# Midterm Exam II Answer key Math 212 Section 4

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Formulas you can use:
1. Cylindrical coordinates: $x = rcos\theta$, $y = rsin\theta$,$z = z$, $0 \leq r$, $0 \leq \theta \leq 2\pi$.
2. Spherical coordinates: $x = \rho sin\phi cos\theta$, $y = \rho sin\phi sin\theta$, $z = \rho cos\phi$, $0 \leq \rho$,$0 \leq \phi \leq \pi$,$0 \leq \theta \leq 2\pi$.

1.Evaluate $\int_0^2 \int_{x^3}^8 x^5 e^{y^3} dydx$
Hint: Change order of integration.

Soln:

$$\begin{aligned} \int_0^2 \int_{x^3}^8 x^5 e^{y^3} dydx &= \int_0^8 \int_0^{\sqrt[3]{y}} x^5 e^{y^3} dxdy = \int_0^8 e^{y^3} \cdot \frac{x^6}{6} |_0^{\sqrt[3]{y}} \ dy \quad = \\ \frac{1}{6} \times \frac{1}{3} \int_0^8 e^{y^3} dy^3 &= \frac{1}{18}(e^{512} - 1) \end{aligned}$$

2.Calculate the volume of the region under the graph of $f(x, y) = ysinx$ and on the region bounded by $y = 0; y = cosx; x = 0; x = \pi/2$.

Soln:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_0^{cosx} ysinxdydx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} sinx(x^2)|_0^{cosx} dydx = -\frac{1}{2} \int_0^{\frac{\pi}{2}} cos^2 xd(cosx) = \\ -\frac{1}{2} \int_1^0 u^2 du &= -\frac{1}{6}(-1) = \frac{1}{6} \end{aligned}$$

3.Find the volume enclosed by the cylinder $x^2 + y^2 = 4$, bounded below by the plane $z = 0$ and above by $z = x^2 + y^2$ (Hint: what coordinates should you use?).

Soln: Use cylindrical coordinate system: Jacobian=r, $r^2 = 4, z = 0, z = r^2$ are the boundaries, hence, $0 \leq z \leq 4, 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{z}$.

$$\int_0^4 \int_0^{\sqrt{z}} \int_0^{2\pi} 1 \cdot rd\theta drdz = 2\pi \int_0^4 (\int_0^{\sqrt{z}} rdr)dz = 2\pi \int_0^4 \frac{z}{2} dz = \pi \times \frac{z^2}{2} |_0^4 = 8\pi \quad (1)$$

4. Set up the integral to calculate the volume enclosed by the surfaces $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 = z$. Evaluate for extra credit.

Soln:Use spherical system to change variables. Hence,$\rho^2 = 2, \rho^2 sin^2\Phi = \rho cos\Phi$ is the boundary condition, which happens when $\Phi = \frac{\pi}{4}$. Therefore,

$$\int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} 1 \cdot \rho^2 sin\Phi d\Phi d\theta d\rho$$

$$= \int_0^{\sqrt{2}} \rho^2 \cdot 2\pi \cdot (-cos\Phi) \mid_0^{\frac{\pi}{4}} d\rho$$

$$= 2\pi \int_0^{\sqrt{2}} \rho^2 (1 - \frac{\sqrt{2}}{2}) d\rho = \frac{4\sqrt{2} - 4}{3}\pi$$

5.A tank is in the shape of a half-cylinder of radius 2 and height 3. It is situated in $R^3$ by the following inequalities:
$\sqrt{x^2 + y^2} \leq 2$,$y \geq 0$,$0 \leq z \leq 3$
The temperature at a point inside the tank is given by $T(x, y, z) = 2yz^2\sqrt{x^2 + y^2}$.

(a) What is volume of the tank? You can do this in multiple ways. $6\pi$
(b) What is $\iiint_S T(x, y, z)dxdydz$ where S is the tank? (Hint: change to cylindrical coordinates).
Soln:Use cylindrical coordinates,

$$\int_0^2 \int_0^3 \int_0^\pi 2r^3 sin\theta z^2 d\theta dz dr = 4 \int_0^2 \int_0^3 r^3 z^2 dz dr = \frac{4}{3} \cdot 27 \int_0^2 r^3 dr = \frac{27}{3}(r^4)|_0^2 = 144$$

(c) Calculate the average temperature inside the tank, i.e. $\frac{answer(b)}{answer(a)}$
Soln: $\frac{144}{6\pi}$

6.Evaluate

$$\iiint_W \frac{z^2}{x^2 + y^2 + z^2} dxdydz$$

where W is the solid shell bounded by the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$

Soln: Use spherical coordinates, then W is described by $1 \leq \rho \leq 2$, as usual,we have the boundaries, $0 \leq \phi \leq \pi$,$0 \leq \theta \leq 2\pi$, with Jacobian $=\rho^2 sin\Phi$
Hence, the integral is ,

$$\iiint_W \frac{z^2}{x^2 + y^2 + z^2} dxdydz = \int_0^\pi \int_0^{2\pi} \int_1^2 \frac{\rho^2 cos^2(\Phi)}{\rho^2} \cdot \rho^2 sin\Phi d\rho d\theta d\Phi$$

$$= 2\pi \cdot \int_1^2 \rho^2 d\rho \cdot \int_0^\pi cos^2(\Phi) sin(\Phi) d\Phi = 2\pi \cdot \frac{7}{3} \cdot \frac{t^3}{3} \mid_{-1}^1 = \frac{28}{9}\pi$$

7.Consider the region D bounded between the circle $x^2+y^2=1$ and $x^2+y^2=4$, $x \geq 0, y \geq 0$.
(a) Write the double integral of a general function f(x, y) over the region D as an iterated integral over x and y.
Soln: $\int_0^2 \int_{\sqrt{1-x^2}}^{\sqrt{1-4x^2}} f(x,y)dydx$
(b) Write an equivalent integral after changing to polar coordinates
Soln: $\int_0^{\frac{\pi}{2}} \int_1^2 f(x(r,\theta), y(r,\theta)) \cdot rdrd\theta$
(c) Calculate

$$\iint_D \frac{sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} dA$$

Soln: Using polar coordinates:

$$\iint_D \frac{sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} dA = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{sin(\pi r)}{r} \cdot rdrd\theta = \int_0^{\frac{\pi}{2}} \int_1^2 sin(\pi r)drd\theta$$

$$= \frac{\pi}{2} \int_0^2 sin(\pi r)dr = -\frac{1}{2}(cos(\pi r)|_1^2 = -1$$

8(**bonus**)Find the volume of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

(Hint: Change coordinates twice, first by a suitable scaling of each variable)
Soln: Since a,b,c are lengths of semiaxes of the ellipsoid,then use a rescaling of the variables, such that, $x = au, y = bv, z = cw$. Then we can compute the relative Jacobian as: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$.
Hence, we get $\iiint_v dxdydz = \iiint_{v*} abcdudvdw$, where V*:

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \leq 1 \Longrightarrow u^2 + v^2 + w^2 \leq 1 \qquad (2)$$

Since this is just the standard defining equation of a sphere in uvw-space, convert to spherical coordinates to solve the original integral, we have:
$\frac{\partial(u,v,w)}{\partial(\rho,\Phi,\theta)} = \rho^2 sin\Phi$
Then, $\iiint_{v*} abc\ dudvdw = \iiint_{v**} \rho^2 sin\Phi \cdot abc \cdot d\rho d\Phi d\theta$
$= abc\times$ volume of the sphere $= abc \cdot \frac{4\pi}{3}$