RESEARCH STATEMENT

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1. Overview

My research interests lie in computational arithmetic geometry. I am especially interested in the study of solutions to Diophantine equations or equivalently, the study of rational points on projective algebraic varieties, over finite fields of small cardinality. Much of my work so far has focused on del Pezzo surfaces of degree 1 and 2 and cubic surfaces. In my thesis, I have characterized del Pezzo surfaces of degree 1 with a unique rational point(§3.1), and del Pezzo surfaces of degree 2 with a unique rational point(§3.2). For cubic surfaces, I observe that the vanishing of all H^1 groups govern the birational triviality. But there are counterexamples of del Pezzo surface of degree 1 and 2 where this criterion fails. Also I show that there is no minimal dP1 with all vanishing H^1's(§3.3).

These results lead to several other problems in a natural way. Using the explicit methods I have developed, I intend to give a list of del Pezzo surfaces of degree 1 and degree 2 with exactly $2q + 1$ rational points over finite field $\mathbb{F}_q$, which naturally improves the Weil bound(§4.1). More generally, I plan to consider additional section of the universal del Pezzo surfaces of degree 1 over the moduli space(§4.2). On the geometric side, I plan to finish the realization problem that is related to our counterexamples of del Pezzo surfaces of degree 1 and 2 to our previous observations(§4.3). I also intend to prove the existence of rational curves passing through rational points on cubic surfaces over finite fields with fewer than 8 elements(§4.4).

2. Introduction

Let F be a homogeneous polynomial over $\mathbb{F}_q$ of degree d in $n > d$ variables; the Chevalley-Warning theorem says F has a nontrivial zero. In geometric terms, the set of $\mathbb{F}_q$-rational points $X(\mathbb{F}_q)$ of the hypersurface $X = \{F = 0\}$ has positive cardinality (in fact, $|X(\mathbb{F}_q)| \equiv 1 \mod q$).

In my thesis, I characterize geometrically rational surfaces of low degree for which this result is sharp, i.e. $|X(\mathbb{F}_q)| = 1$. For example, Swinnerton-Dyer and Corti-Kollár-Smith [KSC04] have shown that there is a unique smooth cubic surface in $\mathbb{P}^3$ with $|X(\mathbb{F}_q)| = 1$ and $q = 2$. I generalize this to del Pezzo surfaces of smaller degree.

A **del Pezzo surface** is a smooth projective surface X over a field k for which the anticanonical class $-K_X$ is ample; the **degree** of X is the integer $d = K_X^2$. Examples include $X = \mathbb{P}^2$ and cubic surface, which is a hypersurface of degree 3 in $\mathbb{P}^3$. Geometrically, a degree d del Pezzo surface is obtained by blowing up $9 - d$ points in general positions in $\mathbb{P}^2$. So over $\bar{k}$, all del Pezzo surfaces admit a birational map to $\mathbb{P}^2$.

From the arithmetic and geometric points of view, the high degree del Pezzo surfaces are simpler than the low degree ones. For instance, when $\deg X = 5$ the surface X is always

$\mathbb{F}_q$-rational. Kollár showed that cubic surfaces over finite fields are always unirational. In the case $\deg X \leq 2$, no equivalent general theorems are known. In my thesis, I give a complete list of del Pezzo surfaces of degree 1 and 2 with few rational points over corresponding finite fields. We expect proving unirationality for these surfaces should be especially challenging.

3. Current work

3.1. **del Pezzo surfaces of degree 1 with few rational points.** Let $k = \mathbb{F}_q$ be a finite field of characteristic p and let $k[x, y, z, w]$ be the weighted graded ring where the variables x, y, z, w have weights 1, 1, 2, 3, respectively. Set $\mathbb{P}(1, 1, 2, 3) := \text{Proj } k[x, y, z, w]$. Every del Pezzo surface of degree 1(dP1) over k is isomorphic to a smooth sextic in $\mathbb{P}(1, 1, 2, 3)$, and conversely [Kol96]. Thus, a dP1 is given by an equation of the form

$$w^2 + H(x, y)w + z^3 + Q(x, y)z^2 + G(x, y)z + F(x, y) = 0$$

where G and F are binary homogeneous forms of degrees 4 and 6, respectively. Over fields of char $k \neq 2$ or 3, we may assume that $H(x, y) = Q(x, y) = 0$.

Every dP1 comes endowed with a $\mathbb{F}_q$-rational point–the base point of the anticanonical linear system. A natural question arises, "For what q and what del Pezzo surfaces of degree 1 X, is it true that $|X(\mathbb{F}_q)| = 1$"?

Blowing up the base point of $|-K_X|$, we obtain an elliptic fibration map $\rho : X \dashrightarrow \mathbb{P}^1$. The surface X has a unique $\mathbb{F}_q$ rational point if and only if every fiber of ρ contains exactly one $\mathbb{F}_q$ point. Using the notation above, the fiber above $[m : n] \in \mathbb{P}^1$ is isomorphic to the elliptic curve $w^2 + z^3 + G(t, 1)z + F(t, 1) = 0$ where $t = \frac{m}{n}$.

Hasse showed that any elliptic curve $E/\mathbb{F}_q$ satisfies

$$|E(\mathbb{F}_q)| \geq q + 1 - 2\sqrt{q}$$

Hence, when $q \geq 5$, we have at least two $\mathbb{F}_q$-points on each smooth fiber. If the fiber is singular, it must be a cuspidal or a nodal curve, and thus isomorphic to a compacification of $\mathbb{G}_a$ and $\mathbb{G}_m$. Consequently, each singular fiber has $q + 1$ rational points. In all cases, X will contain more than 1 points on fibers, hence on the surface. Therefore, $|X(\mathbb{F}_q)| = 1$ could only happen over fields $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_4$.

A systematic, computer-aided search then allows us to determine all dP1s with a unique $\mathbb{F}_q$-point. Our main result is:

Theorem 3.1. *Let X be a del Pezzo surface of degree 1 defined over $\mathbb{F}_q$. We have:*

- *Over $\mathbb{F}_2$, there is a unique del Pezzo surfaces of degree 1 up to projective equivalence with a unique rational point, and its defining equation is:*

$$w^2 + z^3 + w(x^3 + x^2y + y^3) + z(x^4 + x^3y + y^4) + (x^6 + xy^5 + y^6) = 0$$

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- *Over $\mathbb{F}_3$, there is a unique del Pezzo surfaces of degree 1 up to projective equivalence with a unique rational point, and its defining equation is:*

$$w^2 + z^3 + z(2x^4 + x^2y^2 + 2y^4) + x^6 + 2x^2y^4 + y^6 = 0$$

  .

- *Over $\mathbb{F}_q$, where $q \geq 4$, del Pezzo surfaces of degree 1 has at least $2q + 1$ rational points. Furthermore, when $q > 7$, del Pezzo surfaces of degree 1 has at least $3q + 1$ rational points.*

3.2. **del Pezzo surfaces of degree 2 with few rational points.** A del Pezzo surface of degree 2(dP2) can be realized as a surface in weighted projective space $\mathbb{P}(1,1,1,2)$ given by an equation of the form

$$w^2 + Q(x,y,z)w + G_4(x,y,z) = 0$$

where $Q(x,y,z)$ and $G_4(x,y,z)$ are homogeneous polynomials of degree 2 and 4, respectively. If char $k \neq 2$, we could assume that $Q(x,y,z) = 0$, and a dP2 is a double cover of $\mathbb{P}^2$ ramified over the (smooth) quartic curve $G_4(x,y,z) = 0$.

Let k be a finite field of q elements. Let X be a smooth projective surface defined over k. Let Γ denote the Galois group $\mathrm{Gal}(\bar{k}/k)$ and $F \in \Gamma$ denote the Frobenius k-endomorphism of $\bar{k}$ given by

$$F : z \longrightarrow z^q$$

Since the exceptional curves on $\bar{X} = X \otimes_k \bar{k}$ are defined over $\bar{k}$ and they generate $N(X)$ =Pic$(X \otimes \bar{k})$, F induces a permutation F^* acting on the lines. The number of $\mathbb{F}_q$-points on the surface is given by $|X(\mathbb{F}_q)| = q^2 + qTrF^* + 1$, where TrF^* denotes the trace of F in the representation of $\mathrm{Gal}(\bar{k}/k)$ on the Picard group $N(X)$.

Therefore, a necessary condition for the existence of a dP2 with a unique $\mathbb{F}_q$-rational point is the existence of a conjugacy class $\sigma \in W(E_7)$ (Weyl group) with $Tr\sigma = -q$ (here $\sigma = F^*$).

Applying the Urabe's table [Ura96] for del Pezzo surfaces of degree 2, we get that a unique $\mathbb{F}_q$-rational point situation could only happen over the fields $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_4$. Using the fact that X is a double cover of $\mathbb{P}^2$ ramified along a smooth quartic curve, we can show that if $|X(\mathbb{F}_q)| = 1$ then we must have that $q = 2$ or 3.

A computer-aided search allows us to give all dP2s with unique rational point. Our result is:

Theorem 3.2. *Let X be a del Pezzo surface of degree 2 defined over $\mathbb{F}_q$. We have:*

- *Over $\mathbb{F}_2$, there is a unique del Pezzo surfaces of degree 2 up to projective equivalence with a unique rational point, and its defining equation is:*

$$w^2 + w(x^2 + xy + y^2) + (x^4 + x^2y^2 + x^2yz + xyz^2 + y^4) = 0$$

- *Over $\mathbb{F}_3$, there is a unique del Pezzo surfaces of degree 2 up to projective equivalence-with a unique rational point, and its defining equation is:*

$$w^2 + x^4 + 2x^3z + 2x^2y^2 + xy^2z + y^4 + y^2z^2 + z^4 = 0$$

- *Over $\mathbb{F}_q$, where $q \geq 4$, del Pezzo surfaces of degree 2 has at least $2q+1$ rational points.*

3.3. **A Criterion of birational triviality of cubic surfaces.** Let X be a nonsingular cubic surface over a finite field k; it contains precisely 27 lines defined over $\bar{k}$. The classes of the lines generate the group $N(X)$ and the action of the Galois group Γ =Gal$(\bar{k}/k)$ on $N(X)$ preserves incidence relations. Results in Manin's book [Man86] shows that the representation of Γ in $N(X)$ is described by a pair consisting of a finite extension of k and the image of Γ in the group Aut$(N(X))$; this image corresponds to a conjugacy class σ in the Weyl group $W(E_6)$. The 25 classes $C_1, ..., C_{25}$ in $W(E_6)$ have been enumerated by Frame [Fra51], and representative elements of each conjugacy class are given by H.P.F Swinnerton-Dyer [SD67].

Since a conjugacy class $\sigma \in W(E_6)$ acting on $N(X)$ is realized as an automorphism of a cyclic extension K/k, we may consider the corresponding cyclic subgroup G in $W(E_6)$ and the value of the cohomology group $H^1(G, N(X))$. Then our main result is as follows.

Observation 3.3. *Let X be a nonsingular cubic surfaces defined over a finite field k. Then X is birationally trivially iff $H^1(\langle\sigma^m\rangle, N(X)) = 0$ for each m.*

The proof uses a concrete calculation of all the powers of all 25 conjugate classes $C_1, ..., C_{25}$, and the corresponding H^1, which gives explicit information of the number of skew lines and their configurations. These allow us to show that cubic surfaces corresponding to 9 classes $C_3, C_4, C_{10}, C_{11}, C_{19}, C_{20}, C_{12}, C_{13}, C_{14}$ are not birationally trivial, and all the other 16 classes are rational.

When we move on to lower degree del Pezzo surfaces, this is not true as we have examples of minimal dP2 and dP1 with all of their first cohomology groups to all the powers vanish. But an interesting thing is there are no minimal dP1's with all H^1 vanishing. Maps between minimal del Pezzo surfaces are studied by Iskovskih [Isk79].

Example 3.4. *(Counterexample of minimal del Pezzo surface of degree 2 to our criterion)*
Let V be the del Pezzo surface of degree 2 represented by Carter symbol as $(4A_1)'$, which is the first conjugacy class enumerated by Urabe [Ura96]. *V is a minimal dP2, so V is irrational. But the H^1 groups of all powers of its conjugacy class in the Weyl group $W(E_7)$ vanish.*

Example 3.5. *(Counterexample of a non-minimal del Pezzo surface of degree 1 to our criterion) Let V be the minimal dP2 given in Example 3.4, and V' be the blow up of V. Thus V' is a non-minimal del Pezzo surface of degree 1 with all H^1 groups vanish, since H^1 is a birational invariant. And V' is irrational since V is. Therefore, V' is a counterexample of dP1 to our criterion.*

Observation 3.6. *There is no minimal del Pezzo surface of degree 1 with all H^1 groups vanishing.*

These examples show that the rationality of cubic surface is totally determined by H^1 groups, but the rationality of dP1's and dP2's are not. We also expect that proving rationality of dP1 and dP2 should be relatively hard.

4. **Future Work**

4.1. **Improve Weil's bound of $|X(\mathbb{F}_q)|$ where $q \geq 4$.** In my thesis, I give a list of del Pezzo surfaces of degree 1 and 2 with a unique rational point over some finite field $\mathbb{F}_q$. Excluding these q's, the Weil estimate [Wei56]of $|X(\mathbb{F}_q)|$ only gives $1 + q$ as a lower bound.

I have showed that for $q \geq 5$, they always have at least $2q + 1$ rational points. I intend to keep improving on this lower bound of $|X(\mathbb{F}_q)|$, and I conjecture that the lower bound should be $3q+1$ when $q \geq 5$. As a first step, I am working on the special equations of explicit fibers when all of the fibers with exactly $2q + 1$ points. This could only happen in two cases:

- When one and only one of the fibers is singular and all the others are smooth but defined over some field extension of $\mathbb{F}_q$.
- All fibers are smooth with one $\mathbb{F}_q$ point besides the origin.

Do such del Pezzo surfaces of degree 1 and 2 with exactly $2q+1$ rational points exist? Using the strategy mentioned above I could also determine the dP1's and dP2's with $|X(\mathbb{F}_q)| = 3q + 1$. Can they be enumerated up to k-isomorphism? These could be a very good source of examples to a variety of arithmetic problems.

Furthermore, for some small $n \equiv 1 (\mathrm{mod}\ q)$ that satisfies Weil's bound, I hope to construct examples of del Pezzo surfaces with $|X(\mathbb{F}_q)| = n$. I intend to do this by generalizing the method of constructing proper fibers with a given number of rational points.

4.2. **Can $|X_k| = 1$ when k is not a finite field?** We have given the list of dP1's with few rational points over the finite field $\mathbb{F}_q$. But over a more general field k, there are results showing that we often have more rational points.

Let X be a del Pezzo surface of degree 1. Let $k = \mathbb{C}(\mathfrak{B})$, where $\mathcal{B}$ is a curve. A theorem of Kollar, Miyaoka, and Mori [KMM92] shows that if X/k is rationally connected and $X(k) \neq \emptyset$, which holds by a result of Graber Harris and Starr [GHMS05], then $X(k)$ is dense. Could we generalize the argument of the existence and the density of rational points when the base field is a function field in two variables t_1, t_2 over the complex number? In other words, when consider the universal del Pezzo surface of degree 1 over the moduli space, we have a section which is a dP1 fibration. When do we have another section over Spec $\mathbb{C}(t_1, t_2)$?

4.3. **Realizations of the special dP1's and dP2 in our counterexamples.** I have given some counterexamples in dP1's and minimal dP2 to the criterion of birational triviality. As the minimal dP2 surface V in the counterexample is represented by its Frame Symbol, how to realize it in explicit equations? Furthermore, for the counterexample of dP1 surface V', we only know it exists. How to compute its defining equation and orbit decompostion?

4.4. **Finding Rational curves on cubic surfaces.** Swinnerton-Dyer has already proved that only one degree 3 del Pezzo surface has a unique rational point. So instead of looking for rational points, we can look for rational curves. A classical question could be asked: which cubic surfaces contain many rational curves?

Let X $\in \mathbb{P}^{n+1}$ be a smooth cubic hypersurface over $\mathbb{F}_q$. Assume that $n \geq 2$ and $q \geq 8$. János Kollár[Kol08] showed that for cubic hypersurfaces, that every map of set $\phi : \mathbb{P}^1(\mathbb{F}_q) \to X(\mathbb{F}_q)$ can be extended to a map of $\mathbb{F}_q$-varieties $\Phi : \mathbb{P}^1 \to X$.

How about the case where $q \leq 7$? We know for sure that $|X(\mathbb{F}_q) \geq 1 + q$ when $q \neq 2$, but are there any rational curves passing through these rational points? How many rational curves are there up to projective equivalence?

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