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STABILIZED FINITE ELEMENT METHODS OF GLS TYPE FOR MAXWELL-B AND OLDROYD-B VISCOELASTIC FLUIDS

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Abstract. *We evaluate the stabilized three-field stress-velocity-pressure Galerkin/Least-Squares finite element formulation for viscoelastic fluids, using a benchmark problem of Oldroyd-B flow past a cylinder at various Weissenberg numbers. To address the issue of weak consistency exhibited by low-order velocity interpolations in the context of stabilized formulations, we also employ velocity gradient recovery, and study how such an approach affects the quality of computed results. We show that characteristic flow quantities obtained with the new formulation are in good agreement standard DAVSS and DEVSS results, while the cost of fully-implicit velocity gradient computation may be in some cases avoided.*

1 INTRODUCTION

Numerical modeling of viscoelastic fluids of rate type is fraught with difficult issues. The viscoelastic stress, or the conformation tensor, must be represented as an additional flow unknown. The advective nature of the constitutive equations, and the interaction of multiple discrete unknown fields (viscoelastic stress, velocity and pressure) both present obstacles to numerical method development. Yet, as has been the case with Newtonian fluid modeling before, these obstacles are being gradually overcome. In the finite element arena, on which we concentrate our attention, adding the Streamline-Upwind/Petrov-Galerkin (SUPG) terms [1, 2] to the traditional Galerkin formulation was instrumental in overcoming the difficulties associated with the advective terms present in the constitutive equation. The Discontinuous-Galerkin (DG) [3] approach provided similar benefits, albeit with a marked increase in the complexity of the implementation.

Dealing with instabilities related to the interplay of discrete interpolations of the viscoelastic stress and the velocity fields has been even more challenging. Compatibility conditions on these interpolations have been since formulated, and appear to be well understood [4–6]. These requirements are analogous to the LBB condition [7] for mixed formulations of incompressible flow. In order to satisfy such conditions, complex combinations of interpolation functions were needed, such as the 4×4 stress sub-element [2] complementing quadratic velocity interpolation.

In time, alternative methods were developed which admitted simpler, more efficient, and easier to implement, equal-order interpolations of the viscoelastic stress and velocity. Recent reviews by Baaijens [8] and Keunings [9, 10] outline the development of the Elastic Viscous Stress Splitting (EVSS [11]) approaches, in which the original Galerkin, SUPG, or DG formulation is modified and new variables or terms are introduced. The state-of-the-art methods are the so-called Discrete EVSS [12, 13] methods (DEVSS-G/SUPG [14, 15] and DEVSS-G/DG [16]), which include the following features:

- the velocity gradient is an independent variable approximated with continuous interpolation functions;
- the viscous stress is split into two contributions, associated with the continuous velocity gradient and with the discontinuous gradient of the velocity field;
- the constitutive equation of the viscoelastic stress (or the conformation) is transformed into weak form with the streamline-upwind (SUPG) or discontinuous Galerkin (DG) method.

Baaijens [8] points out that even in the relatively simple two-dimensional flow in an abrupt contraction, mesh-converged results cannot be obtained downstream of corner (location of the singularity) with the DEVSS-G/SUPG and DEVSS-G/DG methods.

The common characteristics of all the EVSS-based methods is the addition to the weak form of the momentum equation of an elliptic term which represents the difference between the continuous and discontinuous velocity gradient, and vanishes with increasing mesh resolution. A similar stabilizing term arises naturally when the Petrov-Galerkin approach is replaced with a Least-Squares (LS) method, as was done by Carey's group [17, 18]. Yet, the LS approach carries the usual penalty of high condition numbers of the resulting equation system matrices, and stringent continuity requirements for the interpolation functions (although the orders of interpolation are not higher than those required by the EVSS, DEVSS, and DEVSS-G methods). Moreover, the divergence of the stress is not integrated by parts in the LS form of the momentum equation; thus, the traction at the boundary does not appear naturally in the weak form and imposing boundary conditions at free surfaces and deformable boundaries is difficult.

One approach that has been quite successful in circumventing the LBB condition in the case of the Navier-Stokes equations has been the Galerkin/Least-Squares (GLS) method [19]. Here, the stabilizing least-squares form of the governing equations is added to the Galerkin weak form, in the element interiors. The resulting formulation recovers SUPG terms, and also includes pressure-stabilizing terms that alleviate the need to satisfy the LBB condition. Moreover, the traction term is present in the GLS formulation, which allows imposing free-surface boundary conditions naturally. This particular approach has been used by Behr and Franca [20] to design and analyze a Galerkin/Least-Squares variational formulation of Navier-Stokes equations involving viscous stress, velocity and pressure as the primary variables, without the usual restrictions on the interpolation function spaces. The GLS approach was extended by Behr [21] to the Maxwell-B and Oldroyd-B constitutive models—hereafter referred to as three-field GLS formulation, or GLS3. The method showed a number of desirable characteristics:

- SUPG stabilization terms in the constitutive equation were obtained immediately from the GLS terms;
- equal-order interpolations for all flow field variables (viscoelastic stress, velocity and pressure) were admissible;
- hence, the implementation was straightforward and the computational cost was modest.

The formulation has been then used to compute a benchmark problem of flow in a contraction. The present work represents an effort to resume and complete the evaluation of the GLS3 approach, and to improve upon it by developing several variants of the method.

We will begin by introducing the equations of motion for the Oldroyd-B fluid and its Maxwell-B limit case in Section 2. In Section 3, we present the three-field stress-velocity-pressure Galerkin/Least-Squares finite element formulation. We comment on the weak consistency of this formulation when used with low-order interpolation functions, and introduce two variants that improve the consistency of the method in such situations. In Section 4, we present an example of flow past a cylinder placed in a slit, and compare obtained drag coefficient at various Weissenberg numbers with published results.

2 PROBLEM STATEMENT

We consider a viscous, incompressible fluid occupying at an instant $t \in (0, T)$ a bounded region $\Omega_t \subset \mathbb{R}^{n_{\mathrm{sd}}}$, with boundary Γ_t, where n_{sd} is the number of space dimensions. The velocity and pressure, $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$, are governed by the momentum and mass balance equations:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \mathbf{0} \qquad \text{on } \Omega_t \quad \forall t \in (0, T), \tag{1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad \text{on } \Omega_t \quad \forall t \in (0, T), \tag{2}$$

where ρ is the fluid density, assumed to be constant, and $\mathbf{f}(\mathbf{x}, t)$ is an external, e.g., gravitational, force field. The closure is obtained with a constitutive equation relating the stress tensor $\boldsymbol{\sigma}$ to velocity and pressure fields. Both the Dirichlet and Neumann-type boundary conditions are taken into account, represented as:

$$\mathbf{u} = \boldsymbol{g} \qquad \text{on } (\Gamma_t)_{\boldsymbol{g}}, \tag{3}$$

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \boldsymbol{h} \qquad \text{on } (\Gamma_t)_{\boldsymbol{h}}, \tag{4}$$

where $(\Gamma_t)_{\boldsymbol{g}}$ and $(\Gamma_t)_{\boldsymbol{h}}$ are complementary subsets of the boundary Γ_t. The vector subscripts signify that this decomposition of Γ_t may be different for each of the velocity components. The initial condition consists of a divergence-free velocity field specified over the entire domain:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \boldsymbol{\nabla}\cdot\mathbf{u}_0 = 0 \quad \text{on } \Omega_0. \tag{5}$$

Viscoelastic fluids exhibit dependence of the stress not only on the instantaneous rate of strain, but also on the strain history. For the upper-convected Maxwell fluid, also called the Maxwell-B fluid, the constitutive equation acquires an evolutionary character:

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{T}, \\ \mathbf{T} + \lambda\overset{\triangledown}{\mathbf{T}} &= 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \end{aligned} \tag{6}$$

where the $\overset{\triangledown}{\mathbf{T}}$ denotes an upper-convected derivative:

$$\overset{\triangledown}{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{T} - \left(\boldsymbol{\nabla}\mathbf{u}\,\mathbf{T} + \mathbf{T}\,(\boldsymbol{\nabla}\mathbf{u})^T\right), \tag{7}$$

and the rate-of-strain tensor is defined as:

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right). \tag{8}$$

Note that, in the case of steady flows considered in the remainder of this article, the time derivative in (7) is dropped, and domain Ω_t is replaced simply by a constant region Ω.

The upper-convected Oldroyd model, also known as the Oldroyd-B model, includes the Newtonian and Maxwell models, and covers the cases in which an elastic fluid obeying the Maxwell relation is mixed with a fluid governed by a Newtonian law. This corresponds to a situation in which an elastic polymer with viscosity μ_1 is dissolved in a viscous solvent with viscosity μ_2:

$$
\begin{aligned}
\boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{T}, \\
\mathbf{T} &= \mathbf{T}_1 + \mathbf{T}_2, \\
\mathbf{T}_1 + \lambda \overset{\triangledown}{\mathbf{T}}_1 &= 2\mu_1 \boldsymbol{\varepsilon}(\mathbf{u}), \\
\mathbf{T}_2 &= 2\mu_2 \boldsymbol{\varepsilon}(\mathbf{u}), \\
\mu_1 + \mu_2 &= \mu.
\end{aligned} \tag{9}
$$

The Maxwell fluid is extremely difficult to handle numerically, partially because of the convective character of the stress evolution equation (6). The difficulty is lessened considerably by even a small addition of the Newtonian solvent in an Oldroyd-B fluid. Even so, one may expect the problems normally associated with an advective systems to arise when the relevant constitutive equation is discretized.

3 THREE-FIELD GLS FORMULATION

Due to the non-trivial nature of the constitutive equation, the components of the extra stress have to be treated as additional degrees of freedom, complementing the velocity and pressure fields. A mixed formulation is naturally extended to accommodate the added equation and unknown. However, an arbitrary choice of the stress interpolation often leads to failure, as documented in [2]; possible remedies have been discussed in Section 1.

Also, the values of the Weissenberg number for which a Galerkin formulation would remain convergent are extremely small, as seen, e.g., in [22]. The remedy follows the path of the various upwinding methods developed for the advection-diffusion equation, or its advective limit. In [2] both the consistent SUPG and an inconsistent Streamline Upwind (SU) methods are considered. The SU approach is found to stabilize the Galerkin method sufficiently, but the various coupling effects render the potentially more accurate SUPG method ineffective. Similar conclusion is reached in [3], where a consistent upwinding inherent in the Lesaint-Raviart method has to be augmented by an inconsistent SU addition.

We recall here the stress-velocity-pressure formulation introduced in [20] for Stokes equations, and generalized to Maxwell-B and Oldroyd-B flows in [21]. The $n_{\mathrm{tc}} = n_{\mathrm{sd}}(n_{\mathrm{sd}}+1)/2$ independent tensor components of the extra stress $\mathbf{T}_1$ are necessarily treated as additional unknowns, and equation (9) enters the variational formulation directly. We drop the subscript from $\mathbf{T}_1$, as this is the only stress component entering the formulation explicitly. The interpolation function spaces for the velocity, pressure and extra stress tensor are given as:

$$\mathcal{S}^h_{\mathbf{u}} = \left\{ \mathbf{u}^h \mid \mathbf{u}^h \in \left[H^{1h}(\Omega)\right]^{n_{\rm sd}}, \mathbf{u}^h \doteq \boldsymbol{g}^h \quad \text{on } \Gamma_{\boldsymbol{g}} \right\}, \tag{10}$$

$$\mathcal{V}^h_{\mathbf{u}} = \left\{ \mathbf{u}^h \mid \mathbf{u}^h \in \left[H^{1h}(\Omega)\right]^{n_{\rm sd}}, \mathbf{u}^h \doteq \mathbf{0} \quad \text{on } \Gamma_{\boldsymbol{g}} \right\}, \tag{11}$$

$$\mathcal{S}^h_p = \mathcal{V}^h_p = \left\{ p^h \mid p^h \in H^{1h}(\Omega) \right\}, \tag{12}$$

$$\mathcal{S}^h_{\mathbf{T}} = \mathcal{V}^h_{\mathbf{T}} = \left\{ \mathbf{T}^h \mid \mathbf{T}^h \in \left[H^{1h}(\Omega)\right]^{n_{\rm tc}} \right\}. \tag{13}$$

In the $\lambda > 0$ case, the spaces $\mathcal{S}^h_{\mathbf{T}}$ and $\mathcal{V}^h_{\mathbf{T}}$ must also account for the essential boundary conditions for the extra stress at the inflow boundary of the domain.

The three-field Galerkin/Least-Squares, or GLS3, velocity-pressure-stress formulation for Oldroyd-B fluid is written as follows: find $\mathbf{u}^h \in \mathcal{S}^h_{\mathbf{u}}$, $p^h \in \mathcal{S}^h_p$ and $\mathbf{T}^h \in \mathcal{S}^h_{\mathbf{T}}$ such that:

$$\begin{aligned}
&\int_\Omega \mathbf{w}^h \cdot \rho \left(\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h - \mathbf{f}\right) d\Omega - \int_\Omega \boldsymbol{\nabla} \cdot \mathbf{w}^h p^h d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \mathbf{T}^h d\Omega \\
&+ 2\mu_2 \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^h) : \boldsymbol{\varepsilon}(\mathbf{u}^h) - \int_{\Gamma_{\boldsymbol{h}}} \mathbf{w}^h \cdot \boldsymbol{h}^h d\Gamma d\Omega + \int_\Omega q^h \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega \\
&+ \frac{1}{2\mu_1} \int_\Omega \mathbf{S}^h : \mathbf{T}^h d\Omega + \frac{\lambda}{2\mu_1} \int_\Omega \mathbf{S}^h : \overset{\triangledown}{\mathbf{T}}{}^h d\Omega - \int_\Omega \mathbf{S}^h : \boldsymbol{\varepsilon}(\mathbf{u}^h) d\Omega \\
&+ \sum_{e=1}^{n_{\rm el}} \int_{\Omega^e} \tau_{\rm MOM} \frac{1}{\rho} \left[\rho \left(\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{w}^h\right) + \boldsymbol{\nabla} q^h - \boldsymbol{\nabla} \cdot \mathbf{S}^h - 2\mu_2 \boldsymbol{\nabla} \cdot \boldsymbol{\varepsilon}(\mathbf{w}^h)\right] \\
&\qquad \cdot \left[\rho \left(\mathbf{u}^h \cdot \boldsymbol{\nabla}\mathbf{u}^h - \mathbf{f}\right) + \boldsymbol{\nabla} p^h - \boldsymbol{\nabla} \cdot \mathbf{T}^h - 2\mu_2 \boldsymbol{\nabla} \cdot \boldsymbol{\varepsilon}(\mathbf{u}^h)\right] d\Omega \\
&+ \sum_{e=1}^{n_{\rm el}} \int_{\Omega^e} \tau_{\rm CONS} 2\mu_1 \left[\frac{1}{2\mu_1} \mathbf{S}^h + \frac{\lambda}{2\mu_1} \overset{\triangledown}{\mathbf{S}}{}^h - \boldsymbol{\varepsilon}(\mathbf{w}^h)\right] \\
&\qquad : \left[\frac{1}{2\mu_1} \mathbf{T}^h + \frac{\lambda}{2\mu_1} \overset{\triangledown}{\mathbf{T}}{}^h - \boldsymbol{\varepsilon}(\mathbf{u}^h)\right] d\Omega \\
&+ \sum_{e=1}^{n_{\rm el}} \int_{\Omega^e} \tau_{\rm CONT} \boldsymbol{\nabla} \cdot \mathbf{w}^h \rho \boldsymbol{\nabla} \cdot \mathbf{u}^h d\Omega = 0, \quad \forall \mathbf{w}^h \in \mathcal{V}^h_{\mathbf{u}}, \quad \forall q^h \in \mathcal{V}^h_p, \quad \forall \mathbf{S}^h \in \mathcal{V}^h_{\mathbf{T}}.
\end{aligned} \tag{14}$$

The stabilization parameters $\tau_{\rm MOM}$ and $\tau_{\rm CONT}$ follow standard definitions given, e.g., in [23]. The parameter $\tau_{\rm CONS}$ is taken here as:

$$\tau_{\rm CONS} = \begin{cases} \max\left(1, \frac{h^e}{2\lambda|\mathbf{u}^h|_2}\right), & \lambda|\mathbf{u}^h|_2 > 1 \\ \max\left(1, h^e\right), & \lambda|\mathbf{u}^h|_2 \leq 1 \end{cases} \tag{15}$$

Remark 1

The addition of the least-squares form of the momentum equation, i.e., the $\tau_{\rm MOM}$-term in (14), stabilizes the method against well-known adverse effects of under-diffusivity of the

Galerkin discretization of the momentum equation at high element-level Peclet numbers, and a possible lack of compatibility between velocity and pressure function spaces.

Remark 2
The least-squares form of the continuity equation, i.e., the τ_{CONT}-term in (14), improves the convergence of non-linear solvers at high Reynolds numbers.

Remark 3
The least-squares form of the constitutive equation, i.e. the τ_{CONS}-term in (14), stabilizes the method against two further causes of spurious numerical oscillations: the under-diffusivity of the Galerkin discretization of the constitutive equation at high Weissenberg numbers, and a possible lack of compatibility between velocity and stress function spaces. As shown in [20], this term allows for arbitrary combinations of interpolation functions for velocity and the extra stress, which do not have to satisfy a specific LBB condition.

The combination of the stabilization terms circumventing the inf-sup conditions provides absolute freedom in selecting the interpolation function spaces, allowing in particular convenient equal-order interpolation for velocity, pressure and the stress. In the example that follows, a piecewise bi-linear interpolation is in fact used for all three fields (so called $Q1Q1Q1$ element).

Although simple to implement and computationally efficient, linear or bi-linear velocity interpolation suffers from one important drawback. In the presence of the Newtonian solvent (Oldroyd-B fluid), the second derivatives of velocity in the stabilization terms are not adequately represented—they are either zero in the linear case, or small and unrelated to actual second derivative of velocity field in the bi-linear case. The GLS3 formulation, normally considered to be consistent, is only *weakly consistent* in such cases as pointed out in [24]. Solutions to this problem in general involve obtaining a continuous velocity gradient, with well-defined derivatives inside element domain. To this end, one can employ either of two strategies: a simplified recovery of nodal velocity gradient using a least-squares approach, or introduction of a continuous velocity gradient field, resulting in a four-field velocity-pressure-stress-gradient stabilized formulation. The former approach is taken here; the latter approach leads to a GLS4 formulation which will be explored in a future article. The GLS4 bears similarities to DEVSS approach, although the extra velocity gradient field is used only for the proper representation of second derivatives of velocity in the case of Oldroyd-B fluid, i.e., to improve accuracy, whereas in DEVSS, such field is *necessary* for bypassing the compatibility condition on velocity and stress interpolations, i.e., to prevent a breakdown of discretization.

The least-squares recovery procedure solves, in a decoupled manner, the weak problem: given $\mathbf{u}^h$—piecewise linear or bi-linear computed velocity field—find $\mathbf{L}^h \in \mathcal{S}^h_{\mathbf{L}}$ such that:

$$\int_\Omega \mathbf{K}^h : \mathbf{L}^h d\Omega = \int_\Omega \mathbf{K}^h : \boldsymbol{\varepsilon}(\mathbf{u}^h) \quad \forall \mathbf{K}^h \in \mathcal{V}^h_{\mathbf{L}}, \tag{16}$$

where the function spaces:

$$\mathcal{S}_{\mathbf{L}}^h = \mathcal{V}_{\mathbf{L}}^h = \left\{ \mathbf{L}^h \mid \mathbf{L}^h \in \left[H^{1h}(\Omega) \right]^{n_{\rm sd} \times n_{\rm sd}} \right\}, \tag{17}$$

are by definition composed of *continuous* functions, with non-trivial derivatives in the element interior. The recovered values $\mathbf{L}^h$ are then used to form the residual of (14) at the next iteration of the non-linear iteration procedure, namely, in the computation of the $\tau_{\rm MOM}$-terms involving second derivatives of velocity. The mass matrix of the equation system originating from (16) can be either lumped, for increased computational efficiency, or solved with a direct or iterative solver for increased accuracy. Depending on this choice, we arrive at two GLS3 variants:

GLS3-L Base GLS3 method with recovery of continuous velocity gradient using lumped mass matrix,

GLS3-M Base GLS3 method with recovery of continuous velocity gradient using consistent mass matrix.

4 NUMERICAL EXAMPLE

Flow of Oldroyd-B fluid past a circular cylinder placed between parallel fixed plates is a standard benchmark problem for two-dimensional viscoelastic flow simulation. The results for the ratio of cylinder diameter to slit width of 1/8 have been reported by Sun et al. [16] using DAVSS-G/DG formulation, and by Pasquali [25] using DEVSS-G/SUPG formulation, and are reported here for the GLS3 formulation variants.

The flow domain is shown in Figure 1. The center of the cylinder is assumed to be located at $(0, 0)$, the radius of the cylinder is taken as $R = 1$, and the slit half-width as $h = 8$. The distance from the cylinder center to the inflow boundary $\Gamma_{\rm i}$ and the outflow boundary $\Gamma_{\rm o}$ is 10 and 20, respectively. A parabolic flow profile is prescribed on the inflow and outflow boundaries:

$$\begin{aligned} u_1 &= 1.5\,(Q/h)\,(1 - x_2^2/h^2), \\ u_2 &= 0, \\ T_{11} &= 2\,\lambda\,\mu_1\,\left(-1.5 x_2/h^2\right)^2, \\ T_{12} &= \mu_1\,\left(-1.5 x_2/h^2\right), \\ T_{22} &= 0. \end{aligned} \tag{18}$$

with flow rate per unit thickness $Q = 8$ reflecting the value used in [16, 25]. A no-slip condition was prescribed at the cylinder wall $\Gamma_{\rm c}$ and the slit wall $\Gamma_{\rm s}$. Finally, a symmetry condition was applied at the upstream and downstream symmetry lines $\Gamma_{\rm u}$ and $\Gamma_{\rm d}$.

The fluid and solvent viscosities are $\mu_1 = 0.41$ and $\mu_2 = 0.59$, respectively, giving the ratio of solvent viscosity to total viscosity of 0.59. The relaxation time λ is varied to arrive at the desired Weissenberg number:

$$\mathrm{Ws} = Q\lambda/hR. \tag{19}$$

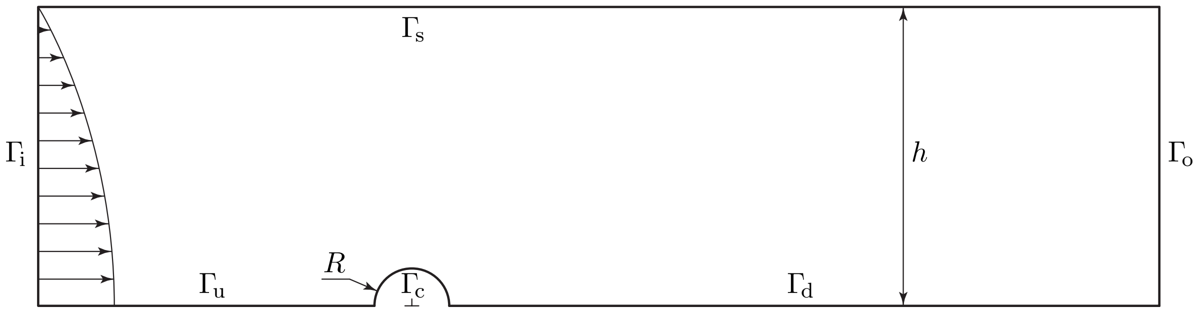

Figure 1: Flow past a circular cylinder: domain description.

A semi-structured mesh with 25,025 nodes and 24,619 bi-linear elements covers the domain, as shown in Figure 2. The size of the smallest elements, i.e., the ones adjacent to the cylinder, is approximately 0.010×0.018 in the radial and circumferential direction, respectively.

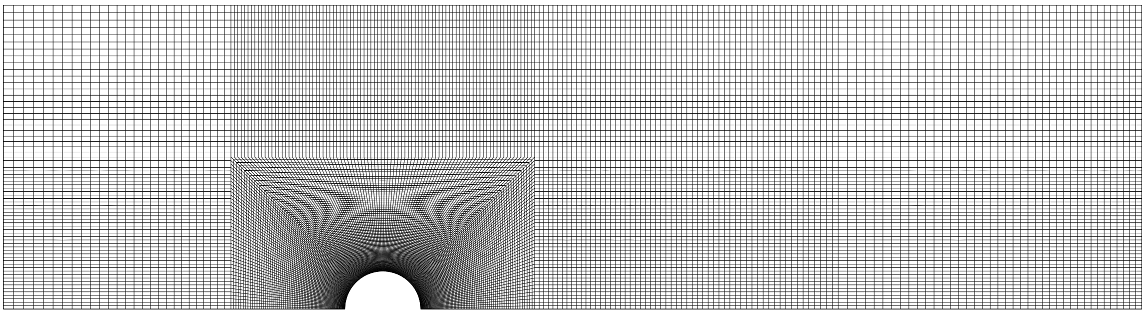

Figure 2: Flow past a circular cylinder: finite element mesh.

A characteristic quantity for this flow field is the drag force exerted by the fluid on the cylinder, and its variation with the Weissenberg number. The drag force is computed using the integral:

$$F_d = -2 \int_{\Gamma_c} \mathbf{e}_1 \cdot \boldsymbol{\sigma}^h \cdot \mathbf{n} d\Gamma, \tag{20}$$

where $\mathbf{e}_1$ is the horizontal unit vector and $\mathbf{n}$ is the unit normal at the cylinder surface pointing out of the flow domain. Note that this line integral includes contributions from the pressure p^h, viscoelastic stress $\mathbf{T}^h$, and the viscous stress $2\mu_2\boldsymbol{\varepsilon}(\mathbf{u}^h)$. The drag for Weissenberg numbers 0.0 to 2.0 is tabulated in Table 1 and shown in Figure 3. The agreement between GLS3-M and results of Sun et al. [16] is excellent up to Weissenberg number of 1, while GLS3-L under-predicts the drag by less than 1%. For Weissenberg numbers higher than 1, the GLS3-M and GLS3-L formulations depart further from Sun et al. [16] results, especially for Ws > 1.8, where the difference reaches 2.5%. Note that Ws $\simeq$ 2.0 is the limit of convergence for many methods when applied to this problem, including that of Sun et al. [16].

Ws	Sun et al.	GLS3-L	GLS3-M
0.0000	15.722	15.6720	15.7004
0.1000	15.705	15.6748	15.7062
0.2000	15.691	15.6648	15.6960
0.3000	15.682	15.6516	15.6822
0.4000	15.668	15.6384	15.6684
0.5000	15.662	15.6292	15.6584
0.6000	15.652	15.6270	15.6556
0.7000	15.665	15.6346	15.6624
0.8000	15.693	15.6536	15.6808
0.9000	15.728	15.6852	15.7122
1.0000	15.774	15.7306	15.7570
1.1000	15.843	15.7898	15.8160
1.2000	15.924	15.8626	15.8886
1.3000	16.007	15.9502	15.9764
1.4000	16.120	16.0492	16.0754
1.5000	16.235	16.1628	16.1892
1.6000	16.360	16.2854	16.3120
1.7000	16.530	16.4158	16.4428
1.8000	16.689	16.5472	16.5754
1.9000	16.884	16.6630	16.6910
2.0000	17.148	16.7608	16.7912

Table 1: Flow past a circular cylinder: drag as a function of Weissenberg number.

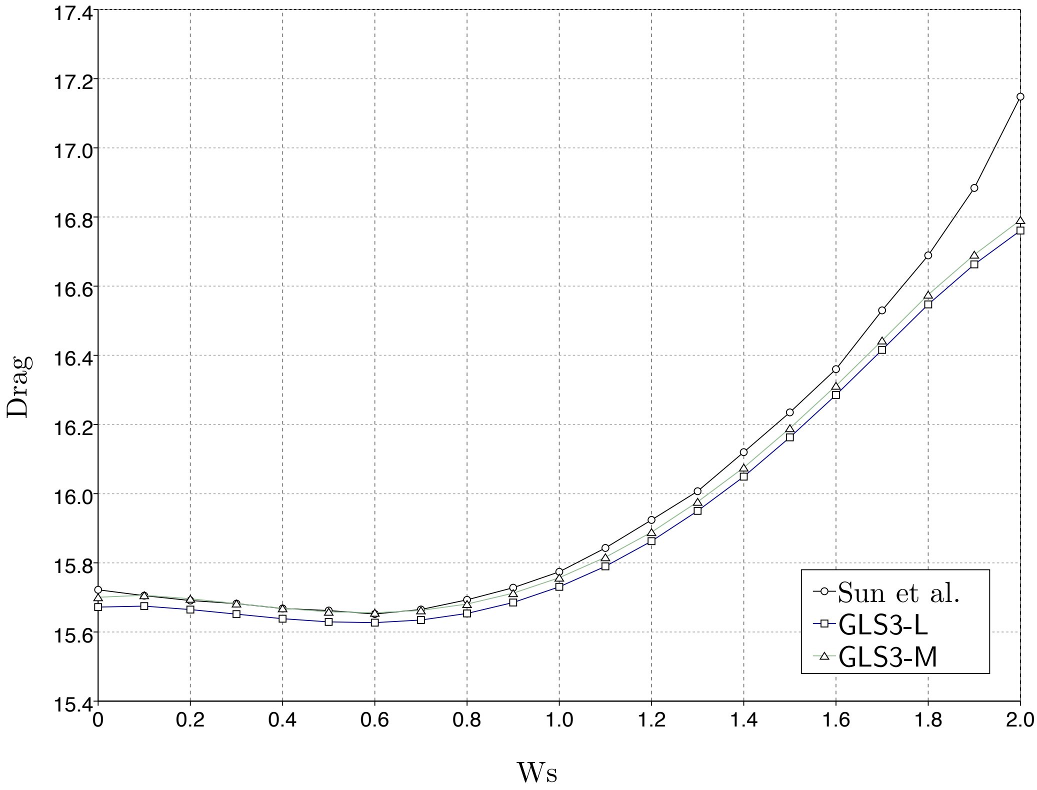

Figure 3: Flow past a circular cylinder: drag as a function of Weissenberg number.

Figures 4–6 show, for comparison with DEVSS-G/SUPG results of Pasquali [25], the three components of the viscoelastic stress $\mathbf{T}$ obtained with the GLS3-M formulation. Qualitatively, the agreement is excellent, and the contours are free of oscillations at this high Weissenberg number.

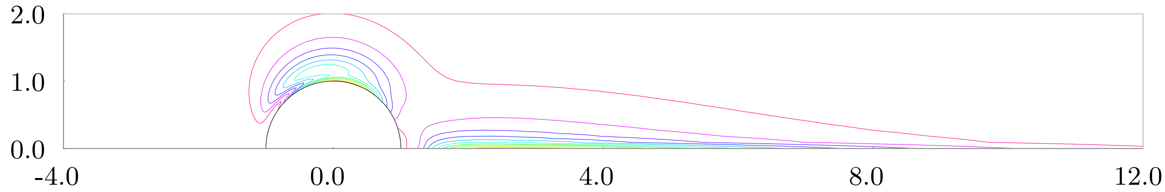

Figure 4: Flow past a circular cylinder: T_{11} contours in the vicinity of the cylinder at Ws = 2.0 obtained with GLS3-M.

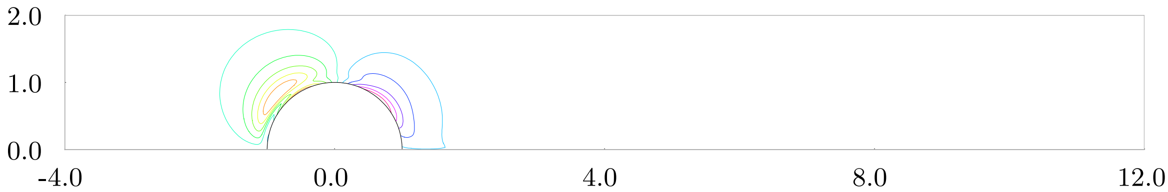

Figure 5: Flow past a circular cylinder: T_{12} contours in the vicinity of the cylinder at Ws = 2.0 obtained with GLS3-M.

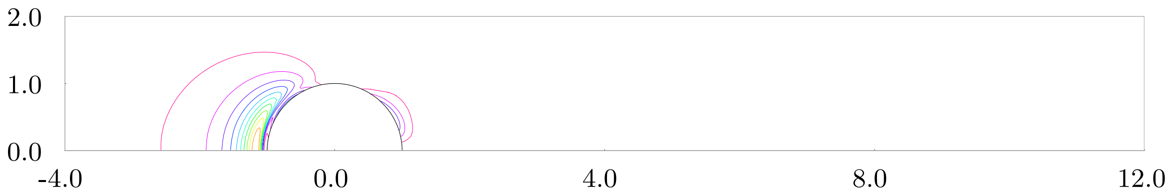

Figure 6: Flow past a circular cylinder: T_{22} contours in the vicinity of the cylinder at Ws = 2.0 obtained with GLS3-M.

5 CLOSING REMARKS

We have presented an evaluation of the stabilized three-field velocity-pressure-stress Galerkin/Least-Squares finite element formulation, designed for efficient and robust computation of flows of viscoelastic fluids. We have outlined the shortcomings of that formulation when used with low-order interpolation functions, and proposed appropriate remedies. The formulation considered allows equal-order interpolation for all three fields involved, and can benefit from, but does not strictly require, a separate interpolation for the velocity gradient, which is a necessary ingredient of contemporary approaches to viscoelastic flow simulation. Comparison with an established approach—DAVSS-G/DG—shows good agreement in measured drag force exerted on a cylinder by an Oldroyd-B fluid. The distribution of the viscoelastic stress also matches closely that reported for DEVSS-G/SUPG method.

A variant of the current method—a four-field GLS formulation—featuring a fully-implicit recovery of the velocity gradient is also under development. More comprehensive set of benchmark results, including quantitative measures of computational efficiency, will be a reported in a future article.

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