

## Conjectures on the braid index and the algebraic crossing number

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We discuss a conjecture of Jones on braid index and algebraic crossing number. We deform it to a stronger conjecture and show many evidences and some ways to approach the conjectures.

*Keywords:* Braid index; Algebraic crossing number; Morton Franks Williams inequality

### 1.

It has been conjectured (see [3] p.357) that the exponent sum in a minimal braid representation is a knot invariant.

To state a stronger form of this conjecture, we need fix some notations:

Let  $\mathcal{K}$  be an oriented topological knot type. Let  $\mathcal{B}_{\mathcal{K}}$  be the set of braid representatives of  $\mathcal{K}$ . For a braid representative  $K \in \mathcal{B}_{\mathcal{K}}$  let  $b_K$  denote the number of braid strands of  $K$  and let  $b_{\mathcal{K}} := \min\{b_K | K \in \mathcal{B}_{\mathcal{K}}\}$  the braid index of  $\mathcal{K}$ . Let  $c_K := \#\{\text{positive crossings of } K\} - \#\{\text{negative crossings of } K\}$ . Then the Jones' conjecture is stated as follows:

**Conjecture 1.1.** *Let  $\mathcal{K}$  be an oriented knot of braid index  $b_{\mathcal{K}}$ . If braid representatives  $K^1$  and  $K^2 \in \mathcal{B}_{\mathcal{K}}$  have  $b_{K^1} = b_{K^2} = b_{\mathcal{K}}$  then their algebraic crossing numbers satisfy  $c_{K^1} = c_{K^2}$ .*

Here is a stronger version of the above conjecture:

**Conjecture 1.2. [4] (Stronger Conjecture):** *Let  $\Phi : \mathcal{B}_{\mathcal{K}} \rightarrow \mathbb{N} \times \mathbb{Z}$  be a map with  $\Phi(K) := (b_K, c_K)$  for  $K \in \mathcal{B}_{\mathcal{K}}$ . Then there exists a unique  $c_{\mathcal{K}} \in \mathbb{Z}$  with*

$$\Phi(\mathcal{B}_{\mathcal{K}}) = \{(b_{\mathcal{K}} + x + y, c_{\mathcal{K}} + x - y) \mid x, y \in \mathbb{N}\}, \quad (1)$$

a subset of the infinite quadrant region in Figure 1.

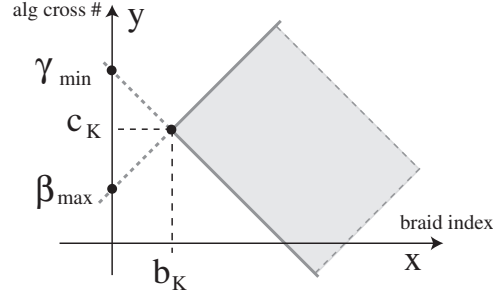


Fig. 1. The region of braid representatives of  $\mathcal{K}$

Obviously, the truth of Conjecture 1.2 implies the truth of Conjecture 1.1.

Thanks to the Markov theorem, we know that any  $K_1, K_2 \in \mathcal{B}_{\mathcal{K}}$  are related by a sequence of braid isotopy,  $(\pm)$ -stabilizations and  $(\pm)$ -destabilizations. Starting from a minimal braid representative, say  $K_{\star}$  with  $\Phi(K_{\star}) = (b_{\mathcal{K}}, c_{\mathcal{K}})$ , for any  $x, y \in \mathbb{N}$  we can obtain a braid representative  $K \in \mathcal{B}_{\mathcal{K}}$  with  $\Phi(K) = (b_{\mathcal{K}} + x + y, c_{\mathcal{K}} + x - y)$  by applying  $(+)$ -stabilization  $x$ -times and  $(-)$ -stabilization  $y$ -times to  $K_{\star}$ .

Both conjectures are naturally motivated by the following inequalities.

**Theorem 1.1.** [5], [2], **The Morton-Franks-Williams inequality.** Let  $d_+$  and  $d_-$  be the maximal and minimal degrees of the variable  $v$  of  $P_{\mathcal{K}}(v, z)$ , the HOMFLYPT polynomial of  $\mathcal{K}$ . If  $K \in \mathcal{B}_{\mathcal{K}}$ , then

$$c_K - b_K + 1 \leq d_- \leq d_+ \leq c_K + b_K - 1. \quad (2)$$

As a corollary,

$$\frac{1}{2}(d_+ - d_-) + 1 \leq b_K, \quad (3)$$

thus,  $(d_+ - d_-)/2 + 1 \leq b_{\mathcal{K}}$ , giving a lower bound for the braid index of  $\mathcal{K}$ .

From (2) we obtain that

$$\Phi(\mathcal{B}_{\mathcal{K}}) \subset \{(x, y) \mid b_{\mathcal{K}} \leq x, -x + (d_+ + 1) \leq y \leq x + (d_- - 1)\}. \quad (4)$$

We say the MFW inequality is *sharp* if there exists  $K \in \mathcal{B}_{\mathcal{K}}$  such that  $c_K - b_K + 1 = d_-$  and  $d_+ = c_K + b_K - 1$ . Then we have

**Theorem 1.2.** *Sharpness of the MFW inequality for  $\mathcal{K}$  implies the truth of Conjectures 1.1 and 1.2. In particular;*

$$b_{\mathcal{K}} = \frac{d_+ - d_-}{2} - 1, \quad c_{\mathcal{K}} = \frac{d_+ + d_-}{2}.$$

Sharpness of the MFW inequality is proved for the following classes of knots and links; the unlinks, any knots with  $\leq 10$  crossings but  $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$  in the standard knot table [3], closed positive braids with a full positive twist (i.e., torus links, Lorenz links) [2], 2-bridge links and alternating fibered links [7].

By a completely different reason, it is proved that (only) Conjecture 1.1 is true for all knots and links with braid index  $\leq 3$ , by using the braid foliation technique [1].

Regardless of the non-sharpness of the MFW inequality, we can prove that:

**Theorem 1.3.** *Conjectures 1.1 and 1.2 are true for all  $9_{42}, 9_{49}, 10_{132}, 10_{150}$  and  $10_{156}$ .*

To prove this theorem, let us fix some notations:

Let  $\mathcal{K}_{p,q}$  be the  $(p, q)$ -cable of  $\mathcal{K}$ . For a closed braid representative  $K \in \mathcal{B}_{\mathcal{K}}$  let  $K_{p,q}$  denote the  $p$ -parallel copies of  $K$  with a  $k/p$ -twist where  $k := q - p \cdot c_K$  (see Figure 2).

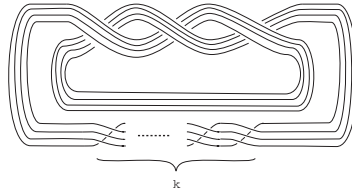


Fig. 2.  $(4, q)$ -cable ( $q = 4 \cdot 3 + k$ ) of the right hand trefoil

Let  $D_{\mathcal{K}} := b_{\mathcal{K}} - \frac{1}{2}(d_+ - d_-) - 1$  be the difference of the numbers in (3), i.e., of the actual braid index and the lower bound for braid index. Call  $D_{\mathcal{K}}$  the *deficit* of the MFW inequality for  $\mathcal{K}$ .

**Proof.** Let  $\mathcal{K} = 9_{42}, 9_{49}, 10_{132}, 10_{150}$  or  $10_{156}$ .

Suppose  $K^1$  and  $K^2$  are braid representatives of  $\mathcal{K}$  with  $b_{K^1} = b_{K^2} = b_{\mathcal{K}} + n \geq b_{\mathcal{K}}$  and with distinct algebraic crossing numbers  $c_{K^1} - c_{K^2} \geq 2(n + 1)$ , (i.e., Conjectures 1.1 and 1.2 do not apply to  $\mathcal{K}$ ).

For fixed  $(p, q)$ , let  $k_i$  ( $i = 1, 2$ ) be an integer such that  $q = p \cdot c_{K^i} + k_i$ . Then we have

$$c_{K_{p,q}^i} = p^2 c_{K^i} + k_i(p-1)$$

and  $c_{K_{p,q}^1} - c_{K_{p,q}^2} = p^2(c_{K^1} - c_{K^2}) + (k_1 - k_2)(p-1) = p(c_{K^1} - c_{K^2})$ . Thanks to the MFW inequality, we have

$$c_{K_{p,q}^1} - (b_{\mathcal{K}_{p,q}} + pn) + 1 \leq d_- \leq d_+ \leq c_{K_{p,q}^2} + (b_{\mathcal{K}_{p,q}} + pn) - 1.$$

Therefore,

$$\begin{aligned} D_{\mathcal{K}_{p,q}} &= b_{\mathcal{K}_{p,q}} - (d_+ - d_-)/2 - 1 \\ &\geq (c_{K_{p,q}^1} - c_{K_{p,q}^2})/2 - pn \\ &= p(c_{K^1} - c_{K^2})/2 - pn \geq p. \end{aligned}$$

However, when  $p = 2$  for each  $\mathcal{K} = 9_{42}, \dots, 10_{156}$ , we have that  $D_{\mathcal{K}_{2,q}} = 1 < 2 = p$  for some  $q$ , which is a contradiction.  $\square$

Here is another property of cabling:

**Theorem 1.4.** *Let  $\mathcal{K}$  be a non-trivial knot type. If Conjecture 1.1 (resp. 1.2) is true for  $\mathcal{K}$  then Conjecture 1.1 (resp. 1.2) is also true for  $\mathcal{K}_{p,q}$  when  $p \geq 2$ .*

*In particular, if  $c_{\mathcal{K}}$  and  $c_{\mathcal{K}_{p,q}}$  denote the unique algebraic crossing numbers of  $\mathcal{K}$  and  $\mathcal{K}_{p,q}$  respectively in their minimal braid representatives then we have*

$$c_{\mathcal{K}_{p,q}} = (p-1)q + p \cdot c_{\mathcal{K}}.$$

Similar results on links also hold, see [4].

As corollary to Theorem 1.4 we have:

**Corollary 1.1.** *Conjectures 1.1 and 1.2 apply to iterated torus knots.*

We can apply this corollary to the  $(2, 7)$ -cable of the right hand trefoil. Although the MFW inequality is sharp on the trefoil knot, it is not sharp on the  $(2, 7)$ -cable [6]. However, Corollary 1.1 applies.

Finally, the following theorem reduces the problem into prime knots.

**Theorem 1.5.** [4] *If Conjecture 1.1 (res. 1.2) is true for  $\mathcal{K}^1, \mathcal{K}^2$  then Conjecture 1.1 (res. 1.2) is also true for the connect sum  $\mathcal{K}^1 \sharp \mathcal{K}^2$ .*

## References

1. Birman, J. S. and Menasco, W. W., *Studying links via closed braids III: Classifying links which are closed 3-braids*, Pacific J. Math. **161**, (1993), no. 1, 25–113.
2. Franks, J. and Williams, R. F., *Braids and the Jones Polynomial*, Trans. Amer. Math. Soc., **303**, (1987), 97–108.
3. Jones, V. F. R., *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math., **126**, (1987), 335–388.
4. Kawamuro, K., *The algebraic crossing number and the braid index of knots and links*, to appear in Algebr. Geom. Topol.
5. Morton, H. R., *Seifert circles and knot polynomials*, Math. Proc. Cambridge Philos. Soc., **99**, (1986), 107–109.
6. Morton, H. R. and Short, H. B., *The 2-variable polynomial of cable knots*, Math. Proc. Cambridge Philos. Soc. 101 (1987), no. 2, 267–278.
7. Murasugi, K., *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326** (1991), no. 1, 237–260.