dividend and
consumption growth, including their volatility dynamics and many salient asset market phenomena.
While the variation in price-dividend ratios and the equity risk premium is primarily driven by risk
aversion, uncertainty plays a large role in the term structure and is the driver of counter-cyclical
volatility of asset returns.

1 Introduction

Without variation in discount rates, it is difficult to explain the behavior of aggregate stock prices within the confines of rational pricing models. There are two main sources of asset price fluctuations and risk premiums: changes in the conditional variance of fundamentals (either consumption growth or dividend growth) or changes in risk aversion and risk preferences. An old literature [Poterba and Summers (1986), Barsky (1989), Abel (1988), Kandel and Stambaugh (1990)] and recent work by Bansal and Yaron (2004) and Bansal and Lundblad (2002) focus primarily on the effect of changes in economic uncertainty on stock prices and risk premiums. However, the work of Campbell and Cochrane (1999) [CC henceforth] has made changes in risk aversion the main focus of current research. They show that a model with counter-cyclical risk aversion can account for a large equity premium, substantial variation in equity returns and price-dividend ratios and long-horizon predictability of returns.

In this article, we try to identify the relative importance of changes in the conditional variance of fundamentals (which we call "uncertainty") and changes in risk aversion [1]. We build on the external habit model formulated in Bekaert, Engstrom and Grenadier (2004) which features stochastic risk aversion and introduce persistent time-varying uncertainty in the fundamentals. We explore the effects of both on price dividend ratios, equity risk premiums, the conditional variability of equity returns and the term structure, both theoretically and empirically. To differentiate time-varying uncertainty from stochastic risk aversion empirically, we use information on higher moments in dividend and consumption growth and the conditional relation between their volatility and a number of instruments.

The model is consistent with the empirical volatility dynamics of dividend and consumption growth and matches a large number of salient asset market features, including a large equity premium and low risk free rate and the volatilities of equity returns, price-dividend ratios and interest rates. We find that variation in the equity premium is driven by both risk aversion and uncertainty with risk aversion dominating. However, variation in asset prices (consol prices and price-dividend ratios) is primarily due to changes in risk aversion. These results arise because risk aversion acts primarily as a level factor in the term structure while uncertainty affects both the level and the slope of the

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[1] Hence, the term uncertainty is used in a different meaning than in the growing literature on Knightian uncertainty, see for instance Epstein and Schneider (2004). However, economic uncertainty is the standard term to denote heteroskedasticity in the fundamentals in both the asset pricing and macroeconomic literature. It is also consistent with a small literature in international finance which has focused on the effect of changes in uncertainty on exchange rates and currency risk premiums, see Hodrick (1989, 1990) and Bekaert (1996). The Hodrick (1989) paper provided the obvious inspiration for the title to this paper. While "risk" is short for "risk aversion" in the title, we avoid confusion throughout the paper contrasting economic uncertainty (amount of risk) and risk aversion (price of risk).

real term structure and also governs the riskiness of the equity cash flow stream. Consequently, our work provides a new perspective on recent advances in asset pricing modelling. We confirm the importance of economic uncertainty as stressed by Bansal and Yaron (2004) and Kandel and Stambaugh (1990) but show that changes in risk aversion are critical too. However, the main channel through which risk aversion affects asset prices in our model is the term structure, a channel shut off in the original CC paper while stressed by the older partial equilibrium work of Barsky (1989). We more generally demonstrate that information in the term structure has important implications for the identification of structural parameters.

The remainder of the article is organized as follows. The second section sets out the theoretical model and motivates the use of our state variables to model time-varying uncertainty of both dividend and consumption growth. In the third section, we derive closed-from solutions for price-dividend ratios and real and nominal bond prices as a function of the state variables and model parameters. In the fourth section, we set out our empirical strategy. We use the General Method of Moments (Hansen (1982), GMM henceforth) to estimate the parameters of the model. The fifth section reports parameter estimates and discusses how well the model fits salient features of the data. The sixth section reports various variance decompositions and dissects how uncertainty and risk aversion affect asset prices. Section 7 examines the robustness of our results to the use of post-war data and clarifies the link and differences between our model and those of Abel (1988), Wu (2001), Bansal and Yaron (2004) and CC. Section 8 concludes.

2 Theoretical Model

2.1 Fundamentals and Uncertainty

To model fundamentals and uncertainty, we start by modelling dividend growth as an AR(1) process with stochastic volatility:

$$\Delta d_t = \mu_d + \rho_{du} u_{t-1} + \sqrt{v_{t-1}} \left(\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v \right) \tag{1}$$

$$v_t = \mu_v + \rho_{vv} v_{t-1} + \sigma_{vv} \sqrt{v_{t-1}} \varepsilon_t^v$$

where $d_t = \log(D_t)$ denotes log dividends, u_t is the demeaned and detrended log consumption-dividend ratio (described further below) and v_t represents "uncertainty," and is proportional to the

conditional volatility of the dividend growth process. All innovations in the model, including ε_t^d and ε_t^v follow independent $N(0,1)$ distributions. Consequently, covariances must be explicitly parameterized. With this specification, the conditional mean of dividend growth varies potentially with past values of the consumption-dividend ratio, which is expected to be a slowly moving stationary process. Uncertainty itself follows a square-root process and may be arbitrarily correlated with dividend growth through the σ_{dv} parameter.[2] Because it is a latent factor, v_t can be scaled arbitrarily without empirical consequence and we therefore fix its unconditional mean at unity.

While consumption and dividends coincide in the original Lucas (1978) framework and many subsequent studies, recent papers have emphasized the importance of recognizing that consumption is financed by sources of income outside of the aggregate equity dividend stream [see for example Santos and Veronesi (2006)]. We model consumption as stochastically cointegrated with dividends, in a fashion similar to Bansal, Dittmar and Lundblad (2005), so that the consumption dividend ratio, u_t, becomes a relevant state variable. While there is a debate on whether the cointegrating factor should be (1,-1) (see Hansen, Heaton and Li (2005)), we follow Bekaert, Engstrom and Grenadier (2004) who find the consumption-dividend ratio to be stationary. We model u_t symmetrically with dividend growth,

$$u_t = \mu_u + \rho_{uu} u_{t-1} + \sigma_{ud} \left(\Delta d_t - E_{t-1} \left[\Delta d_t \right] \right) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u. \tag{2}$$

By definition, consumption growth, Δc_t, is

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\begin{aligned}
\Delta c_t &= \delta + \Delta d_t + \Delta u_t \\
&= (\delta + \mu_u + \mu_d) + (\rho_{du} + \rho_{uu} - 1) u_{t-1} + (1 + \sigma_{ud}) \sqrt{v_{t-1}} \left(\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v \right) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u. \tag{3}
\end{aligned}
$$

Note that δ and μ_u cannot be jointly identified. We proceed by setting the unconditional mean of u_t to zero and then identify δ as the difference in means of consumption and dividend growth.[3] Consequently, the consumption growth specification accommodates arbitrary correlation between dividend and consumption growth, with heteroskedasticity driven by v_t. The conditional means of both consumption and dividend growth depend on the consumption-dividend ratio, which is an

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[2] In discrete time, the square root process does not guarantee that v_t is bounded below by zero. However, by imposing a lower bound on u_v, the process rarely goes below zero. In any case, we use $\max[v_t, 0]$ under the square root sign in any simulation. In deriving pricing solutions, we ignore the mass below zero which has a negligible effect on the results.

[3] The presence of δ means that u_t should be interpreted as the demeaned and detrended log consumption-dividend ratio.

$AR(1)$ process. Consequently, the reduced form model for dividend and consumption growth is an $ARMA(1,1)$ which can accommodate either the standard nearly uncorrelated processes widely assumed in the literature, or the Bansal and Yaron (2004) specification where consumption and dividend growth have a long-run predictable component. Bansal and Yaron (2004) do not link the long run component to the consumption-dividend ratio as they do not assume consumption and dividends are cointegrated.

Our specification raises two important questions. First, is there heteroskedasticity in consumption and dividend growth data? Second, can this heteroskedasticity be captured using our single latent variable specification? In section 4, we marshal affirmative evidence regarding both questions.

2.2 Investor Preferences

Following CC, consider a complete markets economy as in Lucas (1978), but modify the preferences of the representative agent to have the form:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\frac{(C_t-H_t)^{1-\gamma}-1}{1-\gamma}\right], \tag{4}$$

where C_t is aggregate consumption and H_t is an exogenous "external habit stock" with $C_t > H_t$.

One motivation for an "external" habit stock is the "keeping up with the Joneses" framework of Abel (1990, 1999) where H_t represents past or current aggregate consumption. Small individual investors take H_t as given, and then evaluate their own utility relative to that benchmark.[4] In CC, H_t is taken as an exogenously modelled subsistence or habit level. In this situation, the local coefficient of relative risk aversion can be shown to be $\gamma\frac{C_t}{C_t-H_t}$, where $\left(\frac{C_t-H_t}{C_t}\right)$ is defined as the surplus ratio. As the surplus ratio goes to zero, the consumer's risk aversion tends toward infinity. In our model, we view the inverse of the surplus ratio as a preference shock, which we denote by Q_t. Thus, we have $Q_t \equiv \frac{C_t}{C_t-H_t}$, in which case local risk aversion is now characterized by γQ_t, and $Q_t > 1$. Risk aversion is the elasticity of the value function with respect to wealth, but the local curvature plays a major role in determining its value, see CC. As Q_t changes over time, the representative consumer investor's "moodiness" changes, which led Bekaert, Engstrom and Grenadier (2004) to label this a "moody investor economy."

The marginal rate of substitution in this model determines the real pricing kernel, which we

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[4]For empirical analyses of habit formation models where habit depends on past consumption, see Heaton (1995) and Bekaert (1996).

denote by M_t. Taking the ratio of marginal utilities of time $t+1$ and t, we obtain:

$$M_{t+1} = \beta \frac{(C_{t+1}/C_t)^{-\gamma}}{(Q_{t+1}/Q_t)^{-\gamma}} \tag{5}$$

$$= \beta \exp \left[-\gamma \Delta c_{t+1} + \gamma (q_{t+1} - q_t) \right],$$

where $q_t = \ln(Q_t)$.

We proceed by assuming q_t follows an autoregressive square root process which is contemporaneously correlated with fundamentals, but also possesses its own innovation,

$$q_t = \mu_q + \rho_{qq} q_{t-1} + \sigma_{qc} \left(\Delta c_t - E_{t-1} \left[\Delta c_t \right] \right) + \sigma_{qq} \sqrt{q_{t-1}} \varepsilon_t^q \tag{6}$$

As with v_t, q_t is a latent variable and can therefore be scaled arbitrarily without economic consequence; we therefore set its unconditional mean at unity. In our specification, Q_t is not forced to be perfectly negatively correlated with consumption growth as in CC. In this sense, our preference shock specification is closer in spirit to that of Brandt and Wang (2003) who allow for Q_t to be correlated with other business-cycle factors, or Lettau and Watcher (2007), who also allow for shocks to preferences uncorrelated with fundamentals. Only if $\sigma_{qq} = 0$ and $\sigma_{qc} < 0$ does a Campbell Cochrane like specification obtain where consumption growth and risk aversion shocks are perfectly negatively correlated. Consequently, we can test whether independent preference shocks are an important part of variation in risk aversion or whether its variation is dominated by shocks to fundamentals. Note that the covariance between q_t and consumption growth and the variance of q_t both depend on v_t and consequently may inherit its cyclical properties.

2.3 Inflation

When confronting consumption-based models with the data, real variables have to be translated into nominal terms. Furthermore, inflation may be important in realistically modeling the joint dynamics of equity returns, the short rate and the term spread. Therefore, we append the model with a simple inflation process,

$$\pi_t = \mu_\pi + \rho_{\pi\pi} \pi_{t-1} + \kappa E_{t-1} \left[\Delta c_t \right] + \sigma_\pi \varepsilon_t^\pi \tag{7}$$

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The impact of expected "real" growth on inflation can be motivated by macroeconomic intuition, such as the Phillips curve (in which case we expect κ to be positive). Because there is no contemporaneous correlation between this inflation process and the real pricing kernel, the one-period short rate will not include an inflation risk premium. However, non-zero correlations between the pricing kernel and inflation may arise at longer horizons due to the impact of $E_{t-1}[\Delta c_t]$ on the conditional mean of inflation. Note that expected real consumption growth varies only with u_t; hence, the specification in Equation (7) is equivalent to one where $\rho_{\pi u} u_{t-1}$ replaces $\kappa E_{t-1}[\Delta c_t]$.

To price nominal assets, we define the nominal pricing kernel, $\widehat{m}_{t+1}$, that is a simple transformation of the log real pricing kernel, m_{t+1},

$$\widehat{m}_{t+1} = m_{t+1} - \pi_{t+1}. \tag{8}$$

To summarize, our model has five state variables with dynamics described by the equations,

$$\Delta d_t = \mu_d + \rho_{du} u_{t-1} + \sqrt{v_{t-1}} \left(\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v \right)$$

$$v_t = \mu_v + \rho_{vv} v_{t-1} + \sigma_{vv} \sqrt{v_{t-1}} \varepsilon_t^v$$

$$u_t = \rho_{uu} u_{t-1} + \sigma_{ud} \left(\Delta d_t - E_{t-1}[\Delta d_t] \right) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u$$

$$q_t = \mu_q + \rho_{qq} q_{t-1} + \sigma_{qc} \left(\Delta c_t - E_{t-1}[\Delta c_t] \right) + \sigma_{qq} \sqrt{q_{t-1}} \varepsilon_t^q$$

$$\pi_t = \mu_\pi + \rho_{\pi\pi} \pi_{t-1} + \rho_{\pi u} u_{t-1} + \sigma_{\pi\pi} \varepsilon_t^\pi \tag{9}$$

with $\Delta c_t = \delta + \Delta d_t + \Delta u_t$.

As discussed above, the unconditional means of v_t and q_t are set equal to unity so that μ_v and μ_q are not free parameters. Finally, the real pricing kernel can be represented by the expression,

$$m_{t+1} = \ln(\beta) - \gamma \left(\delta + \Delta u_{t+1} + \Delta d_{t+1} \right) + \gamma \Delta q_{t+1} \tag{10}$$

We collect the 19 model parameters in the vector, "Ψ,"

$$\Psi = \begin{bmatrix} \mu_d, \mu_\pi, \rho_{du}, \rho_{\pi\pi}, \rho_{\pi u}, \rho_{uu}, \rho_{vv}, \rho_{qq}, \dots \\ \sigma_{dd}, \sigma_{dv}, \sigma_{\pi\pi}, \sigma_{ud}, \sigma_{uu}, \sigma_{vv}, \sigma_{qc}, \sigma_{qq}, \delta, \beta, \gamma \end{bmatrix}'. \tag{11}$$

3 Asset Pricing

In this section, we present exact solutions for asset prices. Our model involves more state variables and parameters than the existing literature, making it difficult to trace pricing effects back to any single parameter's value. Therefore we defer providing part of the economic intuition for the pricing equations to Section 6. There, we discuss the results and their economic interpretation in the context of the model simultaneously.

The general pricing principle in this model follows the framework of Bekaert and Grenadier (2001). Assume an asset pays a real coupon stream $K_{t+\tau}$, $\tau = 1, 2...T$. We consider three assets: a real consol with $K_{t+\tau} = 1$, $T = \infty$, a nominal consol with $K_{t+\tau} = \Pi_{t,\tau}^{-1}$, $T = \infty$, (where $\Pi_{t,\tau}$ represents cumulative gross inflation from t to τ) and equity with $K_{t+\tau} = D_{t+\tau}$, $T = \infty$. The case of equity is slightly more complex because dividends are non-stationary (see below). Then, the price-coupon ratio can be written as

$$PC_t = E_t \left\{ \sum_{n=1}^{n=T} \exp \left[\sum_{j=1}^{n} (m_{t+j} + \Delta k_{t+j}) \right] \right\} \tag{12}$$

By induction, it is straight forward to show that

$$PC_t = \sum_{n=1}^{n=T} \exp \left(A_n + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t \right) \tag{13}$$

with

$$X_n = f^X \left(A_{n-1}, C_{n-1}, D_{n-1}, E_{n-1}, F_{n-1}, \Psi \right)$$

for $X \in [A, C, D, E, F]$. The exact form of these functions depends on the particular coupon stream. Note that Δd_t is not strictly a priced state variable as its conditional mean only depends on u_{t-1}. The Appendix provides a self-contained discussion of the pricing of real bonds (bonds that pay out 1 unit of the consumption good at a particular point in time), nominal bonds and finally equity. Here we provide a summary, with proposition numbers referring to the Appendix.

3.1 Term Structure

The basic building block for pricing assets is the term structure of real zero coupon bonds. The well known recursive pricing relationship governing the term structure of these bond prices is

$$P_{n,t}^{rz} = E_t \left[M_{t+1} P_{n-1,t+1}^{rz} \right] \tag{14}$$

where $P_{n,t}^{rz}$ is the price of a real zero coupon bond at time t with maturity at time $t + n$. The following proposition summarizes the solution for these bond prices. We solve the model for a slightly generalized (but notation saving) case where $q_t = \mu_q + \rho_{qq} q_{t-1} + \sqrt{v_{t-1}} \left(\sigma_{qd} \varepsilon_t^d + \sigma_{qu} \varepsilon_t^u + \sigma_{qv} \varepsilon_t^v \right) + \sqrt{q_{t-1}} \sigma_{qq} \varepsilon_t^q$. Our current model obtains when

$$\sigma_{qd} = \sigma_{qc} \sigma_{dd} \left(1 + \sigma_{ud} \right)$$

$$\sigma_{qu} = \sigma_{qc} \sigma_{uu}$$

$$\sigma_{qv} = \sigma_{qc} \sigma_{dv} \left(1 + \sigma_{ud} \right). \tag{15}$$

Proposition 1 *For the economy described by Equations (9) and (10), the prices of real, risk free, zero coupon bonds are given by*

$$P_{n,t}^{rz} = \exp \left(A_n + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t \right) \tag{16}$$

where

$$A_n = f^A \left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi \right)$$
$$C_n = f^C \left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi \right)$$
$$D_n = 0$$
$$E_n = f^E \left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi \right)$$
$$F_n = f^F \left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi \right)$$

And the above functions are represented by

$$f^A = \ln \beta - \gamma \left(\delta + \mu_d \right) + A_{n-1} + E_{n-1} \mu_v + \left(F_{n-1} + \gamma \right) \mu_q$$
$$f^C \equiv -\gamma \rho_{du} + C_{n-1} \rho_{uu} + \gamma \left(1 - \rho_{uu} \right)$$
$$f^E \equiv E_{n-1} \rho_{vv}$$
$$\quad + \frac{1}{2} \left(-\gamma \sigma_{dd} + \left(C_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dd} + \left(F_{n-1} + \gamma \right) \sigma_{qd} \right)^2$$
$$\quad + \frac{1}{2} \left(\left(C_{n-1} - \gamma \right) \sigma_{uu} + \left(F_{n-1} + \gamma \right) \sigma_{qu} \right)^2$$
$$\quad + \frac{1}{2} \left(-\gamma \sigma_{dv} + \left(C_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dv} + \left(F_{n-1} + \gamma \right) \sigma_{qv} + E_{n-1} \sigma_{vv} \right)^2$$
$$f^F \equiv F_{n-1} \rho_{qq} + \gamma \left(\rho_{qq} - 1 \right) + \frac{1}{2} \left(\left(F_{n-1} + \gamma \right) \sigma_{qq} \right)^2$$

and $A_0 = C_0 = E_0 = F_0 = 0$. (Proof in Appendix).

Note that inflation has zero impact on real bond prices, but will, of course, affect the nominal term structure. Because interest rates are a simple linear function of bond prices, our model features a three-factor real interest rate model, with the consumption-dividend ratio, risk aversion, and uncertainty as the three factors. The pricing effects of the consumption-dividend ratio, captured by the C_n term, arise because the lagged consumption-dividend ratio enters the conditional mean of both dividend growth and itself. Either of these channels will in general impact future consumption growth given Equation (3). The volatility factor, v_t, has important term structure effects captured by the f^E term because it affects the volatility of both consumption growth and q_t. As such, v_t affects the volatility of the pricing kernel, thereby creating precautionary savings effects. In times of high uncertainty, investors desire to save more but they cannot. For equilibrium to obtain, interest rates must fall, raising bond prices. Note that the second, third and fourth lines of the E_n terms are positive, as is the first line if v_t is persistent: increased volatility unambiguously drives up bond prices. Thus the model features a classic "flight to quality" effect. Finally, the f^F term captures the effect of the risk aversion variable, q_t, which affects bond prices through offsetting utility smoothing and precautionary savings channels. Consequently, the effect of q_t cannot be signed and we defer further discussion to Section 6.

From Proposition 1, the price-coupon ratio of a hypothetical real consol (with constant real coupons) simply represents the infinite sum of the zero coupon bond prices. The nominal term structure is analogous to the real term structure, but simply uses the nominal pricing kernel, $\widehat{m}_{t+1}$, in the recursions underlying Proposition 1. The resulting expressions also look very similar to those obtained in Proposition 1 with the exception that the A_n and C_n terms carry additional terms reflecting inflation effects and D_n is non-zero[5]. Because the conditional covariance between the real kernel and inflation is zero, the nominal short rate rf_t satisfies the Fisher hypothesis,

$$rf_t = rrf_t + \mu_\pi + \rho_{\pi\pi}\pi_t + \rho_{\pi u}u_t - \frac{1}{2}\sigma_{\pi\pi}^2 \tag{17}$$

where rrf_t is the real rate. The last term is the standard Jensen's inequality effect and the previous three terms represent expected inflation.

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[5] The exact formulas for the price-coupon ratio of a real consol and for a nominal zero coupon bond are given in Propositions 3 and 4 respectively, in the Appendix.

3.2 Equity Prices

In any present value model, under a no-bubble transversality condition, the equity price-dividend ratio (the inverse of the dividend yield) is represented by the conditional expectation,

$$\frac{P_t}{D_t} = E_t \left[\sum_{n=1}^{\infty} \exp \left(\sum_{j=1}^{n} (m_{t+j} + \Delta d_{t+j}) \right) \right] \tag{18}$$

where $\frac{P_t}{D_t}$ is the price dividend ratio. This conditional expectation can also be solved in our framework as an exponential-affine function of the state vector, as is summarized in the following proposition.

Proposition 4 *For the economy described by Equations (9) and (10), the price-dividend ratio of aggregate equity is given by*

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp \left(\widehat{A}_n + \widehat{C}_n u_t + \widehat{E}_n v_t + \widehat{F}_n q_t \right) \tag{19}$$

where

$$\widehat{A}_n = f^A \left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \mu_d$$

$$\widehat{C}_n = f^C \left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right) + \rho_{du}$$

$$\widehat{E}_n = f^E \left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right)$$

$$+ \left(\frac{1}{2} \sigma_{dd}^2 + \sigma_{dd} \left((-\gamma) \sigma_{dd} + \left(\widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dd} + \left(\widehat{F}_{n-1} + \gamma \right) \sigma_{qd} \right) \right)$$

$$+ \left(\frac{1}{2} \sigma_{dv}^2 + \sigma_{dv} \left((-\gamma) \sigma_{dv} + \left(\widehat{C}_{n-1} - \gamma \right) \sigma_{ud} \sigma_{dv} + \left(\widehat{F}_{n-1} + \gamma \right) \sigma_{qv} + \widehat{E}_{n-1} \sigma_{vv} \right) \right)$$

$$\widehat{F}_n = f^F \left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi \right)$$

where the functions $f^X(\cdot)$ are given in Proposition 1 for $X \in (A, C, E, F)$ and $A_0 = C_0 = E_0 = F_0 = 0$. (Proof in appendix)

It is clear upon examination of Propositions 1 and 4 that the price-coupon ratio of a real consol and the price-dividend ratio of an equity claim share many reactions to the state variables. This makes perfect intuitive sense. An equity claim may be viewed as a real consol with stochastic coupons. Of particular interest in this study is the *difference* in the effects of state variables on the two financial instruments.

Inspection of C_n and $\widehat{C}_n$ illuminates an additional impact of the consumption-dividend ratio, u_t, on the price-dividend ratio. This marginal effect depends positively on ρ_{du}, describing the feedback from u_t to the conditional mean of Δd_t. When $\rho_{du} > 0$, a higher u_t increases expected cash flows

and thus equity valuations.

Above, we established that higher uncertainty decreases interest rates and consequently increases consol prices. Hence a first order effect of higher uncertainty is a positive "term structure" effect. Two channels govern the differential impact of v_t on equity prices relative to consol prices, reflected in the difference between E_n and $\widehat{E}_n$. First, the terms $\frac{1}{2}\sigma_{dd}^2$ and $\frac{1}{2}\sigma_{dv}^2$ arise from Jensen's Inequality and tend towards an effect of higher cash flow volatility increasing equity prices relative to consol prices. While this may seem counterintuitive, it is simply an artifact of the log-normal structure of the model. The second channel is the conditional covariance between cash flow growth and the pricing kernel, leading to the other terms on the second and third lines in the expression for $\widehat{E}_n$. As in all modern rational asset pricing models, a negative covariance between the pricing kernel and cash flows induces a positive risk premium and depresses valuation. The "direct effect" terms (those excluding lagged functional coefficients) can be signed, they are,

$$-\gamma\left(1+\sigma_{ud}\right)\left(1-\sigma_{qc}\right)\left(\sigma_{dd}^2+\sigma_{dv}^2\right)$$

If the conditional covariance between consumption growth and dividend growth is positive, $(1+\sigma_{ud})>0$, and consumption is negatively correlated with q_t, $\sigma_{qc}<0$, then the dividend stream is negatively correlated with the kernel and increases in v_t exacerbate this covariance risk. Consequently, uncertainty has two primary effects on stock valuation: a positive term structure effect and a potentially negative cash flow effect.

Interestingly, there is no marginal pricing difference in the effect of q_t on riskless versus risky coupon streams: the expressions for F_n and $\widehat{F}_n$ are functionally identical. This is true by construction in this model because the preference variable, q_t, affects neither the conditional mean nor volatility of cash flow growth, nor the conditional covariance between the cash flow stream and the pricing kernel at any horizon. We purposefully excluded such relationships for two reasons. Economically, it does not seem reasonable for investor preferences to affect the productivity of the proverbial Lucas tree. Secondly, it would be empirically very hard to identify distinct effects of v_t and q_t without exactly these kinds of exclusion restrictions.

Finally, note that inflation has no role in determining equity prices for the same reason that it has no role in determining the real term structure. While such effects may be present in the data, we do not believe them to be of first order importance for the question at hand.

3.3 Sharpe Ratios

CC point out that in a lognormal model the maximum attainable Sharpe ratio of any asset is an increasing function of the conditional variance of the log real pricing kernel. In our model, this is given by,

$$V_t \left(m_{t+1} \right) = \gamma^2 \sigma_{qq}^2 q_t + \gamma^2 \left(\sigma_{qc} - 1 \right)^2 \sigma_{cc}^2 v_t$$

where

$$\sigma_{cc}^2 = \left(1 + \sigma_{ud} \right)^2 \left(\sigma_{dd}^2 + \sigma_{dv}^2 \right) + \sigma_{uu}^2.$$

The Sharpe ratio is increasing in preference shocks and uncertainty. Thus, counter-cyclical variation in v_t may imply counter-cyclical Sharpe ratios. The effect of v_t on the Sharpe ratio is larger if risk aversion is itself negatively correlated with consumption growth. In CC, the kernel variance is a positive function of q_t only.

4 Empirical Implementation

In this section, we describe the data and estimation strategy.

4.1 Data

We measure all variables at the quarterly frequency and our base sample period extends from 1927:1 to 2004:3. Use of the quarterly rather than annual frequency is crucial to help identify heteroskedasticity in the data.

4.1.1 Bond Market and Inflation

We use standard Ibbotson data (from the SBBI Yearbook) for Treasury market and inflation series. The short rate, rf_t, is the (continuously compounded) 90-day T-Bill rate. The log yield spread, spd_t, is the average log yield for long term government bonds (maturity greater than ten years) less the short rate. These yields are dated when they enter the econometrician's data set. For instance, the 90-day T-Bill return earned over January-March 1990 is dated as December 1989, as it entered the data set at the end of that month. Inflation, π_t, is the continuously compounded end of quarter change in the CPI.

4.1.2 Equity Market

Our stock return measure is the standard CRSP value-weighted return index. To compute excess equity returns, r_t^x, we subtract the 90-day continuously compounded T-Bill yield earned over the same period. To construct the dividend yield, we proceed by first calculating a (highly seasonal) quarterly dividend yield series as,

$$DP_{t+1} = \left(\frac{P_{t+1}}{P_t}\right)^{-1} \left(\frac{P_{t+1} + D_{t+1}}{P_t} - \frac{P_{t+1}}{P_t}\right)$$

where $\frac{P_{t+1}+D_{t+1}}{P_t}$ and $\frac{P_{t+1}}{P_t}$ are available directly from the CRSP dataset as the value weighted stock return series including and excluding dividends respectively. We then use the four-period moving average of $\ln\left(1 + DP_t\right)$ as our observable series,

$$dp_t^f = \frac{1}{4}\left[\ln\left(1 + DP_t\right) + \ln\left(1 + DP_{t-1}\right) + \ln\left(1 + DP_{t-2}\right) + \ln\left(1 + DP_{t-3}\right)\right].$$

This dividend yield measure differs from the more standard one, which sums dividends over the course of the past four quarters and scales by the current price. We prefer our filter because it represents a linear transformation of the underlying data which we can account for explicitly when bringing the model to the data. As a practical matter, the properties of our filtered series and the more standard measure are very similar with nearly identical means and volatilities and an unconditional correlation between the two of approximately 0.95.

For dividend growth, we first calculate real quarterly dividend growth,

$$\Delta d_{t+1} = \ln\left[\frac{DP_{t+1}}{DP_t}\frac{P_{t+1}}{P_t}\right] - \pi_{t+1}$$

Then, to eliminate seasonality, we use the four-period moving average as the observation series,

$$\Delta d_t^f = \frac{1}{4}\left(\Delta d_t + \Delta d_{t-1} + \Delta d_{t-2} + \Delta d_{t-3}\right). \tag{20}$$

4.1.3 Consumption

To avoid the look-ahead bias inherent in standard seasonally adjusted data, we obtain nominal non-seasonally adjusted (NSA) aggregate non-durable and service consumption data from the website of the Bureau of Economic Analysis (BEA) of the United States Department of Commerce for the

period 1946-2004. We deflate the raw consumption growth data with the inflation series described above. We denote the continuously compounded real growth rate of the sum of non-durable and service consumption series as Δc_t. From 1929-1946, consumption data from the BEA is available only at the annual frequency. For these years, we use repeated values equal to one-fourth of the compounded annual growth rate. Because this methodology has obvious drawbacks, we repeat our analysis using an alternate consumption interpolation procedure which presumes the consumption-dividend ratio, rather than consumption growth is constant over the year. Results using this alternate method are very similar to those reported. Finally, for 1927-1929, no consumption data are available from the BEA. For these years, we obtain the growth rate for real per-capita aggregate consumption from the website of Robert Shiller at www.yale.edu, and compute aggregate nominal consumption growth rates using the inflation data described above in addition to historical population growth data from the United States Bureau of the Census. Then, we use repeated values of the annual growth rate as quarterly observations. Analogous to our procedure for dividend growth, we use the four-period moving average of Δc_t as our observation series, which we denote by Δc_t^f.

4.1.4 Heteroskedasticity in Consumption and Dividend Growth

Many believe that consumption growth is best described as an i.i.d. process. However, Ferson and Merrick (1987), Whitelaw (2000) and Bekaert and Liu (2004) all demonstrate that consumption growth volatility varies through time. For our purposes, the analysis in Bansal, Khatchatrian and Yaron (2005) and Kandel and Stambaugh (1990) is most relevant. The former show that price-dividend ratios predict consumption growth volatility with a negative sign and that consumption growth volatility is persistent. Kandel and Stambaugh (1990) link consumption growth volatility to three state variables, the price-dividend ratio, the AAA versus T-Bill spread, and the BBB versus AA spread. They also find that, price-dividend ratios negatively affect consumption growth volatility. We extend and modify this analysis by estimating the following model by GMM,

$$VAR_t (y_{t+1}) = b_0 + b_1 x_t \tag{21}$$

where y_t is, alternatively, Δd_t^f or Δc_t^f. We explore asset prices as well as measures of the business cycle and a time trend as instruments. The asset prices we entertain as elements of x_t include, rf_t, the risk free rate, dp_t^f, the (filtered) dividend yield (the inverse of the price-dividend ratio), and spd_t, the nominal term spread. We also allow, but do not report results, for time-variation in

the conditional mean using a linear projection onto the consumption-dividend ratio, u_t^f. Because consumption and dividend growth display little variation in the conditional mean, the results are quite similar for specifications wherein the conditional mean is a constant.

The results are reported in Table 1. Panel A focuses on univariate tests while Panel B reports multivariate tests. Wald tests in the multivariate specification reject the null of no time variation for the volatility of both consumption and dividend growth at conventional significance levels. Moreover, all three instruments are mostly significant predictors of volatility in their own right: high interest rates are associated with low volatility, high term spreads are associated with high volatility as are high dividend yields. Hence, the results in Bansal et al. (2005) and Kandel and Stambaugh (1990) regarding the dividend yield predicting economic uncertainty are also valid for dividend growth volatility.

Note that the coefficients on the instruments for the dividend growth volatility are 5-25 times as high as for the consumption growth equation. This suggests that one latent variable may capture the variation in both. We test this conjecture by estimating a restricted version of the model where the slope coefficients are proportional across the dividend and consumption equations. This restriction is not rejected, with a p-value of 0.11. We conclude that our use of a single latent factor for both fundamental consumption and dividend growth volatility is appropriate. The proportionality constant (not reported), is about 0.08, implying that the dividend slope coefficients are about 12 times larger than the consumption slope coefficients.

Table 1 (Panel A) also presents similar predictability results for excess equity returns. We will later use these results as a metric to judge whether our estimated model is consistent with the evidence for variation in the conditional volatility of returns. While the signs are the same as in the fundamentals' equations, none of the coefficients are significantly different from zero at conventional significance levels.

Panel C examines the cyclical pattern in the fundamentals' heteroskedasticity, demonstrating a strong counter-cyclical pattern. This is an important finding as it intimates that heteroskedasticity may be the driver of the counter-cyclical Sharpe ratios stressed by CC and interpreted as counter-cyclical risk aversion.

Lettau, Ludvigson and Watcher (2006) consider the implications of a downward shift in consumption growth volatility for equity prices. Using Post-war data, they find evidence for a structural break in consumption growth volatility somewhere between 1983 or 1993 depending on the data used. Given our very long sample, the assumption of a simple AR(1) process for volatility is

definitely strong. If non-stationarities manifest themselves through a more persistent process than the true model reflecting a break, a regime change, or a trend; the robustness of our results is in doubt and we may over-estimate the importance of economic uncertainty.

Therefore, we examine various potential forms of non-stationary behavior for dividend and consumption growth volatility. We start by examining evidence of a trend in volatility. Its conceivable that a downward trend in volatility can cause spurious counter-cyclical behavior as recessions have become milder and less frequent over time. While there is some evidence for a downward trend (see Panel C in Table 1) in dividend and consumption growth volatility, there is still evidence for counter-cyclicality in volatility, although it is weakened for dividend growth volatility. Yet, a trend model is not compelling for various reasons. First, the deterministic nature of the model suggests the decline is predictable, which we deem implausible. Second, using post-war data there is no trend in dividend growth volatility and the downward trend for consumption growth volatility is no longer statistically significant. Finally, the models with and without a trend yield highly correlated conditional volatility estimates. For example, for dividend growth, this correlation is 0.87.

A more compelling model, is a model with parameter breaks. We therefore conduct Bai and Perron (1998) multiple break tests separately for consumption and dividends based on the following regression equation

$$\left(\Delta d_t^f\right)^2 = a_0 + a_1 rf_{t-4} + a_2 dp_{t-4}^f + a_3 spd_{t-4} + u_t \tag{22}$$

and analogously for consumption. Following Bai and Perron (1998), we first test the null hypothesis of no structural breaks against an alternative with an unknown number of breaks. For both dividend and consumption growth volatility, we reject at the 5 percent level. Having established the presence of a break, we use a BIC-based criterion to estimate the number of breaks. In the case of dividend growth, this yielded one break. This break is estimated to be located at 1939Q2 with a 95 percent confidence interval extending through 1947Q1. For consumption growth, the BIC criterion selects two breaks with the most recent one estimated at 1948Q1 with a 95 percent confidence interval extending through 1957Q1. Other criteria suggested by Bia and Perron (1998) also found two or fewer breaks for both series. These results are robust to various treatments of autocorrelation in the residuals, heteroskedasticity across breaks, etc. Given that pre-war data are also likely subject to more measurement error than post-war data, we therefore consider an alternative estimation using post-war data. Consistent with the existing evidence, including Lettau, Ludvigson and Watcher (2006)'s, we continue to find that dividend yields predict macro-economic

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volatility, but the coefficients on the instruments are indeed smaller than for the full sample.

4.2 Estimation and Testing Procedure

4.2.1 Parameter Estimation

Our economy has five state variables, which we collect in the vector $Y_t = [\Delta d_t, v_t, u_t, q_t, \pi_t]$. While $u_t, \Delta d_t$ and π_t are directly linked to the data, v_t and q_t are latent variables. We are interested in the implications of the model for seven variables: filtered dividend and consumption growth, Δd_t^f and Δc_t^f, inflation, π_t, the short rate, r_t^f, the term spread, spd_t, the dividend yield, or dividend price ratio, dp_t, and log excess equity returns, rx_t. For all these variables we use the data described above. The first three variables are (essentially) observable state variables; the last four are endogenous asset prices and returns. We collect all the observables in the vector W_t.

The relation between term structure variables and state variables is affine, but the relationship between the dividend yield and excess equity returns and the state variables is non-linear. In the Computational Appendix, we linearize this relationship and show that the approximation is quite accurate. Note that this approach is very different from the popular Campbell-Shiller (1988) linearization method, which linearizes the return expression itself before taking the linearized return equation through a present value model. We first find the correct solution for the price-dividend ratio and linearize the resulting equilibrium.

Conditional on the linearization, the following property of W_t obtains,

$$W_t = \mu^w(\Psi) + \Gamma^w(\Psi) Y_t^c \tag{23}$$

where Y_t^c is the companion form of Y_t containing five lags and the coefficients superscripted with "w" are nonlinear functions of the model parameters, Ψ. Because Y_t follows a linear process with square-root volatility dynamics, unconditional moments of Y_t are available analytically as functions of the underlying parameter vector, Ψ. Let $X(W_t)$ be a vector valued function of W_t. For the current purpose, $X(\cdot)$ will be comprised of first, second, third and fourth order monomials, unconditional expectations of which are uncentered moments of W_t. Using Equation (23), we can also derive the analytic solutions for uncentered moments of W_t as functions of Ψ. Specifically,

$$E[X(W_t)] = f(\Psi) \tag{24}$$

where $f(\cdot)$ is also a vector valued function (subsequent appendices provide the exact formulae)[6]. This immediately suggests a simple GMM based estimation strategy. The GMM moment conditions are,

$$g_T(W_t; \Psi_0) = \frac{1}{T}\sum_{t=1}^{T} X(W_t) - f(\Psi_0). \tag{25}$$

Moreover, the additive separability of data and parameters in Equation (25) suggests a "fixed" optimal GMM weighting matrix free from any particular parameter vector and based on the data alone. Specifically, the optimal GMM weighting matrix is the inverse of the spectral density at frequency zero of $g_T(W_t; \Psi_0)$, which we denote as $S(W_T)$. To reduce the number of parameters estimated in calculating the optimal GMM weighting matrix, we use a procedure that exploits the structure implied by the model, and then minimize the standard GMM objective function, as described in Appendix D.

4.2.2   Moment Conditions

We use a total of 34 moment conditions, listed in the notes to Table 2, to estimate the model parameters. They can be ordered into five groups. The first set is simply the unconditional means of the W_t variables; the second group includes the second uncentered moments of the state variables. In combination with the first moments above, these moments ensure that we are matching the unconditional volatilities of the variables of interest. The third set of moments is aimed at identifying the autocorrelation of the fundamental processes. The moving average filter applied to dividend and consumption growth makes it only reasonable to look at the fourth order autocorrelations. Because our specification implies complicated ARMA behavior for inflation dynamics, we attempt to fit both the first and fourth order autocorrelation of this series. The fourth set of moments concerns contemporaneous cross moments of fundamentals with asset prices and returns. As was pointed out by Cochrane and Hansen (1992), the correlation between fundamentals and asset prices implied by standard implementations of the consumption CAPM model is often much too high. We also include cross moments between inflation, the short rate, and consumption growth to help identify the $\rho_{\pi u}$ parameter in the inflation equation.

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[6]In practice, we simulate the unconditional moments of order three and four during estimation. While analytic solutions are available for these moments, they are extremely computationally expensive to calculate at each iteration of the estimation process. For these moments, we simulate the system for roughly 30,000 periods (100 simulations per observation) and take unconditonal moments of the simulated data as the analytic moments implied by the model without error. Due to the high number of simulations per observation, we do not correct the standard errors of the parameter estimates for the simulation sampling variability. To check that this is a reasonable strategy, we perform a one-time simulation at a much higher rate (1000 simulations / observation) at the conclusion of estimation. We check that the identified parameters produce a value for the objective function close to that obtained with the lower simulation rate used in estimation.

Next, the fifth set of moments includes higher order moments of dividend and consumption growth. This is crucial to help ensure that the dynamics of v_t are identified by, and consistent with, the volatility predictability of the fundamental variables in the data, and to help fit their skewness and kurtosis.

Note that there are $34 - 19 = 15$ over-identifying restrictions and that we can use the standard J-test to test the fit of the model.

5   Estimation Results

This section describes the estimation results of the structural model, and characterizes the fit of the model with the data.

5.1   GMM Parameter Estimates

Table 2 reports the parameter estimates. We start with dividend growth dynamics. First, u_t significantly forecasts dividend growth. Consequently, as in Lettau and Ludvigson (2005) and Menzly, Santos and Veronesi (2004), there is a slow moving variable that simultaneously affects dividend growth and potentially equity risk premiums. Second, the conditional volatility of dividend growth, v_t, is highly persistent with an autocorrelation coefficient of 0.9795 and itself has significant volatility (σ_{vv}, is estimated as 0.3288 with a standard error of 0.0785). This confirms that dividend growth volatility varies through time. Further, the conditional covariance of dividend growth and v_t is positive and economically large: σ_{dv} is estimated at 0.0413 with a standard error of 0.0130.

The results for the consumption-dividend ratio are in line with expectations. First, it is very persistent, with an autocorrelation coefficient of 0.9826 (standard error 0.0071). Second, the contemporaneous correlation of u_t with Δd_t is sharply negative as indicated by the coefficient σ_{ud} which is estimated at -0.9226. In light of Equation (3), this helps to match the low volatility of consumption growth. However, because $(1 + \sigma_{ud})$ is estimated to be greater than zero, dividend and consumption growth are positively correlated, as is true in the data. Finally, the idiosyncratic volatility parameter for the consumption dividend ratio σ_{uu} is 0.0127 with a standard error of just 0.0007, ensuring that the correlation of dividend and consumption growth is not unrealistically high.

The dynamics of the stochastic preference process, q_t, are presented next. It is estimated to be quite persistent, with an autocorrelation coefficient of 0.9787 (standard error 0.0096) and it has significant independent volatility as indicated by the estimated value of σ_{qq} of 0.1753 (standard

error 0.0934). Of great importance is the contemporaneous correlation parameter between q_t and consumption growth, σ_{qc}. While σ_{qc} is negative, it is not statistically different from zero. This indicates that risk is indeed moving countercyclically, in line with its interpretation as risk aversion under a habit persistence model such as that of CC. What is different in our model is that the correlation between consumption growth and risk aversion[7] is -0.37 instead of -1.00 in CC. The impatience parameter $\ln(\beta)$ is negative as expected and the γ parameter (which is not the same as risk aversion in this model) is positive, but not significantly different from zero. The wedge between mean dividend growth and consumption growth, δ, is both positive and significantly different from zero.

Finally, we present inflation dynamics. As expected, past inflation positively affects expected inflation with a coefficient of 0.2404 (standard error 0.1407) and there is negative and significant predictability running from the consumption-dividend ratio to inflation.

5.2 Model Moments Versus Sample Data

Table 2 also presents the standard test of the overidentifying restrictions, which fails to reject, with a p-value of 0.6234. However, there are a large number of moments being fit and in such cases, the standard GMM overidentification test is known to have low power in finite samples. Therefore, we examine the fit of the model with respect to specific moments in Tables 3 and 4.

Table 3 focuses on linear moments of the variables of interest: mean, volatilities and autocorrelations. The model matches the unconditional means of all seven of the endogenous variables. This includes generating a realistic low mean for the nominal risk free rate of about 1% and a realistic equity premium of about 1.2% (all quarterly rates). Analogously, the implied volatilities of both the financial variables and fundamental series are within one standard error of the data moment. Finally, the model is broadly consistent with the autocorrelation of the endogenous series. The (fourth) autocorrelation of filtered consumption growth is somewhat too low relative to the data. However, in unreported results we verified that the complete autocorrelograms of dividend and consumption growth implied by the model are consistent with the data. The model fails to generate sufficient persistence in the term spread but this is the only moment not within a two standard bound around the data moment. However, it is within a 2.05 standard error bound!

As explored below, the time varying volatility of dividend growth is an important driver of equity

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[7]More specifically, the conditional correlation between Δc_{t+1} and q_{t+1} when v_t and q_t are at their unconditional mean of unity.

returns and volatility, and it is therefore important to verify that the model implied nonlinearities in fundamentals are consistent with the data. In Table 4, we determine whether the estimated model is consistent with the reduced form evidence presented in Table 1, and we investigate skewness and kurtosis of fundamentals and returns. In Panel A, we find that the volatility dynamics for fundamentals are quite well matched. The model produces the correct sign in forecasting dividend and consumption growth volatility with respect to all instruments. Mostly the simulated coefficients are within or close to two standard errors of the data coefficients with the coefficient on the spread for dividend growth being the least accurate (2.78 standard errors too large). However, for return volatility, the sign with respect to the short rate is incorrect.

With respect to multivariate regressions (not reported), the model deos not perform well. This is understandable, as it represents a very tough test of the model. Implicitly, such test requires the model to also fit the correlation among the three instruments.

Panel B focuses on skewness and kurtosis. The model implied kurtosis of filtered dividend growth is consistent with that found in the data and the model produces a bit too much kurtosis in consumption growth rates. Equity return kurtosis is somewhat too low relative to the data, but almost within a 2 standard error bound. The model produces realistic skewness numbers for all three series. We conclude that the nonlinearities in the fundamentals implied by the model are reasonably consistent with the data.

6 Risk Aversion, Uncertainty and Asset Prices

In this section, we explore the dominant sources of time variation in equity prices (dividend yields), equity returns, the term structure, expected equity returns and the conditional volatility of equity returns. We also investigate the mechanisms leading to our findings.

Tables 5 and 6 contain the core results in the paper. Table 5 reports basic properties of some critical unobserved variables, including v_t and q_t. Table 6 reports variance decompositions with standard errors for several endogenous variables of interest and essentially summarizes the response of the endogenous variables to each of the state variables. Rather than discussing these tables in turn, we organize our discussion around the different variables of interest using information from the two tables.

6.1 Risk Aversion and Uncertainty

Table 5, Panel A presents properties of unobservable variables under the estimated model. The properties of "uncertainty", v_t, which is proportional to the conditional volatility of dividend growth, and q_t, which drives risk aversion, were discussed before [see Section 5]. Because local risk aversion, RA_t, in this model is given by $\gamma \exp(q_t)$, we can examine its properties directly. The median level of risk aversion in the model is 2.52, a level which would be considered perfectly reasonable by most financial economists. However, risk aversion is positively skewed and has large volatility so that risk aversion is occasionally extremely high in this model.

Panel B of Table 5 presents results for means of the above endogenous variables conditional on whether the economy is in a state of expansion or recession. For this exercise, recession is defined as one quarter of negative consumption growth. Both v_t and q_t (and hence local risk aversion) are counter-cyclical.

6.2 Risk Aversion, Uncertainty, and the Term Structure

Panel A of Table 5 also displays the model implied properties of the real interest rate and the real term spread. The average real rate is 17 basis points (68 annualized) and the real interest rate has a standard deviation of around 90 basis points. The real term spread has a mean of 38 basis points, a volatility of only 28 basis points and is about as persistent as the real short rate. In Panel B, we see that real rates are pro-cyclical and spreads are counter-cyclical, consistent with the findings in Ang, Bekaert and Wei (2007).

Panel C of Table 5 shows that uncertainty tends to depress real interest rates, while positive risk aversion shocks tend to increase them. To gain further insight into these effects, let us derive the explicit expression for the real interest rate by exploiting the log-normality of the model:

$$rrf_t = -E_t[m_{t+1}] - \frac{1}{2}V_t[m_{t+1}]. \tag{26}$$

The conditional mean of the pricing kernel economically represents consumption smoothing whereas the variance of the kernel represents precautionary savings effects. To make notation less cumbersome, let us reparameterize the consumption growth process as having conditional mean and

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variance

$$E_t\left[\Delta c_{t+1}\right] = \delta + \mu_d + \left(\rho_{du} + \rho_{uu} - 1\right) u_t \equiv \mu_c + \rho_{cu} u_t$$

$$V_t\left[\Delta c_{t+1}\right] = \sigma_{cc}^2 v_t. \tag{27}$$

Then the real rate simplifies to

$$rrf_t = -\ln\left(\beta\right) + \gamma\left(\mu_c - \mu_q\right) + \gamma\rho_{cu} u_t + \phi_{rq} q_t + \phi_{rv} v_t \tag{28}$$

with $\phi_{rq} = \gamma\left(1 - \rho_{qq}\right) - \frac{1}{2}\gamma^2\sigma_{qq}^2$ and $\phi_{rv} = -\frac{1}{2}\gamma^2\left(\sigma_{qc} - 1\right)^2\sigma_{cc}^2$. Changes in risk aversion have an ambiguous effect on interest rates depending on whether the smoothing or precautionary savings effect dominates (the sign of ϕ_{rq}). At our parameter values, the consumption smoothing effect dominates. As discussed before, v_t represents a precautionary savings motive, so the correlation between real rates and v_t is negative. Overall, real rates are pro-cyclical because v_t is strongly counter-cyclical.

Moving to the real term spread, it displays a positive correlation with both v_t and q_t, but for different reasons. To obtain intuition, let us consider a two period bond and exploit the log-normality of the model. We can decompose the spread into three components:

$$rrf_{2,t} - rrf_t = \frac{1}{2}E_t\left[rrf_{t+1} - rrf_t\right] + \frac{1}{2}Cov_t\left[m_{t+1}, rrf_{t+1}\right] - \frac{1}{4}Var_t\left[rrf_{t+1}\right]$$

The first term is the standard expectations hypothesis (EH) term, the second term represents the term premium and the third is a Jensen's inequality term (which we will ignore). Because of mean reversion, the effects of u_t, v_t, and q_t on the first component will be opposite of their effects on the level of the short rate. Figure 1 decomposes the exposures of both the real interest rate and the spread to v_t and q_t into an expectations hypothesis part and a term premium part and does so for various maturities (to 40 quarters). It shows that q_t has a positive effect on the term spread. Yet, the coefficient on q_t in the EH term is $\phi_{rq}\left(\rho_{qq} - 1\right)$ and thus negative as ϕ_{rq} is positive. However, it is straightforward to show that the coefficient on q_t for the term premium is $\frac{1}{2}\gamma\phi_{rq}\sigma_{qq}^2$ and hence the term premium effect of q_t will counter-balance the EH effect when $\phi_{rq} > 0$. Yields at long maturities feature a term premium that is strongly positively correlated with q_t because higher risk aversion increases interest rates (and lowers bond prices) at a time when marginal utility is high, making bonds risky.

Increased uncertainty depresses short rates and, consequently, the EH effect implies that uncertainty increases term spreads. The effect of v_t on the term premium is very complex because the correlation between q_t and the kernel is also driven by v_t. In fact, straightforward algebra shows that the coefficient on v_t is proportional to

$$(\sigma_{qc} - 1) \left[\sigma_{uu}^2 \left(\gamma \rho_{uc} + \phi_{rq} \sigma_{qc} \right) + (1 + \sigma_{ud}) \left(\gamma \rho_{uc} \sigma_{ud} + \phi_{rq} \sigma_{qc} \left(\sigma_{ud} + 1 \right) \left(\sigma_{dd}^2 + \sigma_{dv}^2 \right) - \phi_{rv} \sigma_{vv} \sigma_{dv} \right) \right].$$

Empirically, we estimate this coefficient to be 0.0020, so that the EH effect is the dominant effect. Hence, when uncertainty increases, the term structure steepens and vice versa.

In Table 6, we report the variance decompositions. While three factors (u_t, v_t, and q_t) affect the real term structure, v_t accounts for the bulk of its variation. An important reason for this fact is that v_t is simply more variable than q_t. The most interesting aspect of the results here is that q_t contributes little to the variability of the spread, so that q_t is mostly a level factor not a spread factor, whereas uncertainty is both a level and a spread factor. When we consider a real consol, we find that q_t dominates its variation. Because consol prices reflect primarily longer term yields, they are primarily driven by q_t, through its effect on the term premium.

For the nominal term structure, inflation becomes an important additional state variable accounting for about 12% of the variation in the nominal interest rates. However, inflation is an even more important spread factor accounting for about 31% of the spread's variability. What may be surprising is that the relative importance of q_t increases going from the real to nominal term structure. The reason is the rather strong positive correlation between inflation and v_t, which arises from the negative relation between inflation and the consumption dividend ratio, that ends up counterbalancing the negative effect of v_t on real interest rates.

6.3 Risk Aversion, Uncertainty and Equity Prices

Here we start with the variance decompositions for dividend yields and equity returns in Table 6. For the dividend yield, q_t dominates as a source of variation, accounting for almost 90% of its variation. To see why, recall first that q_t only affects the dividend yield through its effect on the term structure of real interest rates (see Proposition 4). Under the parameters presented in Table 2, the impact of q_t on real interest rates is positive at every horizon and therefore it is positive for the dividend yield as well. Formally, under the parameters of Table 2, $\widehat{F}_n$ in Proposition 4 is negative at all horizons.

Next, consider the effect of v_t on the dividend yield. Uncertainty has a "real consol effect" and a "cash-flow risk premium" effect which offset each other. We already know that v_t creates a strong precautionary savings motive, which decreases interest rates. All else equal, this will serve to increase price-dividend ratios and decrease dividend yields. However, v_t also governs the covariance of dividend growth with the real kernel. This risk premium effect may be positive or negative, but intuitively the dividend stream will represent a risky claim to the extent that dividend growth covaries negatively with marginal utility. In this case, we would expect high v_t to exacerbate this riskiness and depress equity prices when it is high, increasing dividend yields. As we discussed in section 3, σ_{qc} contributes to this negative covariance. On balance, these countervailing effects of v_t on dividend yields largely cancel out, so that the net effect of v_t on dividend yields is small. This shows up in the variance decomposition of the dividend yield. On balance, q_t is responsible for the overwhelming majority of dividend yield variation, and is highly positively correlated with it. The negative effect of u_t arises from its strong negative covariance with dividend growth.

Looking back to panel C in Table 5, while increases in q_t have the expected depressing effect on equity prices (a positive correlation with dividend yields), increases in v_t do not. This contradicts with the findings in Wu (2001) and Bansal and Yaron (2004) but is consistent with early work by Barsky (1989) and Naik (1994). Because the relation is only weakly negative, there may be instances where our model will generate a classic "flight to quality" effect with uncertainty lowering interest rates, driving up bond prices and depressing equity prices.

Next notice the determinants of realized equity returns in Table 6. First, over 30% of the variation in excess returns is driven by dividend growth and dividend growth is positively correlated with excess returns. This is not surprising in light of the fact that dividend growth enters the definition of stock returns directly and dividend growth has almost half as much variation as returns themselves. The other primary driver of stock returns is q_t. This is a compound statistic which includes the effect of current and lagged q_t. In fact, the contemporaneous effect of q_t on returns is negative (see Table 5) as increases in q_t depress stock valuations. However, the lagged effect of q_t on returns is positive because, all else equal, lower lagged prices imply higher current returns.

6.4 Risk Aversion, Uncertainty and the Equity Premium

We again go back to Table 5 to investigate the properties of the conditional equity premium, $E_t[rx_{t+1}]$. The premium is quite persistent, with an autocorrelation coefficient of 0.9789. In Panel B, we also find it is higher in recessions which is consistent with counter-cyclical risk aversion.

Panel C shows that both v_t and q_t are positively correlated with the equity premium. The risk premium in any model will be negatively correlated with the covariance between the pricing kernel and returns. We already discussed how uncertainty contributes to the negative covariance between the pricing kernel and dividend growth and therefore increases risk premiums. The effect of q_t comes mostly through the capital gain part of the return: increases in q_t both raise marginal utility and decrease prices making stocks risky. Table 6 shows that the point estimate for the share of the equity premium variation due to v_t is about 17% but with a standard error of 13%, with the remainder due to q_t.

The fact that both the dividend yield and expected equity returns are primarily driven by q_t suggest that the dividend yield may be a strong predictor of equity returns in this model. Table 7 shows that this is indeed the case, with a regression of future returns on dividend yields generating a 1.58 coefficient. We also compare the model coefficients with the corresponding statistics in the data. It turns out that the predictability of equity returns during our sample period is rather weak. Table 7 reports univariate coefficients linking equity returns to short rates, dividend yields and spreads. The sign of the coefficients matches well-known stylized facts but none of the coefficients are significantly different from zero. The model produces coefficients within two standard errors of these data coefficients but this is of course a rather weak test. While it is theoretically possible to generate a negative link between current short rates and the equity premium which is observed empirically, our model fails to do so at the estimated parameters. We also report the results of a multivariate regression on the aforementioned instruments. The model here gets all the signs right and is always within two standard errors of the data coefficients. More generally, the ratio, $VAR\left(E_t\left[rx_{t+1}\right]\right)/VAR\left(rx_{t+1}\right)$, from Tables 3 and 5, implies a quarterly R^2 of less than one percent, so the model does not generate much short term predictability of equity returns consistent with recent evidence. There is a large debate on whether predictability increases with the horizon. In our model, the variance ratio discussed above for 10 year returns equals about 12% (not reported).

While we have studied the conditional equity premium, it remains useful to reflect on the success of the model in matching the unconditional equity premium. To make the model's success complete, it also matches the low risk free rate while keeping the correlation between fundamentals (dividend and consumption growth) and returns low. In fact, the correlation between dividend growth and equity returns is 0.28 in the data and 0.33 in the model. For consumption growth, the numbers are 0.07 and 0.11 respectively. In addition, the model matches the correlation between dividend yields and consumption growth, which is -0.14 under the model and in the data. Consequently, this

model performs in general better than the CC model, which had trouble with the fundamentals-return correlation.

6.5   Risk Aversion, Uncertainty, Equity Return Volatility and Sharpe Ratios

To conclude, we investigate the properties of the conditional variance of equity returns, the equity Sharpe ratio and the maximum attainable Sharpe ratio available in the economy discussed in Section 2. We begin with the numbers in Panel A of Table 5. The conditional variance of excess equity returns has a mean of 0.0092, a standard deviation of of 0.0070 and an autocorrelation of 0.9794. The final two columns of Table 5 report results for the conditional Sharpe ratio of equity and the maximum attainable Sharpe ratio available in the economy discussed in Section 2. The mean equity Sharpe ratio attains approximately three quarters of the maximum attainable value. Both Sharpe ratios are strongly persistent and possess significant time variation driven by v_t and q_t. These Sharpe ratios are quarterly, and so their magnitude is roughly half of annualized values.

The conditional variance of equity returns is counter-cyclical. Interestingly, the increase in expected equity returns during recessions is not as large as the increase in the expected variance which contributes to the equity Sharpe ratio being not counter-cyclical. The maximum Sharpe ratio does display counter-cyclical behavior. Moving to Table 6, not surprisingly, the conditional volatility of equity returns is largely governed by v_t, which accounts for 75% of its variation with a standard error of only 32%. Here, q_t contributes 25% to the total volatility variation.

7   Robustness and Related Literature

By introducing a time-varying preference shock not correlated with fundamentals (q_t) and time-varying economic uncertainty (v_t), our model matches a large number of salient asset price features while keeping the correlation between fundamentals and asset returns low. Economic uncertainty acts as both a level and spread factor in the term structure, has a large effect on conditional stock market volatility, but little effect on dividend yields and the equity premium, which are primarily driven by the preference shock. In this section, we first examine whether these results hold up if we estimate the model using post-War data, where macro-economic volatility was decidedly less severe than pre-War. We then provide intuition on why our model yields different results than a number of well-known existing articles.

7.1 Post-War Estimation Results

The model fits the Post-War data well and the test of the over-identifying restrictions does not reject. Table 8 summarizes some of the implications of the Post-War model. The first two columns show how the model continues to match the properties of equity returns and the nominal interest rate, but more generally it does as well as the previous model did. In particular, it also fits the low correlation between asset returns and fundamentals that we continue to observe. Interestingly, it does so with about the same risk aversion as for the full sample. Average risk aversion is only slightly larger than it was for the full sample estimation, but its standard deviation is much smaller reflecting a data sample with fewer extreme observations. The largest change is that v_t is now much less persistent and less volatile than it was for the full sample estimation. This will have implications for the role of v_t in asset pricing, but it does not materially affect the cyclicality of the endogenous variables, or the correlations of v_t with observables. In fact, we do not repeat Panels B and C from the old Table 5 because all the inference is identical for the new model with one exception: the equity Sharpe ratio is now also counter-cyclical confirming the usual finding in the literature. Instead, Panel B reports the variance decompositions for the term structure and equity prices. Here, the reduced persistence of v_t implies that the role of v_t is overall diminished. Interestingly, the role of q_t as a level factor, and v_t as a spread factor is now even sharper, with q_t accounting for the bulk of variation in dividend yields, equity premiums and even the conditional volatility of equity returns. Hence, if there truly was a permanent structural break in macro-economic uncertainty after the War, we need preference shocks to play an even larger role in driving asset prices, than was true in our previous estimation.

7.2 Related Literature

7.2.1 Abel (1988) and Wu (2001)

Abel (1988) creates an economy in which the effect of increased cash flow volatility on equity prices depends on a single parameter, the coefficient of relative risk aversion. His setup is vastly different from ours. Most importantly, Abel (1988) maintains that dividends themselves are stationary and so are prices (at least on a per-capita basis). Also , there is no distinction between consumption and dividends in his model, so that the covariance of cash flows with the pricing kernel and the volatility of the pricing kernel are proportional. Finally, there is no preference shock. In the current framework, we can consider the effects of some of Abel's assumptions by simply shutting

down the dynamics of the consumption dividend ratio ($u_t = 0$) and stochastic risk aversion ($q_t = 0$). However, we do not implement Abel's assumption that dividends and prices are stationary.

Proposition 5 *For the economy described by Equations (9) and (10), and the additional assumption that the following parameters are zero,*

$$\mu_u, \mu_q, \rho_{du}, \rho_{uu}, \rho_{qq}, \sigma_{ud}, \sigma_{uu}, \sigma_{qc}, \sigma_{qq}$$

the equity price-dividend ratio is represented by

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(\overleftrightarrow{A}_n + \overleftrightarrow{E}_n v_t\right)$$

where

$$\overleftrightarrow{A}_n = \ln\beta + \overleftrightarrow{A}_{n-1} + (1-\gamma)\mu_d + \overleftrightarrow{E}_{n-1}\mu_v$$

$$\overleftrightarrow{E}_n = \overleftrightarrow{E}_{n-1}\rho_{vv} + \frac{1}{2}\left(\overleftrightarrow{E}_{n-1} + 1 - \gamma\right)^2 \sigma_{dd}^2 + \frac{1}{2}\left(\left(\overrightarrow{B}_{n-1} + 1 - \gamma\right)\sigma_{dv} + \overleftrightarrow{E}_{n-1}\sigma_{vv}\right)^2$$

with $\overleftrightarrow{A}_0 = \overleftrightarrow{E}_0 = 0$ (Proof available upon request.)

The effect of volatility changes on the price dividend ratio is given by the $\overleftrightarrow{E}_n$ coefficient. When volatility is positively autocorrelated, $\rho_{vv} > 0$, $\overleftrightarrow{E}_n > 0$ and increases in volatility always increase equity valuation, essentially because they depress the interest rate. In comparison to the effects of v_t in Proposition 4, only the Jensen's Inequality terms remain. There is no scope for v_t to alter the riskiness of the dividend stream beyond the real term structure effects because cash flows and the pricing kernel are proportional. Clearly, this simplified framework is too restrictive for our purposes.

Wu (2001) develops a model wherein increases in volatility unambiguously depress the price-dividend ratio. The key difference between his model and ours is that Wu models the interest rate as exogenous and constant. To recover something like Wu's results in our framework requires making the real interest rate process exogenous and maintaining the volatility process of Equation (9). Assume for example that we introduce a stochastic process x_t and modify the specification of the dividend growth process to be:

$$\Delta d_t = \frac{\ln\beta}{\gamma} + \frac{1}{\gamma}x_{t-1} + \frac{\gamma}{2}v_{t-1} + \sqrt{v_{t-1}}\varepsilon_t^d$$

$$x_t = \mu_x + \rho_{xx}x_{t-1} + \sigma_x\varepsilon_t^x + \sigma_{xv}\sqrt{v_{t-1}}\varepsilon_t^v + \sigma_{xd}\sqrt{v_{t-1}}\varepsilon_t^d \tag{29}$$

It is easily verified that under these specifications and the additional assumptions of Proposition 5, x_t is equal to the one-period real risk free rate. The solution for the price-dividend ratio in this economy is described in the following proposition

Proposition 6 *For the economy described in Proposition 5 with the dividend process modified as in Equations (29) the equity price-dividend ratio can be expressed as*

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(\overrightarrow{A}_n + \overrightarrow{G}_n x_t + \overrightarrow{E}_n v_t\right)$$

where

$$\overrightarrow{A}_n = \overrightarrow{A}_{n-1} + \frac{\ln\beta}{\gamma} + \overrightarrow{G}_{n-1}\mu_x + \overrightarrow{E}_{n-1}\mu_v + \frac{1}{2}\left(\overrightarrow{G}_{n-1}\sigma_x\right)^2$$

$$\overrightarrow{G}_n = \left(-1 + \frac{1}{\gamma} + \overrightarrow{G}_{n-1}\rho_{xx}\right)$$

$$\overrightarrow{E}_n = -\frac{\gamma^2}{2} + \frac{\gamma}{2} + \overrightarrow{E}_{n-1}\rho_{vv} + \frac{1}{2}\left(-\gamma+1\right)^2 + \frac{1}{2}\left(\overrightarrow{G}_{n-1}\sigma_{xv} + \overrightarrow{E}_{n-1}\sigma_v\right)^2$$

with $\overrightarrow{A}_0 = \overrightarrow{E}_0 = \overrightarrow{G}_0 = 0$. *(Proof available upon request.)*

By considering the expression for $\overrightarrow{E}_n$, we see that the direct effect of an increase in v_t is $\frac{1}{2}\gamma(1-\gamma)$. Therefore, only when $\gamma > 1$ will an increase in volatility depress the price-dividend ratio, but this ignores equilibrium term structure effects. In the context of a model with an endogenous term structure, Wu's results appear not readily generalizable.

7.2.2 Relation to Bansal and Yaron

In generating realistic asset return features, Bansal and Yaron (2004) [BY henceforth] stress the importance of a small persistent expected growth component in consumption and dividend growth, fluctuations in economic uncertainty and Epstein and Zin (1989) preferences, which allow for separation between the intertemporal elasticity of substitution (IES) and risk aversion. In fact, many of their salient results, including the non-trivial effects of economic uncertainty on price-dividend ratios and the equity premium rely explicitly on preferences being non-CRRA and the IES being larger than 1. While it is of course conceivable that the presence of q_t alone drives enough of a wedge between the Bansal -Yaron framework and ours, it may still come as a surprise that we find such an important role for economic uncertainty in what is essentially a power utility framework. Moreover, empirically, our estimation does not yield as large a role for the persistent expected cash flow and consumption growth component as in Bansal and Yaron (2004)'s calibrations. In this section, we resolve this conundrum by showing that the critical importance of the Epstein -Zin preferences in the BY framework stems from a rather implausibly strong assumption in the dynamics for the exogenous variables. To conserve space, we defer all proofs and derivations to an Appendix available upon request, and provide the key results and intuition here. BY use a log-linearized framework

where the pricing kernel and equity return innovations can be written as follows:

$$m_{t+1} - E_t m_{t+1} = \lambda_{mc}\sqrt{v_t}\varepsilon_{t+1}^c - \lambda_{mu}\sqrt{v_t}\varepsilon_{t+1}^u - \lambda_{mv}\sigma_{vv}\varepsilon_{t+1}^v \tag{30}$$

$$rx_{t+1} - E_t(rx_{t+1}) = \beta_{xd}\sqrt{v_t}\varepsilon_{t+1}^d + \beta_{xu}\sqrt{v_t}\varepsilon_{t+1}^u + \beta_{xv}\sigma_{vv}\varepsilon_{t+1}^v \tag{31}$$

where we transformed BY's notation as much as possible into ours. Here c, d, u and v refer respectively to the consumption growth, dividend growth and unobserved expected consumption growth and volatility processes, and the λ's (β's) measure the exposure of the pricing kernel (equity return) to these shocks. Note that the shocks to consumption/dividend growth and to the persistent expected growth process, u_t, are heteroskedastic, but the shock to volatility is not. Moreover, $\lambda_{mc} \propto [(1-1/IES)-1]$, $\lambda_{mu} = (1-\theta)\widehat{\lambda}_{mu}$, $\lambda_{mv} = (1-\theta)\widehat{\lambda}_{mv}$ where $\theta = [1-\gamma]/[1-1/IES]$ and γ is the usual risk aversion parameter. Hence, for a CRRA utility function $\theta = 1$, and $\lambda_{mu} = \lambda_{mv} = 0$.

From Equations (30) and (31), the equity premium simply follows from the standard expression as the conditional covariance between the return and the kernel corrected for a Jensen's inequality term. So one key assumption is then how the various shocks are correlated. Critically, BY assume total lack of correlation between all these shocks. Consequently, the model must generate the non-trivial correlation between consumption and dividend growth through the joint exposure to the latent u_t variable. Of course u_t represents a persistent predictable component for which we have no direct evidence on its existence. The implications are stark. The general equity premium expression is:

$$E_t[rx_{t+1}] = -\frac{1}{2}var_t(rx_{t+1}) + \beta_{xu}\lambda_{mu}v_t + \beta_{xv}\lambda_{mv}\sigma_v$$

Hence, if $\theta = 1$, as it is for CRRA utility, only the Jensen's inequality term remains. Similarly, it is then also the case that the price dividend ratio is actually increasing in economic uncertainty.

However, once you assume that dividend and consumption growth shocks are correlated, these knife-edge implications of CRRA utility disappear. As a simple example, assume that dividend growth equals consumption growth (we are pricing a claim to consumption), then it is straightforward to show that, under CRRA utility ($\theta = 1$):

$$E_t[rx_{t+1}] = -\frac{1}{2}var_t(rx_{t+1}) + \gamma v_t$$

That is, we recover an intuitive result, also an implication of Wu (2001) 's model, that the equity premium is proportional to the conditional variance of cash flow growth.

While our model does not assume such strong correlation between consumption and dividend growth as we entertained in the above example, there is nonetheless non-trivial conditional correlation that depends on v_t and by itself gives rise to non-trivial pricing and premium effects. Finally, it is also the case that, even with a BY - like structure governing the dynamics of consumption and dividend growth, the "Moody Investor" preferences we use would still lead to non-trivial pricing effects with v_t and q_t both being priced, and the sign of v_t's effect on premiums and price-dividend ratios depending intimately on the sign of σ_{qc}.

7.2.3   Relation to Campbell and Cochrane (1999)

Our model differs from CC's along numerous dimensions, yet, we would like to discuss the implications of two major differences between our approach and theirs. The first is of course the presence of the q_t shock, which is not correlated with fundamentals. A second major difference lies in our strategy of estimating the structural parameters while matching equity related moments, bond related moments and moments capturing features of fundamentals and their correlation with asset returns. CC instead calibrate their economy and choose a parameterization that yields a constant interest rate. We show below that both these differences are essential in interpreting our results and they are inter-related.

To move the model substantially in CC's direction, we set $\sigma_{qq} = 0$ and re-estimate the parameters. The model is now strongly rejected and we fail to match the high equity premium and the low interest rate[8]. This result may be somewhat surprising as CC appear to do very well with respect to many salient asset price features. One main reason for this result is in fact our estimation approach. In Figure 2, we go back to our original estimation and graph the GMM objective function around the estimated σ_{vv} and σ_{qq} (a three dimensional graph yields similar conclusions). Apart from the GMM objective function, we also show a function that aggregates the bond moments and one that aggregates the equity moments. You can then read off which parameters would maximize these separate objective functions. Clearly, parameters yielding the best "bond-fit" are pretty much indistinguishable from the estimated parameters, however, for a good equity fit the model wants a higher σ_{qq} and a lower σ_{vv}. In other words, the bond moments are very informative about the

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[8]Interestingly, an analogous result appears in Lettau and Wachter (2007) and Santos and Veronesi (2006): when the variable moving the price of risk is perfectly negatively correlated with consumption growth (as in the CC model), their model's performance with respect to the cross-section of expected returns also deteriorates considerably.

structural parameters. When we let σ_{qq} go to zero, it is in fact the fit with the term structure moments and the link between fundamentals and asset returns that cause the bad fit. When we restrict σ_{qq} equal to 0, but only try to fit equity moments, fundamental moments and the mean interest rate, the model is not rejected.

Our emphasis on simultaneously matching these three dimensions of the data (term structure movements, equity moments, and the correlation between fundamentals and asset returns) also distinguishes our work from recent articles by Wachter (2006) and Buraschi-Jiltsov (2007) who show reasonable fits of extensions of the CC model with term structure data. Consequently, while we have formulated a consumption-based asset pricing model that successfully matches many salient asset pricing phenomena, the presence of preference shocks *not* with correlated with fundamental shocks is essential to its success.

8 Conclusion

This paper has attempted to sort out the relative importance of two competing hypotheses for the sources of the magnitude and variation of asset prices. First, one literature has explored the role of cash flow volatility dynamics as a determinant of equity premiums both in the time series and cross section. Recent work in this area includes Bansal and Yaron (2003), Bansal, Khatchatrian and Yaron (2002), and Bansal and Lundblad (2004). A quite separate literature has explored shocks to investors preferences as drivers of equity prices. Prominent papers in this area include Campbell and Cochrane (1999), Abel (1990, 1999), and a large number of elaborations such as Wachter (2006), Bekaert, Engstrom and Grenadier (2004), Brandt and Wang (2003), Gordon and St. Amour (2000), Menzly, Santos and Veronesi (2004)), Wei (2004), and Lustig and van Nieuwerburgh (2005). With some exceptions, the focus has been on equities[9].

We design a theoretical model and empirical strategy which are capable of accommodating both explanations, and then implement an optimal GMM estimation to determine the relative importance of each story. We stress that from a theoretical perspective, it is important to consider the term structure effects on equity prices, a point prominent in the work of Abel (1988) and Barsky (1989). We conclude that both the conditional volatility of cash flow growth and time varying risk aversion

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[9]In a recent paper, Bansal, Gallant and Tauchen (2007) show that both a CC and a Bansal-Yaron type model fit the data equally well.

emerge as important factors driving variation in the term structure, dividend yields, and equity risk premium and the conditional volatility of returns. Not surprisingly, uncertainty is more important for volatility whereas risk aversion is more important for dividend yields and the risk premium.

Our work is indirectly related to two other important literatures. First, there is a large literature on the conditional CAPM which predicts a linear, positive relation between expected excess returns on the market and the conditional variance of the market. Since the seminal work of French, Schwert and Stambaugh (1987), the literature has struggled with the identification of the price of risk, which is often negative in empirical applications (see Scruggs (1998)). Of course, in our model, there are multiple sources of time variation in risk premiums and both the price of risk and the quantity of risk vary through time. Upon estimation of our structural model, we identify a strong positive contemporaneous correlation between expected equity returns and their conditional volatility. However, this relationship varies through time and contains a cyclical component (see Table 5).

Second, the volatility feedback literature has provided a link between the phenomenon of asymmetric volatility (or the leverage effect, the conditional return volatility and price shocks are negatively correlated) and risk premiums. It suggests that prices can fall precipitously on negative news as the conditional volatility increases and hence induces higher risk premiums (when the price of risk is positive). Hence, the literature primarily builds on the conditional CAPM literature (see Campbell and Hentschel (1992) and Bekaert and Wu (2000)). Wu (2001) sets up a present value model in which the variance of dividend growth follows a stochastic volatility process and shows under what conditions the volatility feedback effect occurs. There are two reasons why Wu's (2001) conclusions may not be generally valid. First, he ignores equilibrium considerations—that is the discount rate is not tied to preferences. Tauchen (2005) also shows how the presence of feedback may depend on preference parameters. Second, he assumes a constant interest rate. Within our set up, we can re-examine the validity of an endogenous volatility feedback effect. We intend to explore the implications of our model for these two literatures in the near future.

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A Propositions and Proofs

A.1 Proposition 1: Real Zero Coupon Bonds

For the economy described by Equations (9) and (10), the prices of real, risk free, zero coupon bonds are given by

$$P_{n,t}^{rz} = \exp\left(A_n + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t\right) \tag{A-1}$$

where

$$A_n = f^A\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$C_n = f^C\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$D_n = 0$$
$$E_n = f^E\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$F_n = f^F\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$

And the above functions are represented by

$$f^A = \ln\beta - \gamma\left(\delta + \mu_d\right) + A_{n-1} + E_{n-1}\mu_v + \left(F_{n-1} + \gamma\right)\mu_q$$
$$f^C \equiv -\gamma\rho_{du} + C_{n-1}\rho_{uu} + \gamma\left(1 - \rho_{uu}\right)$$
$$f^E \equiv E_{n-1}\rho_{vv}$$
$$\qquad + \frac{1}{2}\left(-\gamma\sigma_{dd} + \left(C_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dd} + \left(F_{n-1} + \gamma\right)\sigma_{qd}\right)^2$$
$$\qquad + \frac{1}{2}\left(\left(C_{n-1} - \gamma\right)\sigma_{uu} + \left(F_{n-1} + \gamma\right)\sigma_{qu}\right)^2$$
$$\qquad + \frac{1}{2}\left(-\gamma\sigma_{dv} + \left(C_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dv} + \left(F_{n-1} + \gamma\right)\sigma_{qv} + E_{n-1}\sigma_{vv}\right)^2$$
$$f^F \equiv F_{n-1}\rho_{qq} + \gamma\left(\rho_{qq} - 1\right) + \frac{1}{2}\left(\left(F_{n-1} + \gamma\right)\sigma_{qq}\right)^2$$

and $A_0 = C_0 = E_0 = F_0 = 0$.

In these equations we used a notation saving transformations:

$$\sigma_{qd} = \sigma_{qc}\sigma_{dd}\left(1 + \sigma_{ud}\right)$$
$$\sigma_{qu} = \sigma_{qc}\sigma_{uu}$$
$$\sigma_{qv} = \sigma_{qc}\sigma_{dv}\left(1 + \sigma_{ud}\right). \tag{A-2}$$

This effectively means that we are solving the model for a more general q_t process: $q_t = \mu_q + \rho_{qq}q_{t-1} + \sqrt{v_{t-1}}\left(\sigma_{qd}\varepsilon_t^d + \sigma_{qu}\varepsilon_t^u + \sigma_{qv}\varepsilon_t^v\right) + \sqrt{q_{t-1}}\sigma_{qq}\varepsilon_t^q$.

Proof: We start from the bond pricing relationship in equation (14) in the text:

$$P_{n,t}^{rz} = E_t\left[M_{t+1}P_{n-1,t+1}^{rz}\right] \tag{A-3}$$

where $P_{n,t}^{rz}$ is the price of a real zero coupon bond at time t with maturity at time $(t+n)$.

Suppose the prices of real, risk free, zero coupon bonds are given by

$$P_{n,t}^{rz} = \exp\left(A_n + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t\right) \tag{A-4}$$

where

$$A_n = f^A\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$C_n = f^C\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$E_n = f^E\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$
$$F_n = f^F\left(A_{n-1}, C_{n-1}, E_{n-1}, F_{n-1}, \Psi\right)$$

Then we have

$$
\begin{aligned}
&\exp\left(A_n + C_n u_t + D_n \pi_t + E_n v_t + F_n q_t\right)\\
&= E_t\{\exp\left(m_{t+1} + A_{n-1} + C_{n-1} u_{t+1} + D_{n-1}\pi_{t+1} + E_{n-1}v_{t+1} + F_{n-1}q_{t+1}\right)\}\\
&= E_t\{\exp(\ln\left(\beta\right) - \gamma\left(\delta + \Delta u_{t+1} + \Delta d_{t+1}\right) + \gamma\Delta q_{t+1}\\
&\quad + A_{n-1} + C_{n-1}u_{t+1} + D_{n-1}\pi_{t+1} + E_{n-1}v_{t+1} + F_{n-1}q_{t+1})\}
\end{aligned}
$$

After taking expectations, exploiting log-normality, we equate the coefficients on the two sides of the equation to obtain the expression given in (A-1).

A.2 Proposition 2: Real consol

Under the conditions set out in Proposition 1, the price-coupon ratio of a consol paying a constant real coupon is given by

$$P_t^{rc} = \sum_{n=1}^{\infty} \exp\left(A_n + B_n \Delta d_t + C_n u_t + E_n v_t + F_n q_t\right) \tag{A-5}$$

This follows immediately from recognizing that the "normalized" consol is a package zero coupon bonds.

A.3 Proposition 3: Nominal Zero Coupon Bonds

For the economy described by Equations (9) and (10), the time t price of a zero coupon bond with a risk free dollar payment at time $t + n$ is given by

$$P_{n,t}^z = \exp\left(\widetilde{A}_n + \widetilde{B}_n \Delta d_t + \widetilde{C}_n u_t + \widetilde{D}_n \pi_t + \widetilde{E}_n v_t + \widetilde{F}_n q_t\right) \tag{A-6}$$

where

$$\widetilde{A}_n = f^A\left(\widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1}\right) + \left(\widetilde{D}_{n-1} - 1\right)\mu_\pi + \frac{1}{2}\left(\widetilde{D}_{n-1} - 1\right)^2 \sigma_{\pi\pi}^2$$

$$\widetilde{B}_n = 0$$

$$\widetilde{C}_n = f^C\left(\widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1}\right) + \left(\widetilde{D}_{n-1} - 1\right)\rho_{\pi u}$$

$$\widetilde{D}_n = \left(\widetilde{D}_{n-1} - 1\right)\rho_{\pi\pi}$$

$$\widetilde{E}_n = f^E\left(\widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1}\right)$$

$$\widetilde{F}_n = f^F\left(\widetilde{A}_{n-1}, \widetilde{B}_{n-1}, \widetilde{C}_{n-1}, \widetilde{E}_{n-1}, \widetilde{F}_{n-1}\right)$$

where the functions $f^X\left(\cdot\right)$ are given in Proposition 1 for $X \in (A, B, C, E, F)$ and $\widetilde{A}_0 = \widetilde{B}_0 = \widetilde{C}_0 = \widetilde{D}_0 = \widetilde{E}_0 = \widetilde{F}_0 = 0$.

The proof of Proposition 3 follows the same strategy as Proposition 1 and is ommited to conserve space.

A.4 Proposition 4: Equities

For the economy described by Equations (9) and (10), the price-dividend ratio of aggregate equity is given by

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} \exp\left(\widehat{A}_n + \widehat{C}_n u_t + \widehat{E}_n v_t + \widehat{F}_n q_t\right) \tag{A-7}$$

where

$$\widehat{A}_n = f^A\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right) + \mu_d$$

$$\widehat{C}_n = f^C\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right) + \rho_{du}$$

$$\widehat{E}_n = f^E\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right)$$

$$+ \left(\frac{1}{2}\sigma_{dd}^2 + \sigma_{dd}\left((-\gamma)\sigma_{dd} + \left(\widehat{C}_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dd} + \left(\widehat{F}_{n-1} + \gamma\right)\sigma_{qd}\right)\right)$$

$$+ \left(\frac{1}{2}\sigma_{dv}^2 + \sigma_{dv}\left((-\gamma)\sigma_{dv} + \left(\widehat{C}_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dv} + \left(\widehat{F}_{n-1} + \gamma\right)\sigma_{qv} + \widehat{E}_{n-1}\sigma_{vv}\right)\right)$$

$$\widehat{F}_n = f^F\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right)$$

where the functions $f^X(\cdot)$ are given in Proposition 1 for $X \in (A, C, E, F)$ and $A_0 = C_0 = E_0 = F_0 = 0$.

Proof: Let P_t and D_t be the time-t ex-dividend stock price and dividend.
Guess

$$J_{n,t} \triangleq E_t \exp\left[\sum_{j=1}^{n}(m_{t+j} + \Delta d_{t+j})\right] = \exp\left(\widehat{A}_n + \widehat{C}_n u_t + \widehat{E}_n v_t + \widehat{F}_n q_t\right)$$

Then

$$J_{n,t} = E_t\left[\exp\left(m_{t+1} + \Delta d_{t+1}\right) E_{t+1}\sum_{j=1}^{n-1}\exp\left(m_{t+1+j} + \Delta d_{t+1+j}\right)\right]$$

$$= E_t\left[\exp\left(m_{t+1} + \Delta d_{t+1}\right) J_{n-1,t+1}\right]$$

or:

$$\exp\left(\widehat{A}_n + \widehat{C}_n u_t + \widehat{E}_n v_t + \widehat{F}_n q_t\right)$$
$$= E_t\{\exp[\ln\left(\beta\right) - \gamma\left(\delta + \Delta u_{t+1} + \Delta d_{t+1}\right) + \gamma\Delta q_{t+1} + \Delta d_{t+1}$$
$$+ \widehat{A}_{n-1} + \widehat{C}_{n-1} u_{t+1} + \widehat{E}_{n-1} v_{t+1} + \widehat{F}_{n-1} q_{t+1}]\}$$

Using the properties of lognormal distribution and equating coefficients on both sides of the equation gives us:

3

$$\widehat{A}_n = f^A\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right) + \mu_d$$

$$\widehat{C}_n = f^C\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right) + \rho_{du}$$

$$\widehat{E}_n = f^E\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right)$$
$$+ \frac{1}{2}\left(\sigma_{dd}^2 + \sigma_{dd}\left((-\gamma)\,\sigma_{dd} + \left(\widehat{C}_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dd} + \left(\widehat{F}_{n-1} + \gamma\right)\sigma_{qd}\right)\right)$$
$$+ \frac{1}{2}\left(\sigma_{dv}^2 + \sigma_{dv}\left((-\gamma)\,\sigma_{dv} + \left(\widehat{C}_{n-1} - \gamma\right)\sigma_{ud}\sigma_{dv} + \left(\widehat{F}_{n-1} + \gamma\right)\sigma_{qv} + \widehat{E}_{n-1}\sigma_{vv}\right)\right)$$

$$\widehat{F}_n = f^F\left(\widehat{A}_{n-1}, \widehat{C}_{n-1}, \widehat{E}_{n-1}, \widehat{F}_{n-1}, \Psi\right)$$

where the functions $f^X\left(\cdot\right)$ are given in Proposition 1 for $X \in (A, C, E, F)$ and $A_0 = C_0 = E_0 = F_0 = 0$.

For the purposes of estimation the coefficient sequences are calculated out 200 years. If the resulting calculated value for PD_t has not converged, then the sequences are extended another 100 years until either the PD_t value converges, or becomes greater than 1000 in magnitude.

B   Log Linear Approximation of Equity Prices

In the estimation, we use a linear approximation to the price-dividend ratio. From Equation (19), we see that the price dividend ratio is given by

$$\frac{P_t}{D_t} = \sum_{n=1}^{\infty} q_{n,t}^0$$
$$= \sum_{n=1}^{\infty} \exp\left(b_n^0 + b_n' Y_t\right) \tag{A-8}$$

and the coefficient sequences, $\{b_n^0\}_{n=1}^{\infty}$ and $\{b_n'\}_{n=1}^{\infty}$, are given above. We seek to approximate the log price-dividend ratio using a first order Taylor approximation of Y_t about $\overline{Y}$, the unconditional mean of Y_t. Let

$$\overline{q}_n^0 = \exp\left(b_n^0 + b_n'\overline{Y}\right) \tag{A-9}$$

and note that

$$\frac{\partial}{\partial Y_t}\left(\sum_{n=1}^{\infty} q_{n,t}^0\right) = \sum_{n=1}^{\infty} \frac{\partial}{\partial Y_t} q_{n,t}^0 = \sum_{n=1}^{\infty} q_{n,t}^0 \cdot b_n' \tag{A-10}$$

Approximating,

$$pd_t \simeq \ln\left(\sum_{n=1}^{\infty} \overline{q}_n^0\right) + \frac{1}{\sum_{n=1}^{\infty} \overline{q}_n^0}\left(\sum_{n=1}^{\infty} \overline{q}_n^0 \cdot b_n'\right)\left(Y_t - \overline{Y}\right)$$
$$= d_0 + d' Y_t \tag{A-11}$$

where d_0 and d' are implicitly defined. Similarly,

$$gpd_t \equiv \ln\left(1 + \frac{P_t}{D_t}\right) \simeq \ln\left(1 + \sum_{n=1}^{\infty} \overline{q}_n^0\right) + \frac{1}{1 + \sum_{n=1}^{\infty} \overline{q}_n^0}\left(\sum_{n=1}^{\infty} \overline{q}_n^0 \cdot b_n'\right)\left(Y_t - \overline{Y}\right)$$
$$= h_0 + h' Y_t \tag{A-12}$$

where h_0 and h' are implicitly defined. Note also that the dividend yield measure used in this study can be expressed as follows

$$dp_t \equiv \ln\left(1 + \frac{D_t}{P_t}\right) = gpd_t - pd_t \tag{A-13}$$

so that it is also linear in the state vector under these approximations. Also, log excess equity returns can be represented follows. Using the definition of excess equity returns,

$$
\begin{aligned}
rx_{t+1} &= -rf_t - pd_t + gd_{t+1} + \pi_{t+1} + gpd_{t+1} \\
&\sim (h_0 - d_0) + (e'_d + e'_\pi + h') Y_{t+1} + \left(-e'_{rf} + -d'\right) Y_t \\
&= r_0 + r'_1 Y_{t+1} + r'_2 Y_t
\end{aligned} \tag{A-14}
$$

where r_0, r'_1 and r'_2 are implicitly defined.

B.1 Accuracy of the Equity Approximation

To assess the accuracy of the log linear approximation of the price dividend ratio, the following experiment was conducted. For the model and point estimates reported in Table 2, a simulation was run for 10,000 periods. In each period, the 'exact' price dividend ratio and log dividend yield were calculated in addition to their approximate counterparts derived in the previous subsection. The resulting series for exact and approximate dividend yields and excess stock returns compare as follows (quarterly rates).

	appx dp_t	exact dp_t	appx r_t^x	exact r_t^x
mean	0.0099	0.0100	0.0118	0.0119
std. dev.	0.0032	0.0034	0.0945	0.0891
correlation	0.9948		0.9853	

C Analytic Moments of Y_t and W_t

Recall that the data generating process for Y_t is given by,

$$
\begin{aligned}
Y_t &= \mu + AY_{t-1} + \left(\Sigma_F F_{t-1} + \Sigma_H\right)\varepsilon_t \\
F_t &= sqrt(diag(\phi + \Phi Y_t))
\end{aligned} \tag{A-15}
$$

We can show that the uncentered first, second, and first autocovariance moments of Y_t are given by,

$$
\overline{Y_t} = (I_k - A)^{-1}\mu
$$

$$
vec\left(\overline{Y_t Y_t'}\right) = (I_{k^2} - A \otimes A)^{-1} \cdot vec\left(\mu\mu' + \mu\overline{Y_t}' A' + A\overline{Y_t}\mu' + \Sigma_F \overline{F_t^2}\Sigma_F' + \Sigma_H \Sigma_H'\right)
$$

$$
vec\left(\overline{Y_t Y_{t-1}'}\right) = (I_{k^2} - A \otimes A)^{-1} \cdot vec\left(\mu\mu' + \mu\overline{Y_t}' A' + A\overline{Y_t}\mu' + A\left(\Sigma_F \overline{F_t^2}\Sigma_F' + \Sigma_H \Sigma_H'\right)\right) \tag{A-16}
$$

where overbars denote unconditional means and $\overline{F_t^2} = diag\left(\phi + \Phi\overline{Y_t}\right)$.

Now consider the unconditional moments of a n-vector of observable variables W_t which obey the condition

$$W_t = \mu^w + \Gamma^w Y_{t-1} + \left(\Sigma_F^w F_{t-1} + \Sigma_H^w\right)\varepsilon_t \tag{A-17}$$

where μ^w is an n-vector and Σ_F^w, Σ_H^w and Γ^w are $(n \times k)$ matrices. It is straightforward to show

that the uncentered first, second, and first autocovariance moments of W_t are given by,

$$\overline{W_t} = \mu^w + \Gamma^w \overline{Y_t}$$

$$\overline{W_t W_t'} = \mu^w \mu^{w\prime} + \mu^w \overline{Y_t}' \Gamma^{w\prime} + \Gamma^w \overline{Y_t} \mu^{w\prime} + \Gamma^w \overline{Y_t Y_t'} \Gamma^{w\prime} + \Sigma_F^w \overline{F_t^2} \Sigma_F^{w\prime} + \Sigma_H^w \Sigma_H^{w\prime}$$

$$\overline{W_t W_{t-1}'} = \mu^w \mu^{w\prime} + \mu^w \overline{Y_t}' \Gamma^{w\prime} + \Gamma^w \overline{Y_t} \mu^{w\prime} + \Gamma^w \overline{Y_t Y_{t-1}'} \Gamma^{w\prime} + \Gamma^w \left(\Sigma_F \overline{F_t^2} \Sigma_F^{w\prime} + \Sigma_H \Sigma_H^{w\prime} \right) \quad \text{(A- 18)}$$

It remains to demonstrate that the observable series used in estimation obey Equation (A- 17). This is trivially true for elements of W_t which are also elements of Y_t such as Δd_t, Δc_t, π_t. Using Equations (17), (A- 14) and (A- 11), it is apparent that r_t^f, dp_t and r_t^x satisfy Equation (A- 17) as well.

D   GMM Estimation and Constructing $\widehat{S}\left(W_T\right)$

Armed with an estimate for $S(W_t)$, $\widehat{S}(W_t)$, we minimize

$$J\left(W_T; \widehat{\Psi}\right) = g_T\left(\widehat{\Psi}\right)\ \left(\widehat{S}(W_t)\right)^{-1}\ g_T\left(\widehat{\Psi}\right)' \quad \text{(A- 19)}$$

in a one-step GMM procedure.

To estimate $\widehat{S}(W_t)$, we use the following procedure.

Under the model, we can project $X\left(W_t\right)$ onto the vector of state variables Y_t^c, which stacks the contemporaneous five state variables and a number of lags,

$$X\left(W_t\right) = \widehat{B} Y_t^c + \widehat{\varepsilon}_t$$

where $\widehat{B}$ and $\widehat{\varepsilon}_t$ are calculated using a standard linear projection of $X\left(W_t\right)$ onto Y_t^c. We assume the covariance matrix of the residuals, $\widehat{D}$, is diagonal and estimate it using the residuals, $\widehat{\varepsilon}_t$, of the projection. The projection implies

$$\widehat{S}\left(W_T\right) = \widehat{B} \widehat{S}\left(Y_T^c\right) \widehat{B}' + \widehat{D}$$

where $\widehat{S}\left(Y_T^c\right)$ is the spectral density at frequency zero of Y_t^c. To estimate $\widehat{S}\left(Y_T^c\right)$, we use a standard pre-whitening technique as in Andrews and Monahan (2004). Because Y_t^c contains two unobservable variables, v_t and q_t, we use instead the vector $Y_t^p = \left[\Delta d_t^f, \pi_t, \Delta c_t^f, r f_t, dp_t^f\right]'$ and one lag of Y_t^p to span Y_t^c.[10]

Because the system is nonlinear in the parameters, we take precautionary measures to assure that we find the global minimum. First, over 100 starting values for the parameter vector are chosen at random from within the parameter space. From each of these starting values, we conduct preliminary minimizations. We discard the runs for which estimation fails to converge, for instance, because the maximum number of iterations is exceeded, but retain converged parameter values as "candidate" estimates. Next, each of these candidate parameter estimates is taken as a new starting point and minimization is repeated. This process is repeated for several rounds until a global minimizer has been identified as the parameter vector yielding the lowest value of the objective function. In this process, the use of a fixed weighting matrix is critical. Indeed, in the presence of a parameter-dependent weighting matrix, this search process would not be well defined. Finally, we confirm the parameter estimates producing the global minimum by starting the minimization routine at small perturbations around the parameter estimates, and verifying that the routine returns to the global minimum.

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[10] Strictly speaking, the moving average filters of Δd_t^f Δc_t^f and dp_t^f would require using 3 lags, but the dimensionality of that system is too large.

Table 1: Heteroskedasticity in Fundamentals

Panel A: Univariate Volatility Regression

	Δd_t^f	Δc_t^f	rx_t
rf_{t-1}	−0.0809	−0.0044	−0.4574
	(0.0279)	(0.0020)	(0.3468)
dp_{t-1}^f	0.1155	0.0148	1.0851
	(0.0392)	(0.0061)	(1.1342)
spd_{t-1}	0.1288	0.0052	0.6812
	(0.0735)	(0.0060)	(1.2146)

Panel B: Multivariate Volatility Regression

	baseline		restricted	
	Δd_t^f	Δc_t^f	Δd_t^f	Δc_t^f
rf_{t-1}	−0.0660	−0.0023	−0.0365	−0.0031
	(0.0242)	(0.0010)	(0.0171)	(0.0011)
dp_{t-1}^f	0.0770	0.0135	0.0916	0.0077
	(0.0380)	(0.0058)	(0.0344)	(0.0035)
spd_{t-1}	0.0651	0.0043	0.0280	0.0024
	(0.0473)	(0.0052)	(0.0349)	(0.0029)
joint p-val	0.01	0.04	0.03	0.02
restriction, p-val			0.11	

Panel C: Cyclicality and Trend Regressions

	recession		recession and trend	
	Δd_t^f	Δc_t^f	Δd_t^f	Δc_t^f
$I_{recess,t-1}$	0.0033	0.0003	0.0022	0.0003
	(0.0016)	(0.0001)	(0.0017)	(0.0001)
$\ln(t)$			−0.0456	−0.0027
			(0.0252)	(0.0014)

Table 1: Notes

The symbols Δd_t^f, Δc_t^f, and rx_t refer to log filtered dividend and consumption growth and log excess equity returns. The table presents projections of the conditional mean and volatility of these variables onto a set of instruments; including the log yield on a 90 day T-bill, rf_t, the filtered log dividend yield, dp_t^f, the log yield spread, spd_t, a dummy variable equal to one during NBER-defined U.S. recessions, and a log time trend $\ln(t)$. For all estimations, the generic specification is,

$$E_t\left[\Delta d_{t+1}^f\right] = a_0 + a_1 x_{1t}$$

$$VAR_t\left[\Delta d_{t+1}^f\right] = b_0 + b_1 x_{2t}$$

where x_{1t} and x_{2t} refer to generic vectors of instruments for the conditional mean and volatility equations respectively, and analogous equations are estimated for consumption growth and returns (simultaneously). Throughout, the conditional mean instrument vector, x_{1t}, for dividend and consumption growth includes only the consumption-dividend ratio, u_t^f. For returns, we additionally allow the conditional mean to depend on rf_t, dp_t^f, and spd_t. Results from the conditional mean equations are not reported. Because both the consumption and dividend growth series are effectively four-quarter moving averages, we instrument for all the explanatory variables in both the mean and volatility equations with the variable lagged four quarters and also use GMM standard errors with four New-West lags.

In panel A, the volatility equations are univariate, so that x_{2t} is comprised of only one variable at a time. In panels B and C, the volatility equations are multivariate, so that x_{2t} contains all the listed instruments at once. In Panel B, the row labeled "joint p-val" presents a Wald test for the joint significance of the b_1 parameters. Under the columns labeled "restricted," a restriction is imposed under which the volatility parameters, b_1, are proportional across consumption and dividend growth. The bottom row, reports the p-value for this overidentifying restriction.

Table 2: Dynamic Risk and Uncertainty Model Estimation

Parameter Estimates				
$E\left[\Delta d\right]$	ρ_{du}		σ_{dd}	σ_{dv}
0.0039	0.0214		0.0411	0.0413
(0.0011)	(0.0082)		(0.0116)	(0.0130)
$E\left[v_t\right]$	ρ_{vv}		σ_{vv}	
1.0000	0.9795		0.3288	
$(fixed)$	(0.0096)		(0.0785)	
	ρ_{uu}		σ_{ud}	σ_{uu}
	0.9826		-0.9226	0.0127
	(0.0071)		(0.0233)	(0.0007)
$E\left[q_t\right]$	ρ_{qq}		σ_{qc}	σ_{qq}
1.0000	0.9787		-5.2211	0.1753
$(fixed)$	(0.0096)		(4.5222)	(0.0934)
$\ln\left(\beta\right)$	γ	δ		
-0.0168	1.1576	0.0047		
(0.0042)	(0.7645)	(0.0011)		
$E\left[\pi_t\right]$	$\rho_{\pi\pi}$	$\rho_{\pi u}$	$\sigma_{\pi\pi}$	
0.0081	0.2404	-0.0203	0.0086	
(0.0010)	(0.1407)	(0.0073)	(0.0017)	

Overidentification Test	
$J\left(15\right)$	12.7262
$(pval)$	(0.6234)

9

Table 2: Notes

The model is defined by the equations

$$\Delta d_t = \mu_d + \rho_{du} u_{t-1} + \sqrt{v_{t-1}} \left(\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v \right)$$

$$v_t = \mu_v + \rho_{vv} v_{t-1} + \sigma_{vv} \sqrt{v_{t-1}} \varepsilon_t^v$$

$$u_t = \mu_u + \rho_{uu} u_{t-1} + \sigma_{ud} \left(\Delta d_t - E_{t-1}[\Delta d_t] \right) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u$$

$$q_t = \mu_q + \rho_{qq} q_{t-1} + \sigma_{qc} \left(\Delta c_t - E_{t-1}[\Delta c_t] \right) + \sigma_{qq} \sqrt{q_{t-1}} \varepsilon_t^q$$

$$\pi_t = \mu_\pi + \rho_{\pi\pi} \pi_{t-1} + \rho_{\pi u} u_{t-1} + \sigma_\pi \varepsilon_t^\pi$$

$$\Delta c_t = \delta + \Delta d_t + \Delta u_t$$

$$= (\delta + \mu_d) + (\rho_{ud} + \rho_{uu} - 1) u_t + (1 + \sigma_{ud}) \sqrt{v_{t-1}} \left(\sigma_{dd} \varepsilon_t^d + \sigma_{dv} \varepsilon_t^v \right) + \sigma_{uu} \sqrt{v_{t-1}} \varepsilon_t^u$$

$$m_{t+1} = \ln(\beta) - \gamma \Delta c_{t+1} + \gamma \Delta q_{t+1}$$

The moments used to estimate the model are

$$E \left[\Delta d_t^f, \Delta c_t^f, \pi_t, rf_t, dp_t^f, spd_t, r_t^{ex} \right] \text{ (7)}$$

$$E \left[\left(\Delta d_t^f \right)^2, \left(\Delta c_t^f \right)^2, (\pi_t)^2, (rf_t)^2, \left(dp_t^f \right)^2, (spd_t)^2, (r_t^{ex})^2 \right] \text{ (7)}$$

$$E \left[(\pi_t \pi_{t-1}), (\pi_t \pi_{t-4}), \left(\Delta d_t^f \Delta d_{t-4}^f \right), \left(\Delta c_t^f \Delta c_{t-4}^f \right) \right] \text{ (4)}$$

$$E \left[\left(\Delta d_t^f \Delta c_t^f \right), \left(\Delta d_t^f rf_t \right), \left(\Delta d_t^f dp_t \right), \left(\Delta d_t^f spd_t \right), \left(\Delta c_t^f rf_t \right), \left(\Delta c_t^f dp_t \right), \left(\Delta c_t^f spd_t \right), \left(\pi_t \Delta c_t^f \right), (\pi_t rf_t) \right] \text{ (9)}$$

$$E \left[\left(\Delta d_t^f \right)^2 \otimes \left(rf_{t-4}, dp_{t-4}^f, spd_{t-4} \right), \left(\Delta d_t^f \right)^3, \left(\Delta d_t^f \right)^4, \left(\Delta c_t^f \right)^3, \left(\Delta c_t^f \right)^4 \right] \text{ (7)}$$

The model is estimated by GMM. Data are quarterly US aggregates from 1927:1-2004:3. Δd_t^f, Δc^f, π_t, rf_t, dp_t^f, spd_t, and rx_t refer to filtered log dividend growth, filtered log consumption growth, log inflation, the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, and log excess equity returns (with respect to the 90 day T-bill). See text for data construction and estimation details.

Table 3: The Fit of the Model: Linear Moments

Simulated observable moments							
	Δd_t^f	π_t	Δc_t^f	rf_t	dp_t^f	spd_t	rx_t
mean	[0.0038]	[0.0084]	[0.0085]	[0.0097]	[0.0096]	[0.0038]	[0.0121]
	0.0026	0.0077	0.0080	0.0094	0.0099	0.0040	0.0141
	(0.0029)	(0.0013)	(0.0008)	(0.0010)	(0.0004)	(0.0004)	(0.0062)
std.dev.	[0.0291]	[0.0121]	[0.0068]	[0.0074]	[0.0035]	[0.0033]	[0.0967]
	0.0308	0.0130	0.0075	0.0078	0.0035	0.0032	0.1085
	(0.0034)	(0.0015)	(0.0009)	(0.0007)	(0.0003)	(0.0002)	(0.0126)
autocorr	$[-0.0275]^*$	[0.5837]	$[0.0233]^*$	[0.9170]	$[0.9429]^*$	[0.6840]	[−0.0071]
	0.0699	0.6016	0.2460	0.9582	0.9347	0.8107	−0.0446
	(0.0995)	(0.0802)	(0.2008)	(0.0356)	(0.1751)	(0.0618)	(0.1004)

Simulated moments, in square brackets, are calculated by simulating the system for 100,000 periods using the point estimates from Table 2 and calculating sample moments of the simulated data. Autocorrelations are all at one lag except for series denote with an asterisk (*): dividend growth, consumption growth and the dividend price ratio, which are calculated at 4 lags. The second and third numbers for each entry are the sample moments and corresponding standard errors (in parentheses) computed using GMM with 4 Newey West lags. Data are quarterly US aggregates from 1927:1-2004:3. Δd_t^f, π_t, Δc_t^f, rf_t, dp_t^f, spd_t, and rx_t, refer to filtered log dividend growth, log inflation, filtered log consumption growth, the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, log excess equity returns (with respect to the 90 day T-bill). See text for data construction details.

Table 4: The Fit of the Model: Nonlinear Moments

Panel A: Univariate Volatility Regression

	Δd_t^f	Δc_t^f	rx_t
rf_{t-1}	$[-0.0298]$	$[-0.0020]$	$[0.3969]$
	-0.0809	-0.0044	-0.4574
	(0.0279)	(0.0020)	(0.3468)
dp_{t-1}^f	$[0.0439]$	$[0.0019]$	$[1.3486]$
	0.1155	0.0148	1.0851
	(0.0392)	(0.0061)	(1.1342)
spd_{t-1}	$[0.3332]$	$[0.0179]$	$[2.9831]$
	0.1288	0.0052	0.6812
	(0.0735)	(0.0060)	(1.2146)

Panel B: Skewness and Kurtosis

	Δd_t^f	Δc_t^f	rx_t
skew	$[-0.2250]$	$[-0.4574]$	$[0.1494]$
	-0.3287	-0.7537	0.1254
	(0.6339)	(0.4450)	(0.7228)
kurt	$[10.0250]$	$[10.1726]$	$[5.4295]$
	7.9671	6.4593	9.7118
	(1.3668)	(0.9673)	(2.0755)

Panel A repeats the regression model of Table 1 and also reports analogous simulated statistics generated by the model estimated in Table 2. Panel B reports unconditional skewness and kurtosis for the variables in each column. In each panel, the simulated moments (50,000 observations) are reported in square brackets and the corresponding data statistics and standard errors are reported below, with the standard errors in parentheses.

Table 5: Dynamic Properties of Risk, Uncertainty and Asset Prices

Panel A: Unconditional

Simulated unobservable univariate moments

	v_t	q_t	RA_t	rrf_t	$rspd_t$	$E_t[rx_{t+1}]$	$V_t[rx_{t+1}]$	S_t	$MaxS_t$
mean	1.0090	1.0097	7.06	0.0017	0.0038	0.0121	0.0092	0.1396	0.2075
median	0.3611	0.7784	2.52	0.0037	0.0034	0.0103	0.0070	0.1320	0.2095
std.dev.	1.6063	0.9215	36.34	0.0093	0.0028	0.0075	0.0083	0.1265	0.0491
autocorr	0.9788	0.9784	0.9212	0.9784	0.9777	0.9789	0.9794	0.5384	0.9653

Panel B: Cyclicality of Means

Simulated unobservable univariate means

	v_t	q_t	RA_t	rrf_t	$rspd_t$	$E_t[rx_{t+1}]$	$V_t[rx_{t+1}]$	S_t	$MaxS_t$
Expansion	0.8665	0.9893	6.73	0.0024	0.0035	0.0117	0.0085	0.1406	0.2053
Recession	2.7195	1.2544	10.96	−0.0064	0.0076	0.0171	0.0183	0.1283	0.2349

Panel C: Correlations with v_t and q_t

Simulated correlations between v_t, q_t and observables

	rrf_t	$rspd_t$	rf_t	dp_t	rx_t	$E_t[rx_{t+1}]$	$V_t[rx_{t+1}]$
v_t	−0.9232	0.9562	−0.5163	−0.1835	0.1470	0.3428	0.8799
q_t	0.4687	0.1756	0.5375	0.9215	−0.1071	0.8943	0.3758

Simulated moments are calculated by simulating the system for 100,000 periods using the point estimates from Table 2 for a number of variables including: v_t, dividend growth volatility, q_t, the log inverse consumption surplus ratio, RA_t, local risk aversion which is $\gamma \exp(q_t)$,. The variables rrf_t and $rspd_t$ represent the real short rate and real term spread respectively, and $E_t[rx_{t+1}]$ and $V_t[rx_{t+1}]$ denote the conditional mean and conditional variance of excess stock returns. S_t denotes the conditional Sharpe ratio for equity. $MaxS_t$ denotes the maximum attainable Sharpe ratio for any asset in the economy which is given by the quantity, $[\exp(V_t(m_{t+1})) - 1]^{1/2}$.

In Panel B, means of simulated data conditional on a binary recession/expansion variable are presented. Recessions are defined in the simulated data as periods of negative real consumption growth. Recessions represent approximately 8% of all observations in the simulated data.

In Panel C, the simulated unconditional correlations among v_t, q_t and other endogenous variables are reported.

Table 6: Variance Decompositions

	Δd_t	π_t	u_t	v_t	q_t

Fraction of variance due to variation in each state element

	Δd_t	π_t	u_t	v_t	q_t
rrf_t	[0.0000]	[0.0000]	[0.0999]	[0.7239]	[0.1761]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.1154⟩	⟨0.1472⟩	⟨0.0698⟩
$rspd_t$	[0.0000]	[0.0000]	[0.0752]	[0.8653]	[0.0596]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.01010⟩	⟨0.0943⟩	⟨0.0510⟩
cp_t^{rcons}	[0.0000]	[0.0000]	[0.0502]	[0.2299]	[0.7199]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.0630⟩	⟨0.1041⟩	⟨0.1296⟩
rf_t	[0.0000]	[0.1230]	[0.0904]	[0.5010]	[0.2856]
	⟨0.0000⟩	⟨0.0765⟩	⟨0.2222⟩	⟨0.1216⟩	⟨0.1796⟩
spd_t	[0.0000]	[0.3148]	[0.0035]	[0.6019]	[0.0797]
	⟨0.0000⟩	⟨0.3407⟩	⟨0.0599⟩	⟨0.3413⟩	⟨0.0745⟩
dp_t^f	[0.0000]	[0.0000]	[0.0655]	[0.0544]	[0.8801]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.0901⟩	⟨0.0798⟩	⟨0.0627⟩
rx_t	[0.3605]	[0.0091]	[−0.1593]	[0.1640]	[0.6257]
	⟨0.0733⟩	⟨0.0036⟩	⟨0.0401⟩	⟨0.0895⟩	⟨0.1397⟩
$E_t[rx_{t+1}]$	[0.0000]	[0.0000]	[−0.0167]	[0.1665]	[0.8502]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.0146⟩	⟨0.1281⟩	⟨0.1182⟩
$V_t[rx_{t+1}]$	[0.0000]	[0.0000]	[0.0000]	[0.8029]	[0.1971]
	⟨0.0000⟩	⟨0.0000⟩	⟨0.0000⟩	⟨0.2229⟩	⟨0.2229⟩

The symbols, rrf_t, $rspd_t$, and cp^{rcons} refer to the theoretical real short rate, real term spread, and the coupon-price ratio of a real consol. The table reports the fraction of variation of selected variables due to variation in elements of the state vector.

The variable in each row can be expressed as a linear combination of the current state and lagged vector. Generally, under the model in Table 2, for the row variables, x_t,

$$x_t = \mu + \Gamma' Y_t^c$$

where Y_t^c is the 'companion form' of the $N-$vector, Y_t; that is, Y_t^c is comprised of 'stacked' current and lagged values of Y_t. μ and Γ are constant vectors implied by the model and parameter estimates of Table 2. Let $Var(Y_t^c)$ be the variance covariance matrix of Y_t^c. Based on μ and Γ, the proportion of the variation of each row variable attributed to the nth element of the state vector is calculated as

$$\frac{\Gamma' Var(Y_t^c) \Gamma^{(n)}}{\Gamma' Var(Y_t^c) \Gamma}$$

where $\Gamma^{(n)}$ is a column vector such that $\left\{\Gamma^{(n)}\right\}_i = \{\Gamma\}_i$ for $i = n, N+n, ...$ and zero elsewhere. Standard errors are reported below in angle brackets and are calculated from the variance covariance matrix of the parameters in Table 2 using the Δ-method.

Table 7: Model Implied Reduced Form Return Predictability

Excess Returns

Parameter estimates

	multivariate	univariate
β_0	[−0.0037]	
	−0.0358	
	(0.0256)	
β_1	[−0.3097]	[0.2192]
	−0.1669	−1.1651
	(0.7464)	(0.7839)
β_2	[2.0695]	[1.5770]
	3.7980	3.7260
	(2.0231)	(1.9952)
β_3	[0.7668]	[1.2389]
	3.5376	3.4728
	(1.8900)	(1.8728)

The predictability model for excess returns is defined as,

$$rx_{t+1} = \beta_0 + \beta_1 rf_t + \beta_2 dp_t^f + \beta_3 spd_t + \varepsilon_{t+1}$$

and is estimated by GMM. Data are quarterly US aggregates from 1927:1-2004:3. The symbols rf_t, dp_t^f, spd_t, and rx_t refer to the log yield on a 90 day T-bill, the filtered log dividend yield, the log yield spread, and log excess equity returns (with respect to the 90 day T-bill). Simulated moments, in square brackets, are calculated by simulating the model for 100,000 periods using the point estimates from Table 2 and estimating the above model on the simulated data. The second and third numbers for each entry are the sample moments and corresponding standard errors (in parentheses).

Table 8: Selected Post-War Estimation Results

Panel A: Unconditional Moments

Simulated univariate moments

	rf_t	rx_t	v_t	q_t	RA_t	rrf_t	$E_t[rx_{t+1}]$	$V_t[rx_{t+1}]$	S_t
mean	0.0118	0.0122	1.0038	1.0072	7.70	0.0022	0.0123	0.0073	0.1440
median	0.0114	0.0121	0.8989	0.8898	4.50	0.0020	0.0113	0.0067	0.1411
std.dev.	0.0066	0.0861	0.6128	0.7354	13.48	0.0065	0.0053	0.0036	0.0275
autocorr	0.9028	0.0018	0.9072	0.9854	0.9690	0.9643	0.9827	0.9670	0.9756

Panel B: Variance Decompositions

Fraction of var. due to state element

	v_t	q_t
rrf_t	[0.2825]	[0.5008]
	⟨0.1566⟩	⟨0.1347⟩
$rspd_t$	[0.9252]	[0.0638]
	⟨0.0700⟩	⟨0.0478⟩
dp_t^f	[0.0035]	[0.9960]
	⟨0.0102⟩	⟨0.0125⟩
$E_t[rx_{t+1}]$	[0.0480]	[0.9523]
	⟨0.0540⟩	⟨0.0538⟩
$V_t[rx_{t+1}]$	[0.2843]	[0.7157]
	⟨0.1754⟩	⟨0.1754⟩

This table reports results from estimation of the structural model using data from 1946Q1 through 2004Q3, the post-war era. Panel A reports results analagous to Table 5 (see Table 5 notes) and Panel B reports results similar to Table 6.

Figure 1: Term Structure Determinants
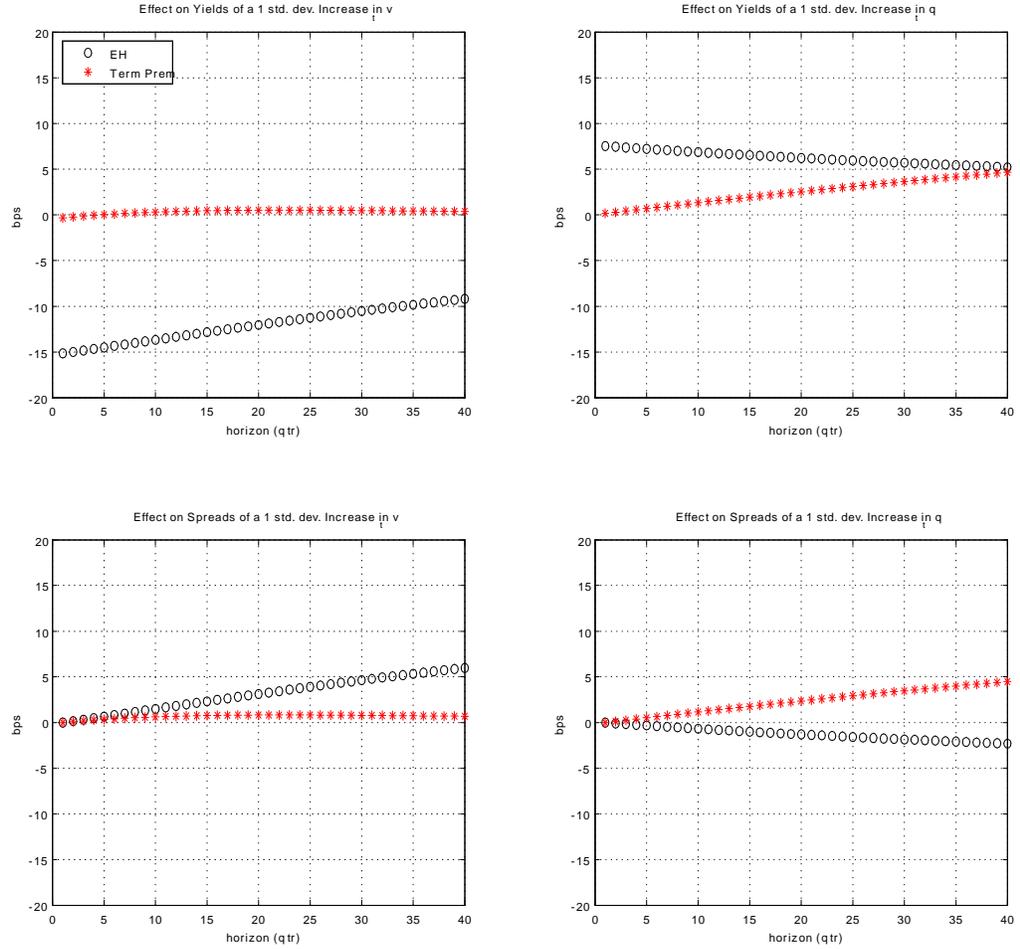
Under the model of Table 2, real risk free yields of horizon, h, have solutions of the form,

$$rrf_{h,t} = a_h + A'_h Y_t$$

where the coefficients above are functions of the 'deep' model parameters. This figure shows the effect on these yields and the associated spreads (relative to the 1 period yield) of 1 standard deviation changes in the latent factors, v_t and q_t using the point estimates in Table 2. At horizons greater than 1, these effects can be further decomposed into parts corresponding to the expectations hypothesis (EH), and term premiums, which are drawn in blue and red bars respectively.

Figure 2: Stock and Bond Moment Sensitivity to Volatility Parameters
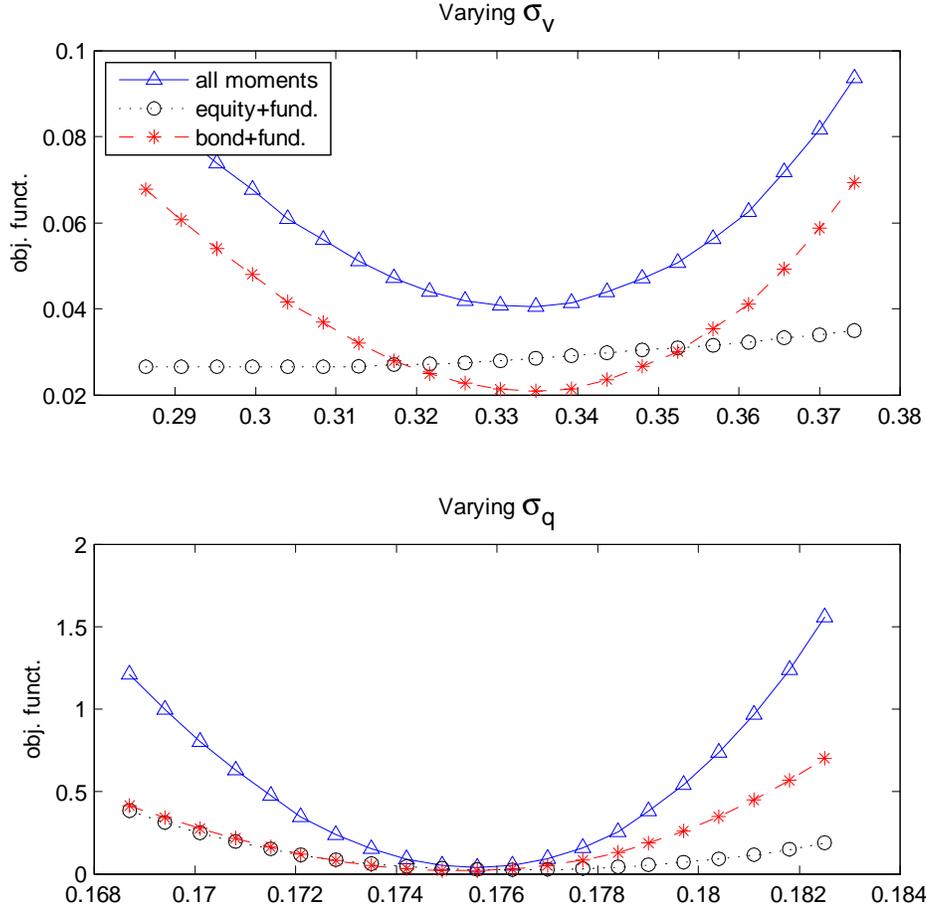
The lines plot the shape of the GMM objective function from the structural model for small perturbations of σ_{qq} and σ_{vv} about their estimated values in Table 2, holding all other parameters at exactly their estimated values.

The blue triangle line plots the values for the overall GMM objective function. The red stars lines plot values for the quadratic,

$$g_T^{bonds}\left(\Psi\right)\ \left(\widehat{S}(W_t^{bonds})\right)^{-1}\ g_{1T}^{bonds}\left(\Psi\right)'$$

as a function of σ_{qq} and σ_{vv} (holding all other parameters at their estimated values). The 'bonds' superscript denotes that we are restricting attention to the 10 moments which involve either the short rate or term spread. The black circle lines plots the location of the analogous quadratic for the equity moments,

$$g_T^{equity}\left(\Psi\right)\ \left(\widehat{S}(W_t^{equity})\right)^{-1}\ g_{1T}^{equity}\left(\Psi\right)'$$

where we now restrict attention the the 7 moments which involve either the dividend yield or excess equity return.