The Algebraic Crossing Number and The Braid Index of Knots and Links

Keiko Kawamuro

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2006

Abstract

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It has been conjectured that the algebraic crossing number of link is uniquely determined in minimal braid representation. This conjecture is true for many classes of knots and links.

The Morton-Franks-Williams inequality gives a lower bound for braid index. And sharpness of the inequality on a knot type implies the truth of the conjecture for the knot type.

We prove that there are infinitely many examples of knots and links on which the inequality is not sharp, but the conjecture is still true in these cases.

We also show that if the conjecture is true for $\mathcal{K}$ and $\mathcal{L}$, then it is also true for $\mathcal{K}_{p,q}$, the (p, q)-cable of $\mathcal{K}$, and for $\mathcal{K}\sharp\mathcal{L}$, the connect sum of $\mathcal{K}$ and $\mathcal{L}$.

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Acknowledgements

First of all, I am indebted to my advisor, Professor Joan S. Birman, for sharing her very deep insights with me and for her strong support and warm encouragement. I wish to express my gratitude to Professor Mikami Hirasawa, for enlightening discussions and teaching me much about fibred knots, including the definition and properties of the enhanced Milnor number. Professor William Menasco kindly invited me to SUNY Buffalo twice and shared his knowledge about the Birman-Menasco diagram and the associated conjecture. I also thank the members of my thesis defense committee, Professors Mikhail Khovanov, Ilya Kofman, Walter Neumann and Dylan Thurston. Professor Kofman gave me many profitable comments. Professor Neumann's suggestion about hyperbolic knots has been critical to Theorem 2.2.2. Professor Thurston listened patiently to my ideas on cable knots and asked me new questions about cabled/composite knots that helped a lot to further my work.

Finally, I thank my family and my friends, Professor Yasuyuki Kawahigashi and Marta Asaeda, for their friendship and support from afar.

To my parents,

Keiroku and Sachiko Kawamuro

Chapter 1

Introduction

The *Braid index* is one of the classical invariants of knots and links. Any knot and link type is presented as a braid closure. The braid index of a link type is the least number of braid strands needed for that.

The *Algebraic crossing number* (or writhe) is a quantity of oriented link diagram counting the crossings with weight $+1$ (resp. -1) for a positive (resp. negative) crossing as shown in the left (resp. middle) sketch of Figure 1.1. Since it is changed under Reidemeister move I, it is *not* an invariant of link types. However, it has been asked (see [11] page 357 for example):

Question: *"Is the algebraic crossing number in a minimal braid representation a link invariant?"*

(Here "minimal" means that the number of braid strands of a link diagram is equal to the braid index of the link type.)

We have known that torus links, closed positive braids with a full twist (including the Lorenz links) [8], 2-bridge links and alternating fibred links [16] and links with braid index ≤ 3 [4] have unique algebraic crossing numbers at minimal braid index.

In the following, we overview Chapter 3 first then go back to Chapter 2.

Our goal in Chapter 3 is to approach the question in three ways:

The first way (Theorem 3.2.1 and its corollaries) is by studying the deficit of the Morton-Franks-Williams (MFW) inequality [15], [8]. It is easy to see that sharpness of the MFW-inequality implies the uniqueness of the algebraic crossing number at minimal index (Theorem 3.1.3). Then how do we answer the question for links on which the inequality is not sharp? In fact this dissertation provides infinitely many examples of non-sharp links having unique algebraic crossing numbers at minimal braid index.

The second way is by studying the behavior of the braid index and the algebraic crossing number under the cabling operation. In Theorem 3.2.7 and Theorem 3.2.9, we will prove that the uniqueness property is preserved under cabling.

The third way is by studying the connect sum operation. In Theorem 3.3.1, we will show that the uniqueness property is preserved under taking the connect sum.

In Chapter 2, we focus on non-sharpness of the MFW-inequality.

To state the MFW inequality, let $\mathcal{K}$ be an oriented knot type and let K be a diagram of $\mathcal{K}$ on a plane. Focus on one crossing of K with sign ε. Denote $K_\varepsilon := K$ and let $K_{-\varepsilon}$ (resp. K_0) be the closed braid obtained from K_ε by changing the the crossing to the opposite sign $-\varepsilon$ (resp. resolving the crossing), see Figure 1.1. The

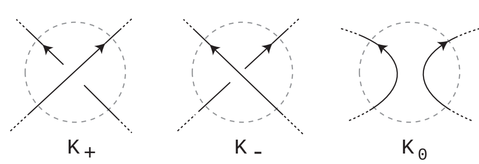

Figure 1.1: Local views of K_+, K_-, K_0.

HOMFLYPT polynomial $P_{\mathcal{K}}(v,z) = P_K(v,z)$ satisfies the following relations (for any choice of a crossing):

$$\frac{1}{v}P_{K_+} - vP_{K_-} = zP_{K_0}. \quad (1.1)$$
$$P_{\text{unknot}} = 1.$$

Now we are ready to state the MFW-inequality.

Theorem 1.0.1 [15],[8], ***The Morton-Franks-Williams inequality.***

Let d_+ and d_- be the maximal and minimal degrees of the variable v of $P_K(v, z)$. If a knot type $\mathcal{K}$ has a closed braid representative K with braid index b_K and algebraic crossing number c_K, then we have

$$c_K - b_K + 1 \le d_- \le d_+ \le c_K + b_K - 1. \quad (1.2)$$

As a corollary,

$$\frac{1}{2}(d_+ - d_-) + 1 \le b_K, \quad (1.3)$$

giving a lower bound for the braid index $b_{\mathcal{K}}$ *of $\mathcal{K}$.*

In general, it is hard to determine the braid index. This inequality was the first known result of a general nature relating to the computation of braid index, and it appeared to be quite effective. Jones notes, in [11], that on all but five knots, $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$ in the standard knot table, up to crossing number 10, the MFW inequality is sharp. Furthermore it has been known that the inequality is sharp on all torus links, closed positive n-braids with a full twist [8], 2-bridge links and fibred alternating links [16].

However, the MFW inequality is not as strong as it appears to be as above. In Theorem 2.2.2 we give an infinite class of prime links in which the deficit $D_{\mathcal{K}} := b_{\mathcal{K}} - \frac{1}{2}(d_+ - d_-) - 1$ of the MFW inequality (1.3) can be arbitrarily large. And in Theorem 2.3.2 we see another infinite class of knots, including $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$, on which the inequality is not sharp.

Then we may ask "why does non-sharpness occur?" Theorem 2.1.2 gives a sufficient condition for non-sharpness of the MFW inequality. In fact all the examples in Theorems 2.2.2 and 2.3.2 satisfy this sufficient condition.

The idea of Theorem 2.1.2 is to find knots K_α of known braid index $= b$ which have a distinguished crossing such that, after changing that crossing to each of the other two possibilities in Figure 1.1, giving knots or links K_β and K_γ, it is revealed that K_β and K_γ each has braid index $< b$.

Thanks to Theorem 2.1.2 one can visually observe the "accumulation" of deficits (for example under the connect sum operation and other linking operation) by looking only at the distinguished crossings which contribute to deficits. See the proof of Theorem 2.2.2 for details.

Chapter 2

Non-sharpness of the Morton-Franks-Williams inequality

2.1 Sufficient conditions for non-sharpness

In this section we define the deficit of MFW inequality (Definition 2.1.1) then give sufficient conditions (Theorem 2.1.2) for a closed braid on which the inequality is not sharp.

Let $b_{\mathcal{K}}$ be the braid index of knot type $\mathcal{K}$, that is the smallest integer $b_{\mathcal{K}}$ such that $\mathcal{K}$ can be represented by a closed $b_{\mathcal{K}}$-braid. Let b_K, c_K denote the braid index and the algebraic crossing number of a braid representative K of $\mathcal{K}$.

Definition 2.1.1 *Let*

$$D_{\mathcal{K}} := b_{\mathcal{K}} - \frac{1}{2}(d_+ - d_-) - 1$$

be the difference of the numbers in (1.3), *i.e., of the actual braid index and the* *lower bound for braid index. Call $D_{\mathcal{K}}$ the* deficit *of the MFW inequality for $\mathcal{K}$.*

If $D_{\mathcal{K}} = 0$, the MFW inequality is sharp on $\mathcal{K}$. If K is a braid representative of $\mathcal{K}$ let $D_K^+ := (c_K + b_K - 1) - d_+$ and $D_K^- := d_- - (c_K - b_K + 1)$. When $b_K = b_{\mathcal{K}}$, we have

$$D_{\mathcal{K}} = \frac{1}{2}(D_K^+ + D_K^-). \tag{2.1}$$

Note that $D_K^{\pm}$ depends on the choice of braid representative K, but the deficit $D_{\mathcal{K}}$ is independent from the choice.

Theorem 2.1.2 *Assume that K is a closed braid representative of $\mathcal{K}$ with $b_K = b_{\mathcal{K}}$. Focus on one site of K and construct K_+, K_-, K_0 (one of the three must be K). Let $\alpha, \beta, \gamma \in \{+, -, 0\}$ and assume that α, β, γ are mutually distinct. If $K_\alpha = K$ and positive destabilization is applicable p-times to each of K_β and K_γ, then*

$$D_K^+ \geq 2p; \tag{2.2}$$

and if $K_\alpha = K$ and negative destabilization is applicable n-times to each of K_β and K_γ, then

$$D_K^- \geq 2n. \tag{2.3}$$

Therefore, by (2.1), the MFW inequality is not sharp on $\mathcal{K}$ if $p + n > 0$.

Here is a lemma to prove Theorem 2.1.2.

Lemma 2.1.3 *Let K be a closed braid. Choose one crossing, and construct K_+, K_-, K_0 (one of the three must be K). We have*

$$d_+(P_{K_+}) \leq \max\{d_+(P_{K_-}) + 2, \quad d_+(P_{K_0}) + 1\} \tag{2.4}$$

$$d_+(P_{K_-}) \leq \max\{d_+(P_{K_+}) - 2, \quad d_+(P_{K_0}) - 1\} \tag{2.5}$$

$$d_+(P_{K_0}) \leq \max\{d_+(P_{K_+}) - 1, \quad d_+(P_{K_-}) + 1\} \tag{2.6}$$

and

$$
\begin{aligned}
d_-(P_{K_+}) &\geq \min\{d_-(P_{K_-})+2, \quad d_-(P_{K_0})+1\} \\
d_-(P_{K_-}) &\geq \min\{d_-(P_{K_+})-2, \quad d_-(P_{K_0})-1\} \\
d_-(P_{K_0}) &\geq \min\{d_-(P_{K_+})-1, \quad d_-(P_{K_-})+1\}.
\end{aligned}
$$

Proof of Lemma 2.1.3. By (1.1), we have $P_{K_+} = v^2 P_{K_-} + vzP_{K_0}$. Thus, $d_+(P_{K_+}) = d_+(v^2 P_{K_-} + vzP_{K_0}) \leq \max\{d_+(v^2 P_{K_-}),\ d_+(vzP_{K_0})\}$ and we obtain (2.4). The other results follow similarly.

□

Table (2.7) shows the changes of c_K, b_K, $c_K - b_K + 1$ and $c_K + b_K - 1$ under stabilization and destabilization of a closed braid.

	c_K	b_K	$c_K - b_K + 1$	$c_K + b_K - 1$
+ stabilization	+1	+1	0	+2
+ destabilization	−1	−1	0	−2
− stabilization	−1	+1	−2	0
− destabilization	+1	−1	+2	0

(2.7)

Note that c_K and b_K are invariant under braid isotopy and exchange moves.

Proof of Theorem 2.1.2. Suppose that $K = K_\alpha = K_+$. Suppose we can apply positive destabilization k-times ($k \geq p$) to K_-. Let $\tilde{K}_-$ denote the closed braid obtained after the destabilization. Then we have:

$$
\begin{aligned}
d_+(P_{K_-}) + 2 &= d_+(P_{\tilde{K}_-}) + 2 \\
&\leq (c_{\tilde{K}_-} + b_{\tilde{K}_-} - 1) + 2 \\
&= \{(c_{K_-} + b_{K_-} - 1) - 2k\} + 2 \\
&= (c_{K_+} - 2) + b_{K_+} - 1 - 2k + 2 \\
&= (c_{K_+} + b_{K_+} - 1) - 2k = (c_K + b_K - 1) - 2k.
\end{aligned}
\tag{2.8}
$$

The first equality holds since K_- and $\tilde{K}_-$ have the same knot type. The first inequality is the MFW inequality. The second equality follows from Table (2.7).

Similarly, if we can apply positive destabilization l-times ($l \geq p$) to K_0, and obtain $\tilde{K}_0$, we have

$$
\begin{aligned}
d_+(P_{K_0}) + 1 &= d_+(P_{\tilde{K}_0}) + 1 \\
&\leq (c_{\tilde{K}_0} + b_{\tilde{K}_0} - 1) + 1 \\
&= (c_{K_0} + b_{K_0} - 1 - 2l) + 1 \\
&= (c_{K_+} - 1) + b_{K_+} - 1 - 2l + 1 \\
&= (c_{K_+} + b_{K_+} - 1) - 2l = (c_K + b_K - 1) - 2l.
\end{aligned}
\tag{2.9}
$$

By (2.4), (2.8) and (2.9) we get

$$
\begin{aligned}
d_+(P_K) &= d_+(P_{K_+}) \leq \max\{d_+(P_{K_-}) + 2, \quad d_+(P_{K_0}) + 1\} \\
&\leq (c_K + b_{\mathcal{K}} - 1) - \min\{2k, 2l\},
\end{aligned}
$$

i.e., $D^+_K \geq \min\{2k, 2l\} \geq 2p$. When $K_\alpha = K_-$ or $K_\alpha = K_0$, the same arguments work (use (2.5) or (2.6) for these cases in the place of (2.4)) and we get (2.2).

The other inequality (2.3) also holds by the identical argument. □

2.2 Deficit growth

Our goal is to exhibit examples (Theorem 2.2.2) of prime links on which the deficit of the inequality can be arbitrary large.

Theorem 2.2.1 *Knot type* $\mathcal{K} = 9_{42}$ *has a braid representative* $K = K_+$ *(see Figure 2.1) satisfying the sufficient condition in Theorem 2.1.2.*

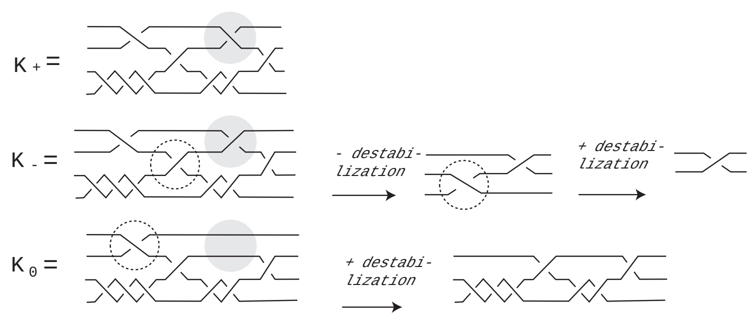

Figure 2.1: Knot 9_{42} satisfies the conditions of Theorem 2.1.2

Proof of Theorem 2.2.1. It is known that 9_{42} has braid index = 4 and deficit $D_{9_{42}} = 1$. Let $K = K_+$ be its braid representative of the minimal braid index as in Figure 2.1. Construct K_-, K_0 by changing the shaded crossing. Sketches show that both K_-, K_0 can be positively destabilized. Thus by Theorem 2.1.2, $D_K^+ \geq 2$ and $D_{9_{42}} \geq 1$. □

Theorem 2.2.2 *For any positive integer* n, *there exists a prime link* L *whose deficit* $D_L \geq n$.

Proof of Theorem 2.2.2. We prove the theorem by exhibiting examples. For $n \in \mathbb{N}$ let $\mathcal{A}^n(9_{42})$ be the closure of n-copies of 9_{42} linked each other by two

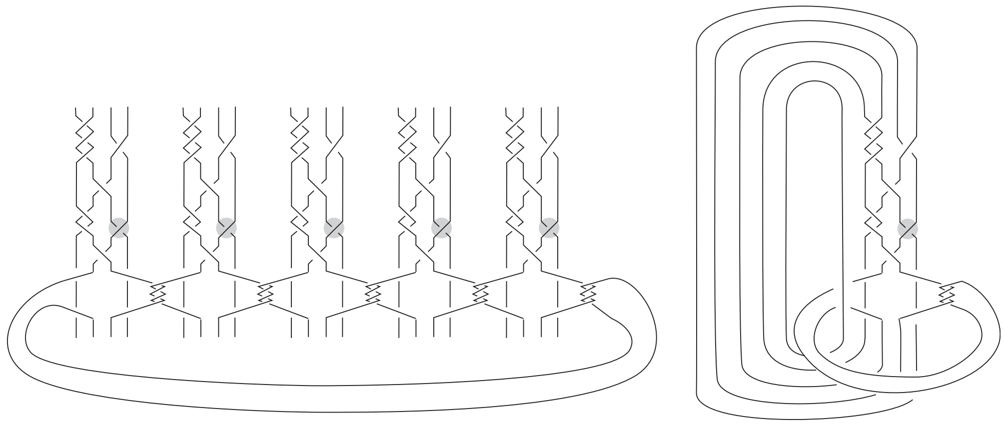

Figure 2.2: Prime link $\mathcal{A}^5(9_{42})$ and 2-component link $\mathcal{A}$.

full twists as in the left sketch of Figure 2.2. Since the braid index $b_{9_{42}} = 4$ and $\mathcal{A}^n(9_{42})$ is an n-component link, we know the braid index of $\mathcal{A}^n(9_{42})$ is $4n$. This construction gives a braid representative with $4n$-strands and n distinguished (shaded in the left sketch) crossings.

In the following we will see that each of the shaded crossing contributes to the deficit.

Let $\mathcal{K} := \mathcal{A}^2(9_{42})$ and let K be the braid representative of $\mathcal{K}$ as in Figure 2.2. Let $K_{--}, K_{-0}, K_{0-}, K_{00}$ be the links obtained from K by changing the two shaded crossings. We repeat the discussion of the proof of Theorem 2.1.2: We have

$$\begin{aligned}
d_+(P_{K_{--}}) + (2+2) &= d_+(P_{\tilde{K}_{--}}) + 4 \\
&\leq (c_{\tilde{K}_{--}} + b_{\tilde{K}_{--}} - 1) + 4 \\
&= \{(c_{K_{--}} + b_{K_{--}} - 1) - 2 \cdot 2\} + 4 \\
&= (c_K - 4) + b_K - 1 - 2 \cdot 2 + 4 \\
&= (c_K + b_K - 1) - 2 \cdot 2.
\end{aligned}$$

Similarly,

$$\begin{aligned} d_+(P_{K_{-0}}) + (2+1) &\leq (c_K + b_K - 1) - 2\cdot 2, \\ d_+(P_{K_{0-}}) + (1+2) &\leq (c_K + b_K - 1) - 2\cdot 2, \\ d_+(P_{K_{00}}) + (1+1) &\leq (c_K + b_K - 1) - 2\cdot 2. \end{aligned}$$

Thus,

$$\begin{aligned} d_+(P_K) &= \max\{d_+(P_{K_{--}}) + 4,\ d_+(P_{K_{-0}}) + 3,\ d_+(P_{K_{0-}}) + 3,\ d_+(P_{K_{00}}) + 2\} \\ &\leq (c_K + b_{\mathcal{K}} - 1) - 2\cdot 2 \end{aligned}$$

and $D_{\mathcal{K}} \geq \frac{1}{2}D_K^+ \geq \frac{1}{2}(2\cdot 2) = 2$.

Similar arguments work when $\mathcal{K} = \mathcal{A}^n(9_{42})$ for $n \geq 3$ and we have $D_{\mathcal{A}^n(9_{42})} \geq \frac{1}{2}D^+_{\mathcal{A}^n(9_{42})} \geq \frac{1}{2}(2\cdot n) \geq n$.

The 2-component link $\mathcal{A}$ of the right sketch is hyperbolic [17]. Pair $(S^3, \mathcal{A}^n(9_{42}) \cup$ z-axis) is an n-fold cover of $(S^3, \mathcal{A})$ branched at z-axis . Therefore, by [19] we can conclude that $\mathcal{A}^n(9_{42})$'s are all prime except for finitely many n's. □

Remark 2.2.3 By taking the connected sum of knots on which the MFW inequality is non-sharp, one can also construct examples of (non-prime) knots with arbitrarily large deficits. This fact follows not only by Theorem 2.1.2 but also by the definition of HOMFLYPT polynomial (1.1) and the additivity of braid indices under connected sums [5].

2.3 Birman-Menasco diagram

In this section as an application of Theorem 2.1.2 we study another infinite class of knots including all the Jones' five knots $(9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156})$ on which the MFW inequality is not sharp. We call the block-strand diagram (see [7] for definition) of Figure 2.3 the Birman-Menasco (BM) block-strand diagram.

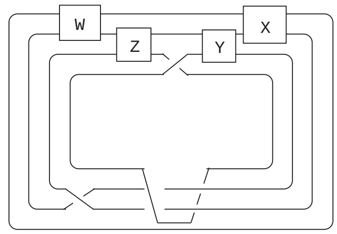

Figure 2.3: The Birman-Menasco diagram $BM_{x,y,z,w}$.

Definition 2.3.1 *Let $BM_{x,y,z,w}$, where $x, y, z, w \in \mathbb{Z}$, be the knot (or the link) type which is obtained by assigning x positive half -twists (resp. y, z, w) to the braid block X (resp. Y, Z, W) of the BM diagram.*

Recall that on all but only five knots $(9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156})$ up to crossing number 10 the MFW inequality is sharp. An interesting property of the BM diagram is that it carries all the five knots. Namely, we have $9_{42} = BM_{-1,1,-2,-1} = BM_{-1,-2,-2,2}$, $9_{49} = BM_{-1,1,1,2}$, $10_{132} = BM_{-1,-2,-2,-2}$, $10_{150} = BM_{3,-2,-2,2} = BM_{-1,2,-2,2} = BM_{-1,-2,2,2} = BM_{-1,1,2,-1} = BM_{3,1,-2,-1}$, and $10_{156} = BM_{-1,1,1,-2}$.

We have the following theorem, which was conjectured informally by Birman and Menasco:

Theorem 2.3.2 *There are infinitely many (x, y, z, w)'s such that the MFW inequality is not sharp on $BM_{x,y,z,w}$.*

Lemma 2.3.3 *We have $D^+_{BM_{x,y,z,w}} \geq 2$.*

Proof of Lemma 2.3.3. Change the BM diagram into the diagram in sketch (1) of Figure 2.4 by braid isotopy and denote it by K. Focus on the crossing shaded in sketch (1). Regard $K = K_-$. We can apply positive destabilization once to K_+

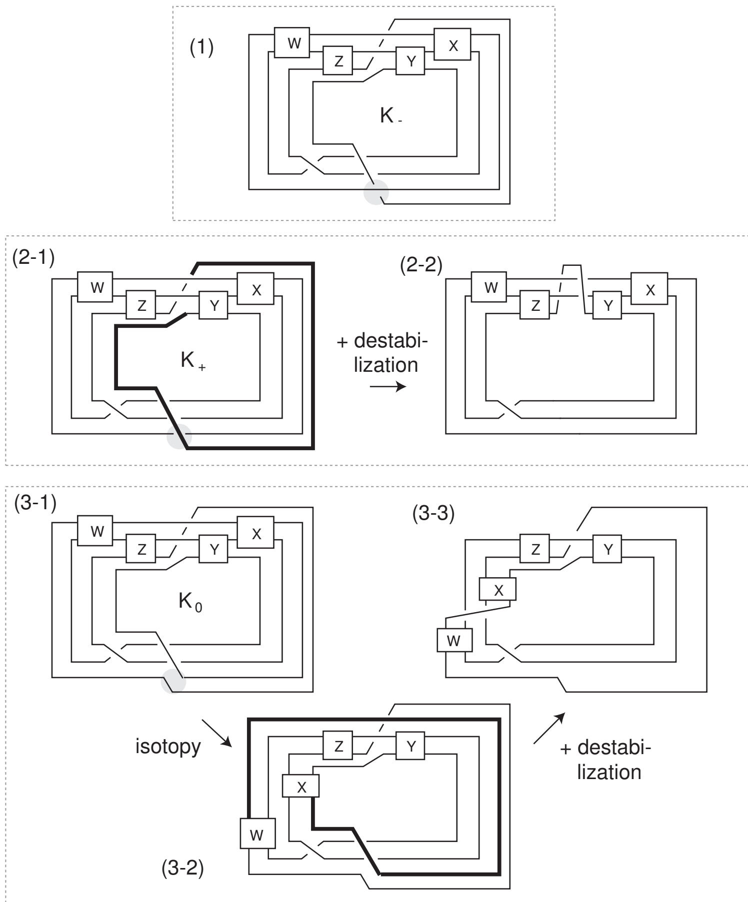

Figure 2.4: BM-diagram satisfies the sufficient condition

and obtain the diagram in sketch (2-2). We also can apply positive destabilization once to K_0 as we can see in the passage sketch (3-1) $\Rightarrow$ (3-2) $\Rightarrow$ (3-3). Therefore by Theorem 2.1.2 we have $D^+_{BM_{x,y,z,w}} \geq 2$ for any (x, y, z, w).

□

It remains to prove that there exist infinitely many (x, y, z, w)'s such that the braid index of $BM_{x,y,z,w}$ is 4. We introduce $\mathcal{K}_n := BM_{-1,-2,n,2}$ and will show that for all $m \geq 1$ the braid index of $\mathcal{K}_{2m}$ is 4. (Note that $\mathcal{K}_{-2} = 9_{42}, \mathcal{K}_2 = 10_{150}$ and $\mathcal{K}_{2m}$ is a knot.) It will then follow, thanks to Lemma 2.3.3, that the MFW inequality cannot be sharp on any $\mathcal{K}_{2m},\ m \geq 1$.

In order to do this, we use the *enhanced Milnor number* λ defined by Neumann and Rudolph [18]. Recall that the fiber surface of a fibre knot is obtained by plumbing and deplumbing Hopf bands [9]. This λ is an invariant of fibred knots and links counting algebraically the number of negative Hopf bands to get the fibre surface.

Lemma 2.3.4 *All $\mathcal{K}_n$ $(n \geq 2)$ are fibred and have the enhanced Milnor number $\lambda = 1$.*

Proof of Lemma 2.3.4. Sketch (1) of Figure 2.5 is the standard Bennequin surface of $\mathcal{K}_n$. We compress it twice as in the passage sketch (1) $\Rightarrow$ (2) along the disks bounded by dotted circles in sketch (1). Next, deplumb positive Hopf bands as much as possible as in the passage sketch (2) $\Rightarrow$ (3) $\Rightarrow$ (4) = (5). Then isotope the surface until we get sketch (8). These operations do not change the enhanced Milnor number.

We apply Melvin and Morton's trick [13] p.167, as in the passage sketch (8) $\Rightarrow$ (9). We remark that the enhanced Milnor number is invariant under this trick.

The surface of sketch (9) = (10), whose boundary is Pretzel link $P(-2, 0, 2)$, is plumbing of a positive Hopf band and a negative Hopf band. Thus it has $\lambda = 1$

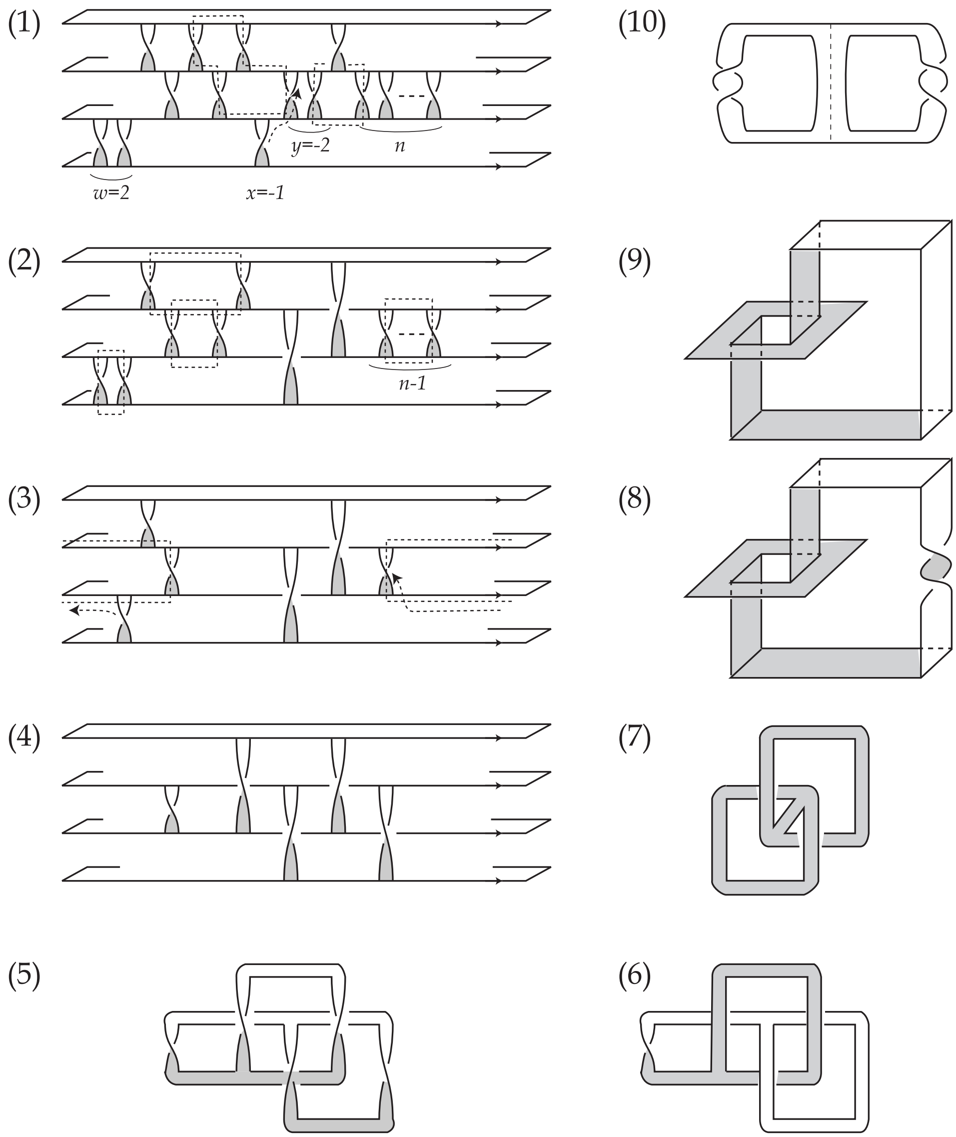

Figure 2.5: Deformation of $\mathcal{K}_n$

so does $\mathcal{K}_n$. □

Here we summarize Xu's classification of 3-braids [22]. Let σ_1, σ_2 be the standard generators of B_3 the braid group of 3-strings satisfying $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Let $a_1 := \sigma_1, a_2 := \sigma_2$ and $a_3 := \sigma_2\sigma_1\sigma_2^{-1}$. The generators are the so called *band generators* of B_n, later introduced in [3]. We can identify them with the twisted bands in Figure 2.6.

Let $\alpha := a_1a_3 = a_2a_1 = a_3a_2$. If $w \in B_3$ let $\overline{w}$ denote w^{-1}.

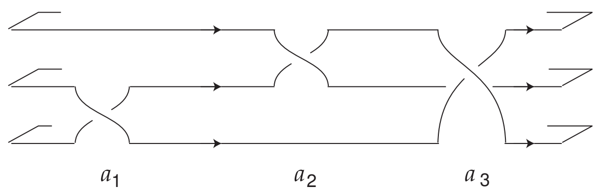

Figure 2.6: Xu's band generators

Theorem 2.3.5 (Xu [22].) *Every conjugacy class in B_3 can be represented by a shortest word in a_1, a_2, a_3 uniquely up to symmetry. And the word has one of the three forms:*

$$(1)\alpha^k P, \quad (2)N\overline{\alpha}^k, \quad (3)NP.$$

where $k \geq 0$ and $\overline{N}, P$ are positive words and the arrays of subscripts of the words are non-decreasing.

The next is another lemma for Theorem 2.3.2:

Lemma 2.3.6 *If a closed 3-braid has $\lambda = 1$ and is a knot, then up to symmetry it has one of the following forms:*

$$
\begin{aligned}
A_x &:= \overline{a_3}\,\overline{a_2}\,(a_1)^x, \quad x \geq 2,\ \textit{even}, \\
B_{x,y} &:= \overline{a_3}\,\overline{a_3}\,(a_1)^x (a_2)^y, \quad x, y \geq 3,\ \textit{odd}, \\
C_{x,y,z} &:= \overline{a_2}\,(a_1)^x (a_2)^y (a_3)^z, \quad x + z = \textit{odd},\ \ y = \textit{even},\ \ x, y, z \geq 1, \\
D_{x,y,z,w} &:= \overline{a_2}\,(a_1)^x (a_2)^y (a_3)^z (a_1)^w, \quad x, y \geq 2,\ \ z, w \geq 1.
\end{aligned}
$$

Proof of Lemma 2.3.6. Assume we have a word $w \in B_3$. By Theorem 2.3.5, w has one of the following forms:

Case (1)-1	$w = \alpha^k$	$k \geq 1$
Case (1)-2	$w = \alpha^k P$	$k \geq 1$
Case (1)-3	$w = P$	no α part
Case (2)-1	$w = \overline{\alpha}^k$	$k \geq 1$
Case (2)-2	$w = N\overline{\alpha}^k$	$k \geq 1$
Case (2)-2	$w = N$	no $\overline{\alpha}$ part
Case (3)	$w = NP$	18 cases to study

In this proof, we use the simplified notations:

symbol	meaning	change in λ
i	a_i for $i = 1, 2, 3$.	—
$=$	same conjugacy class	0
$\longrightarrow$	deplumb positive-Hopf bands	0
$\Longrightarrow$	deplumb negative-Hopf bands	≥ 1
$\approx$	Melvin-Morton trick [13]	0
$\rightsquigarrow$	composition of deplumbings of $\pm$ Hopf bands	≥ 0

These are formulae we use:

$$\alpha^2 \longrightarrow \alpha, \quad \text{Figure 2.7} \tag{2.10}$$

$$\alpha 123 \longrightarrow \alpha, \quad \text{Figure 2.8} \tag{2.11}$$

$$\bar{i}(i-1)i \ \approx \ \bar{i}\ \overline{i-1} i, \quad \text{Melvin-Morton trick} \tag{2.12}$$

$$i(i+1)\bar{i} \ \approx \ i\ \overline{i+1}\ \bar{i}, \quad \text{Melvin-Morton trick} \tag{2.13}$$

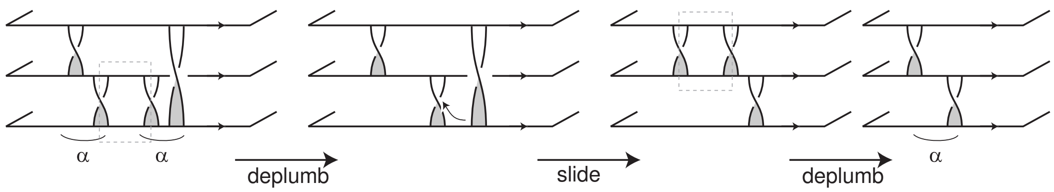

Figure 2.7: $\alpha^2 \longrightarrow \alpha$.

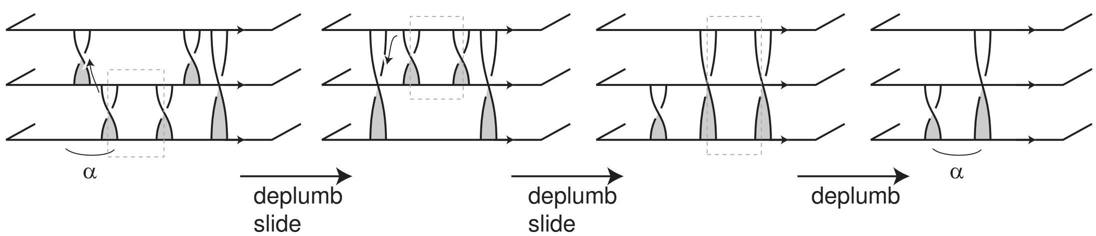

Figure 2.8: $\alpha 123 \longrightarrow \alpha$.

Now we study each case.

Case (1)-1. By (2.10), we have $w = \alpha^k \longrightarrow \alpha(=\text{unknot})$. Thus w has $\lambda = 0$.

Case (1)-2. By (2.10) up to permutation of $\{1,2,3\}$ we have $\alpha^k P \longrightarrow \alpha P \longrightarrow \alpha(123123\cdots\cdots)$. Thanks to (2.11) we have

$$\alpha \overbrace{123123\cdots\cdots}^{\text{length}=l} \longrightarrow \alpha \overbrace{123123\cdots\cdots}^{\text{length}=l-3} \quad \text{for } l \geq 3.$$

If $l = 1, 2$, we have $\alpha 1 = 211 \longrightarrow \alpha$ and $\alpha 12 = 2112 \longrightarrow \alpha$. Thus w has $\lambda = 0$.

Case (1)-3. Assume $w = P$. There are three possible cases:

$$P \longrightarrow (123)^n, \quad P \longrightarrow (123)^n 1 \quad \text{and} \quad P \longrightarrow (123)^n 12 \quad \text{where } n \geq 0.$$

If w satisfies the first case, it is known that $(123)^n$ is not fibred [10][20] i.e., w is not fibred.

The second case can be reduced to the first case, since $(123)^n 1 = 1(123)^n \longrightarrow (123)^n$.

For the third case, since $(123)^n 12 = 2(123)^n 1 = \alpha(231)^n \longrightarrow \alpha$, w has $\lambda = 0$.

Case (2)-1. By (2.10), $w = \overline{\alpha}^k \Longrightarrow \overline{\alpha}$ and w has $\lambda = 2(k-1) \neq 1$.

Case (2)-2. Suppose $w = N\overline{\alpha}^k$ where $k \geq 1$.

If $w = \overline{i}\,\overline{\alpha}$, we have $\overline{i}\,\overline{\alpha} \Longrightarrow \overline{\alpha}$ and w has $\lambda = 1$. However, the closure of w has more than one component and it does not satisfy the condition of the lemma.

If $w \neq \overline{i}\,\overline{\alpha}$, we have $N\overline{\alpha}^k \Longrightarrow \overline{\alpha}$ by (2.11), and w has $\lambda \geq 2$.

Case (2)-3. Suppose $w = N$. There are three possible cases:

$$N \Longrightarrow (\overline{3}\,\overline{2}\,\overline{1})^n, \quad N \Longrightarrow (\overline{3}\,\overline{2}\,\overline{1})^n\overline{3} \quad \text{and} \quad N \Longrightarrow (\overline{3}\,\overline{2}\,\overline{1})^n\overline{3}\,\overline{2} \quad \text{where } n \geq 0.$$

For the first case, w is not fibred [10][20].

For the second case, if $n = 0$ then w has $\lambda = 1$ if and only if $w = \overline{3}\,\overline{3}$. However this has two components. If $n \geq 1$, since $(\overline{3}\,\overline{2}\,\overline{1})^n\overline{3} \Longrightarrow (\overline{3}\,\overline{2}\,\overline{1})^n$ it can be reduced to the first case.

For the third case, if $n = 0$ then w has $\lambda = 1$ if and only if $w = \overline{3}\,\overline{3}\,\overline{2}$. However it has two components. If $n \geq 1$, we have $(\overline{3}\,\overline{2}\,\overline{1})^n\overline{3}\,\overline{2} = \overline{2}\,\overline{3}(\overline{2}\,\overline{1}\,\overline{3})^n = \overline{\alpha}\,(\overline{2}\,\overline{1}\,\overline{3})^n \Longrightarrow \overline{\alpha}$ and w has $\lambda \geq 3n$.

Case (3). Assume $w = NP$. Let w' be a word obtained from w by deplumbing $\pm$ Hopf bands sufficiently enough times, i.e., $w \rightsquigarrow w'$. This w' has one of the following 18 forms up to permutation of $\{1, 2, 3\}$.

case	w'		
i	$(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l$	$k, l \geq 1$	
ii	$(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 1$	$k \geq 1, l \geq 0$	
iii	$(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 12$	$k \geq 1, l \geq 0$	not shortest word
iv	$\overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l$	$k \geq 0, l \geq 1$	not shortest word
v	$\overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 1$	$k, l \geq 0$	
vi	$\overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 12$	$k, l \geq 0$	
vii	$\overline{1}\ \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l$	$k \geq 0, l \geq 1$	
viii	$\overline{1}\ \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 1$	$k, l \geq 0$	not shortest word
ix	$\overline{1}\ \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l 12$	$k, l \geq 0$	

i'	$(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l$	$k, l \geq 1$	
ii'	$(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 1$	$k \geq 1, l \geq 0$	not shortest word
iii'	$(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 12$	$k \geq 1, l \geq 0$	
iv'	$\overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l$	$k \geq 0, l \geq 1$	
v'	$\overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 1$	$k, l \geq 0$	
vi'	$\overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 12$	$k, l \geq 0$	not shortest word
vii'	$\overline{3}\ \overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l$	$k \geq 0, l \geq 1$	not shortest word
viii'	$\overline{3}\ \overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 1$	$k, l \geq 0$	
ix'	$\overline{3}\ \overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l 12$	$k, l \geq 0$	

Since words of case iii, iv, viii, ii', vi' and vii' are not shortest (reducible) we eliminate them from the list.

These are reduction formulae we use:

$$
\begin{aligned}
(\overline{2}\ \overline{1}\ \overline{3})(123) &\overset{(2.19)}{\longrightarrow} \overline{2}(3\overline{2}) = 1\overline{2}\ \overline{2} \Longrightarrow 1\overline{2} && (2.14)\\
(1\overline{2})(123) &\overset{(2.28)}{\longrightarrow} 11\overline{2} \longrightarrow 1\overline{2} && (2.15)\\
(\overline{2}\ \overline{1}\ \overline{3})(1\overline{2}) &\overset{(2.17)}{\longrightarrow} 1\overline{3}\ \overline{2} = \overline{2}\ \overline{2}3 \Longrightarrow = 1\overline{2} && (2.16)\\
(\overline{2}\ \overline{1}\ \overline{3})1 &\approx \overline{2}\ \overline{1}31 = 1\overline{2}\ \overline{2}1 \Longrightarrow = 11\overline{3} \longrightarrow 1\overline{3} && (2.17)\\
(\overline{2}\ \overline{1}\ \overline{3})1\overline{3} &\overset{(2.17)}{\longrightarrow} 1\overline{3}\ \overline{3} \Longrightarrow 1\overline{3} && (2.18)\\
\overline{1}\ \overline{3}(123) &= \overline{1}22\overline{1}3 \longrightarrow \overline{1}2\overline{1}3 = 3\overline{1}\ \overline{1}3 \Longrightarrow = 33\overline{2} \longrightarrow 3\overline{2} && (2.19)\\
3\overline{2}(123) &\overset{(2.28)}{\longrightarrow} 3(1\overline{2}) = \overline{1}33 \longrightarrow \overline{1}3 = 3\overline{2} && (2.20)\\
(123)1\overline{3} &\approx 123\overline{1}\ \overline{3} = 1\overline{3}22\overline{3}\\
&\longrightarrow 1\overline{3}2\overline{3} = 1\overline{3}\ \overline{3}1 \Longrightarrow 1\overline{3}1 \longrightarrow 1\overline{3} && (2.21)\\
12\overline{1}\ \overline{3} &= 1\overline{3}\ \overline{3}1 \Longrightarrow 1\overline{3}1 \longrightarrow \overline{2}1 = 1\overline{3} && (2.22)\\
(\overline{1}\ \overline{3}\ \overline{2})(123) &\overset{(2.26)}{\longrightarrow} 3\overline{1}3 = 33\overline{2} \Longrightarrow 3\overline{2} = \overline{1}3 && (2.23)\\
(\overline{1}\ \overline{3}\ \overline{2})\overline{1}3 &= \overline{1}\ \overline{3}\ 1\overline{2}\ \overline{2} \Longrightarrow \approx \overline{1}31\overline{2} = \overline{1}\ \overline{1}33 \longrightarrow \Longrightarrow \overline{1}3 && (2.24)\\
\overline{1}3(123) &\approx \overline{1}\ \overline{3}123 = \overline{1}22\overline{1}3 \longrightarrow \Longrightarrow 3\overline{1}3 \longrightarrow 3\overline{2} = \overline{1}3 && (2.25)\\
(\overline{1}\ \overline{3}\ \overline{2})12 &= \overline{1}2\overline{3}\ \overline{3}2 \Longrightarrow = \overline{1}22\overline{1} \longrightarrow \overline{1}2\overline{1} = 3\overline{1}\ \overline{1} \Longrightarrow 3\overline{1} && (2.26)\\
(\overline{1}\ \overline{3}\ \overline{2})3\overline{1} &\overset{(2.16)}{\longrightarrow} 3\overline{1} \quad \text{permutation of (2.16)} && (2.27)\\
\overline{2}(123) &\approx \overline{2}\ \overline{1}23 = \overline{2}33\overline{2} \longrightarrow \overline{2}3\overline{2} = 1\overline{2}\ \overline{2} \Longrightarrow 1\overline{2} && (2.28)\\
1\overline{2}\ (\overline{1}\ \overline{3}\ \overline{2}) &\approx 12\overline{1}\ \overline{3}\ \overline{2} = 1\overline{3}\ \overline{3}1\overline{2} \Longrightarrow = \overline{2}11\overline{2} \longrightarrow \overline{2}1\overline{2} \Longrightarrow 1\overline{2} && (2.29)
\end{aligned}
$$

Sublemma 2.3.7 *For $k, l \geq 1$, we have:*

$$
\begin{aligned}
(\overline{2}\ \overline{1}\ \overline{3})^k(123)^l &\rightsquigarrow 1\overline{2} && (2.30)\\
(\overline{1}\ \overline{3}\ \overline{2})^k(123)^l &\rightsquigarrow \overline{1}3 && (2.31)
\end{aligned}
$$

Either case, the increase of λ is ≥ 2.

Proof. From (2.14), (2.16) and (2.17) we obtain (2.30). Similarly, (2.31) follows from (2.23), (2.24) and (2.25). □

Case 3-i. By (2.30), our w is fibred and $\lambda \geq 2$.

Case 3-ii. If $k \geq 1.l = 0$,

$$w \rightsquigarrow w' = (\overline{2}\ \overline{1}\ \overline{3})^k 1 \overset{(2.17)}{\longrightarrow} (\overline{2}\ \overline{1}\ \overline{3})^{k-1} 1\overline{3} \overset{(2.18)}{\longrightarrow} \cdots \overset{(2.18)}{\longrightarrow} 1\overline{3}$$

and w has $\lambda = 1$ if and only if $w = \overline{2}\ \overline{1}\ \overline{3}\ 1^x$ for some $x \geq 1$. However it has 2 or 3 components and it does not satisfy the condition of Lemma 2.3.6. If $k, l \geq 1$,

$$w \rightsquigarrow w' = (\overline{2}\ \overline{1}\ \overline{3})^k (123)^l 1 \overset{(2.30)}{\longrightarrow} (1\overline{2})1 \longrightarrow 1\overline{2}$$

and w has $\lambda \geq 2$.

Case 3-v. When $k = l = 0$, w has $\lambda = 1$ if and only if $w = \overline{3}\ \overline{3}\ 1^x$ for some $x \geq 1$, which has more than 1 component. If $k \geq 1$ and $l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ (\overline{2}\ \overline{1}\ \overline{3})^k 1 \overset{(2.18)}{\longrightarrow} \cdots \overset{(2.18)}{\longrightarrow} 1\overline{3}$$

and $\lambda \geq 2$. If $k = 0, l \geq 1$,

$$w \rightsquigarrow w' = \overline{3}\ (123)^l 1 = (123)^l 1\overline{3} \overset{(2.21)}{\longrightarrow} \cdots \overset{(2.21)}{\longrightarrow} 1\overline{3}.$$

Thus w has $\lambda = 1$ if and only if $w = \overline{3}\ 1^x\ 2^y\ 3^z\ 1^w$ for $x, y, z, w \geq 1$. If $k, l \geq 1$,

$$w \rightsquigarrow w' = \overline{3}\ (\overline{2}\ \overline{1}\ \overline{3})^k (123)^l 1 \overset{(2.30)}{\longrightarrow} \overline{3}(1\overline{2})1 = \overline{3}11\overline{3} \longrightarrow \Longrightarrow 1\overline{3}$$

and $\lambda \geq 2$.

Case 3-vi. If $k = l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ 12 = 22\overline{1} \longrightarrow 2\overline{1}.$$

Therefore, w has $\lambda = 1$ if and only if $w = \overline{3}\ \overline{3}\ 1^x\ 2^y$ for $x, y \geq 1$. If $k = 0, l \geq 1$

$$w \rightsquigarrow w' = \overline{3}(123)^l 12 = (123)^l 12\overline{3} \longrightarrow (123)^l 1\overline{3} \stackrel{(2.21)}{\longrightarrow} \cdots \stackrel{(2.21)}{\longrightarrow} 1\overline{3}$$

and w has $\lambda = 1$ if and only if $w = \overline{3}1^x\ 2^y\ 3^z\ 1^w\ 2^v$ for some $x, y, z, w, v \geq 1$. When $k \geq 1, l = 0$,

$$w \rightsquigarrow w' = \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k 12 \longrightarrow (\overline{2}\ \overline{1}\ \overline{3})^k 1\overline{3} \stackrel{(2.18)}{\longrightarrow} \cdots \stackrel{(2.18)}{\longrightarrow} 1\overline{3}$$

and $\lambda \geq 2$. If $k, l \geq 1$,

$$w \rightsquigarrow w' \stackrel{(2.30)}{\longrightarrow} \overline{3}(1\overline{2})12 = \overline{3}11\overline{3}2 \longrightarrow \overline{3}1\overline{3}2 = 2\overline{3}\ \overline{3}\ 2 \longrightarrow \Longrightarrow 2\overline{3}$$

and $\lambda \geq 2$.

Case 3-vii. If $k = 0, l \geq 1$,

$$w \rightsquigarrow w' = \overline{1}\ \overline{3}\ (123)^l \stackrel{(2.19)}{\longrightarrow} 3\overline{2}(123)^{l-1} \stackrel{(2.20)}{\longrightarrow} \cdots \stackrel{(2.19)}{\longrightarrow} 3\overline{2}$$

and $\lambda = 1$ if and only if $w = \overline{1}\ \overline{3}\ 1^x\ 2^y\ 3^z$ for some $x, y, z \geq 1$. If $k, l \geq 1$,

$$w \rightsquigarrow w' \stackrel{(2.30)}{\longrightarrow} \overline{1}\ \overline{3}(1\overline{2}) = \overline{2}\ \overline{1}\ \overline{3}1 \stackrel{(2.17)}{\longrightarrow} 1\overline{3}$$

and $\lambda \geq 2$.

Case 3-ix. When $k = l = 0$,

$$w \rightsquigarrow w' = \overline{1}\ \overline{3}\ 12 = \overline{1}22\overline{1} \longrightarrow \Longrightarrow \overline{1}2.$$

Thus w has $\lambda = 1$ if and only if $w = \overline{1}\ \overline{3}\ 1^x\ 2^y$ for some $x, y \geq 1$. When $k = 0, l \geq 1$

$$w \rightsquigarrow w' = \overline{1}\ \overline{3}(123)^l 12 \overset{(2.19)}{\longrightarrow} \overset{(2.20)}{\longrightarrow} \overline{1}312 = 3\overline{2}12 \approx 3\overline{2}\ \overline{1}2 = \overline{1}33\overline{1} \longrightarrow \Longrightarrow \overline{1}3$$

and $\lambda \geq 2$. When $k \geq 1, l = 0$

$$w \rightsquigarrow w' = \overline{1}\ \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k 12 = 12\overline{1}\ \overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k \overset{(2.22)}{\longrightarrow} 1\overline{3}(\overline{2}\ \overline{1}\ \overline{3})^k \overset{(2.18)}{\longrightarrow} \cdots \overset{(2.18)}{\longrightarrow} 1\overline{3}$$

and $\lambda \geq 2$. When $k, l \geq 1$

$$w \rightsquigarrow w' \overset{(2.30)}{\longrightarrow} \overline{1}\ \overline{3}(1\overline{2})12 = 12\overline{1}\ \overline{3}1\overline{2} \overset{(2.22)}{\longrightarrow} 1\overline{3}1\overline{2} = \overline{2}11\overline{2} \longrightarrow \Longrightarrow 1\overline{2}$$

and $\lambda \geq 2$.

Case 3-i'. By (2.31) our w is fibred and $\lambda \geq 2$.

Case 3-iii'. When $k > l = 0$,

$$w \rightsquigarrow w' = (\overline{1}\ \overline{3}\ \overline{2})^k 12 \overset{(2.26)}{\longrightarrow} (\overline{1}\ \overline{3}\ \overline{2})^{k-1} 2\overline{1} \overset{(2.27)}{\longrightarrow} \cdots \overset{(2.27)}{\longrightarrow} 3\overline{1}$$

and $\lambda \geq 2$. When $k, l \geq 1$,

$$w \rightsquigarrow w' \overset{(2.31)}{\longrightarrow} (\overline{1}3)12 \approx \overline{1}\ \overline{3}\ 12 = \overline{1}22\overline{1} \longrightarrow \Longrightarrow 3\overline{1}$$

and $\lambda \geq 2$.

Case 3-iv'. When $k = 0, l \geq 1$,

$$w \rightsquigarrow w' = \overline{2}(123)^l \xrightarrow{(2.28)} 1\overline{2}(123)^{l-1} \xrightarrow{(2.15)} \cdots \xrightarrow{(2.15)} 1\overline{2}.$$

Thus w has $\lambda = 1$ if and only if $w = \overline{2}1^x2^y3^z$ for $x, y, z \geq 1$. To make the braid closure one component, we further require $x + z =$ odd. If $k, l \geq 1$,

$$w \rightsquigarrow w' \xrightarrow{(2.31)} \overline{2}\ (\overline{1}3) = 1\overline{2}\ \overline{2} \Longrightarrow 1\overline{2}$$

and $\lambda \geq 2$.

Case 3-v'. When $k = l = 0$,

$$w \rightsquigarrow w' = \overline{2}1.$$

Thus w has $\lambda = 1$ if and only if $w = \overline{2}\ \overline{2}1^x$, which has 2 or 3 components. When $k = 0$ and $l \geq 1$,

$$w \rightsquigarrow w' = \overline{2}(123)^l1 \xrightarrow{(2.28)} 1\overline{2}(123)^{l-1}1 \xrightarrow{(2.15)} \cdots \xrightarrow{(2.15)} 1\overline{2}1 = 11\overline{2} \longrightarrow 1\overline{2}$$

thus w has $\lambda = 1$ if and only if $w = \overline{2}\ 1^x\ 2^y\ 3^z\ 1^w$ for some $x, y, z, w \geq 1$. When $k \geq 1, l = 0$,

$$w \rightsquigarrow w' = \overline{2}\ (\overline{1}\ \overline{3}\ \overline{2})^k1 = 1\overline{2}\ (\overline{1}\ \overline{3}\ \overline{2})^k \xrightarrow{(2.29)} \cdots \xrightarrow{(2.29)} 1\overline{2}$$

and $\lambda \geq 2$. When $k, l \geq 1$

$$w \rightsquigarrow w' \xrightarrow{(2.31)} \overline{2}(\overline{1}3)1 \Longrightarrow 1\overline{2}1 \longrightarrow 1\overline{2}$$

and $\lambda \geq 2$.

Case 3-viii'. When $k = l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ \overline{2}1 = 1\overline{3}\ \overline{2} = \overline{2}1\overline{2} \Longrightarrow 1\overline{2}$$

Thus w has $\lambda = 1$ if and only if $w = \overline{3}\ \overline{2}1^x$ for some $x \geq 1$. When $k = 0$ and $l \geq 1$,

$$w \rightsquigarrow w' = \overline{3}\ \overline{2}(123)^l 1 = 1\overline{3}\ \overline{2}(123)^l \Longrightarrow 1\overline{2}(123)^l \xrightarrow{(2.15)} \cdots \xrightarrow{(2.15)} 1\overline{2}$$

and $\lambda \geq 2$. When $k \geq 1$ and $l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ \overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k 1 \Longrightarrow 1\overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k \xrightarrow{(2.29)} \cdots \xrightarrow{(2.29)} 1\overline{2}$$

and $\lambda \geq 2$. When $k, l \geq 1$

$$w \rightsquigarrow w' \xrightarrow{(2.31)} \overline{3}\ \overline{2}(\overline{1}3)1 \Longrightarrow 1\overline{2}\ \overline{1}3 = \overline{2}33\overline{2} \Longrightarrow \longrightarrow \overline{2}3 = 1\overline{2}.$$

and $\lambda \geq 2$.

Case 3-ix'. When $k = l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ \overline{2}12 = 2\overline{3}\ \overline{3}2 \Longrightarrow \longrightarrow 2\overline{3}.$$

Thus w has $\lambda = 1$ if and only if $w = \overline{3}\ \overline{2}1^x 2^y$ for some $x, y \geq 1$. When $k = 0, l \geq 1$,

$$\begin{aligned} w &\rightsquigarrow w' = \overline{3}\ \overline{2}(123)^l 12 = 12\overline{3}\ \overline{2}(123)^l = \overline{2}11\overline{2}(123)^l \longrightarrow \Longrightarrow \overline{2}3(123)^l \\ &= 1\overline{2}(123)^l \xrightarrow{(2.15)} \cdots \xrightarrow{(2.15)} 1\overline{2} \end{aligned}$$

and $\lambda \geq 2$. When $k \geq 1, l = 0$,

$$w \rightsquigarrow w' = \overline{3}\ \overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k 12 \longrightarrow \Longrightarrow 1\overline{2}(\overline{1}\ \overline{3}\ \overline{2})^k \xrightarrow{(2.29)} \cdots \xrightarrow{(2.29)} 1\overline{2}$$

and $\lambda \geq 2$. When $k, l \geq 1$

$$w \rightsquigarrow w' \overset{(2.31)}{\longrightarrow} \overline{3}\ \overline{2}(\overline{1}3)12 \longrightarrow \Longrightarrow 1\overline{2}(\overline{1}3) = 11\overline{2}\ \overline{2} \longrightarrow \Longrightarrow 1\overline{2}$$

and $\lambda \geq 2$.

Table 2.1 summarizes all the words with $\lambda = 1$.

Table 2.1:

case	word with $\lambda = 1$.
i	none.
ii	$\overline{2}\ \overline{1}\ \overline{3}\ 1^x$ (2 or 3 components.)
v	$\overline{3}\ 1^x\ 2^y\ 3^z\ 1^w = \begin{cases} C_{x+1,y,z} & \text{when } w = 1, \\ D_{x+1,y,z,w-1} & \text{when } w \geq 2. \end{cases}$ $\overline{3}\ \overline{3}\ 1^x$ (2 or 3 components.)
vi	$\overline{3}\ \overline{3}\ 1^x\ 2^y =: B_{x,y}$. $\overline{3}\ 1^x\ 2^y\ 3^z\ 1^w\ 2^v = \begin{cases} C_{x+v+1,y,z} & \text{when } w = 1, \\ D_{x+v+1,y,z,w-1} & \text{when } w \geq 2. \end{cases}$
vii	$\overline{1}\ \overline{3}\ 1^x\ 2^y\ 3^z = \overline{1}\ \overline{3}\ 1^{x+z}\ 2^y$
ix	$\overline{1}\ \overline{3}\ 1^x\ 2^y = \begin{cases} \overline{3}\ \overline{3}1^{x+1} & \text{when } y = 1 \quad \text{(2 or 3 components)} \\ B_{x+1,y-1} & \text{when } y \geq 2 \end{cases}$
i'	none.
iii'	none.
iv'	$\overline{2}\ 1^x\ 2^y\ 3^z =: C_{x,y,z}$.
v'	$\overline{2}\ 1^x\ 2^y\ 3^z\ 1^w =: D_{x,y,z,w}$. $\overline{2}\ \overline{2}\ 1^x$ (2 or 3 components.)
viii'	$\overline{3}\ \overline{2}\ 1^x =: A_x$.
ix'	$\overline{3}\ \overline{2}\ 1^x\ 2^y = \begin{cases} \overline{3}\ \overline{3}\ 2^{y+1} & \text{when } x = 1, \quad \text{(2 or 3 components)} \\ B_{x-1,y+1} & \text{when } x \geq 2. \end{cases}$

Words $A_x, \cdots, D_{x,y,z,w}$ are defined in Table 2.1. We can see that any word with $\lambda = 1$ and having one component has one of the forms; $A_x, \cdots, D_{x,y,z,w}$.

□

Lemma 2.3.8 *The leading terms of the Alexander polynomials of* $\mathcal{K}_n$, A_x, $B_{x,y}$,

$C_{x,y,z}$ and $D_{x,y,z,w}$ are the following:

$$\begin{aligned}
\mathcal{K}_n; &\quad \pm(1-4t-6t^2+8t^3-\cdots) \quad \textit{if } n\geq 2,\\
A_x; &\quad \pm(1-3t+\cdots) \quad \textit{if } x\geq 2,\\
B_{x,y}; &\quad \pm(1-3t+\cdots) \quad \textit{if } x,y\geq 3,\\
C_{x,y,z}; &\quad \pm(1-5t+\cdots) \quad \textit{if } x,y,z\geq 2,\\
C_{1,2,z}, C_{1,y,2}, C_{2,y,1}, C_{x,2,1}; &\quad \pm(1-4t+6t^2-7t^3+\cdots) \quad \textit{if } x,y,z\geq 4,\\
C_{1,y,z}, C_{x,y,1}; &\quad \pm(1-4t+7t^2+\cdots) \quad \textit{if } x,y,z\geq 3\\
D_{x,y,z,w}, D_{x,y,z,1}; &\quad \pm(1-6t+\cdots) \quad \textit{if } x,y,z,w\geq 2,\\
D_{x,y,1,w}; &\quad \pm(1-5t+\cdots) \quad \textit{if } x,y,w\geq 2.
\end{aligned}$$

In particular, $\mathcal{K}_n \neq A_x, B_{x,y}, C_{x,y,z}, D_{x,y,z,w}$.

Proof of Lemma 2.3.8. We prove that the Alexander polynomial of $C_{x,y,z}$ for $x, y, z \geq 2$ is $\pm(1-5t+\cdots)$. Recall that Xu's Bennequin surface is a minimal genus Seifert surface. Let F be the Bennequin surface of $C_{x,y,z}$ and choose a basis

$$\{u^{(1)}, u^{(2)}, u^{(3)}_1, \cdots, u^{(3)}_{x-1}, u^{(4)}_1, \cdots, u^{(4)}_{y-1}, u^{(5)}_1, \cdots, u^{(5)}_{z-1}\}$$

for $H_1(F)$ as in Figure 2.9, where $u^{(k)}$ $(k = 1, \cdots, 5)$ corresponds to loop (k).

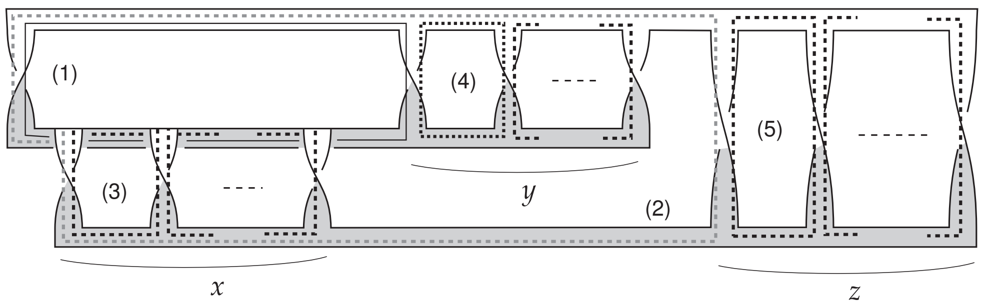

Figure 2.9: The Bennequin surface F of $C_{x,y,z} = \overline{2}1^x2^y3^z$ and a basis for $H_1(F)$

With respect to the basis, let $V_{x,y,z}$ denote the Seifert matrix for $C_{x,y,z}$.

$$V_{x,y,z} = \left[\begin{array}{c|c|cccc|cccc|cccc}
 & 1 & & & & & 1 & & & & & & & \\ \hline
1 & & -1 & & & & & & & & 1 & & & \\ \hline
 & & -1 & 1 & & & & & & & & & & \\
 & & & -1 & \ddots & & & & & & & & & \\
 & & & & \ddots & 1 & & & & & & & & \\
 & & & & & -1 & & & & & & & & \\ \hline
 & & & & & & -1 & 1 & & & & & & \\
 & & & & & & & -1 & \ddots & & & & & \\
 & & & & & & & & \ddots & 1 & & & & \\
 & & & & & & & & & -1 & & & & \\ \hline
 & & & & & & & & & & -1 & 1 & & \\
 & & & & & & & & & & & -1 & \ddots & \\
 & & & & & & & & & & & & \ddots & 1 \\
 & & & & & & & & & & & & & -1
\end{array}\right]$$

The empty spaces contain only 0's. The 3rd (resp. 4th, 5th) diagonal block has size $(x-1) \times (x-1)$ (resp. $(y-1) \times (y-1)$, $(z-1) \times (z-1)$). The Alexander polynomial satisfies:

$$\Delta_{x,y,z}(t) = \det(V_{x,y,z}^T - tV_{x,y,z})$$

$$= \det \left[\begin{array}{c|c|cccc|cccc|cccc}
 & 1-t & & & & & -t & & & & & & & \\ \hline
1-t & & t & & & & & & & & -t & & & \\ \hline
 & -1 & -1+t & -t & & & & & & & & & & \\
 & & 1 & \ddots & \ddots & & & & & & & & & \\
 & & & \ddots & \ddots & -t & & & & & & & & \\
 & & & & 1 & -1+t & & & & & & & & \\ \hline
1 & & & & & & -1+t & -t & & & & & & \\
 & & & & & & 1 & \ddots & \ddots & & & & & \\
 & & & & & & & \ddots & \ddots & -t & & & & \\
 & & & & & & & & 1 & -1+t & & & & \\ \hline
 & 1 & & & & & & & & & -1+t & -t & & \\
 & & & & & & & & & & 1 & \ddots & \ddots & \\
 & & & & & & & & & & & \ddots & \ddots & -t \\
 & & & & & & & & & & & & 1 & -1+t
\end{array} \right].$$

Expanding it in the $(x+1)$th column, we have;

$\Delta_{x,y,z}(t) = (-1+t)\Delta_{x-1,y,z}(t)$

$$-(-t)\det\left[\begin{array}{c|c|cccc|ccc|ccc}
 & 1-t & & & & & -t & & & & & \\ \hline
1-t & & t & & & & & & & -t & & \\ \hline
 & -1 & -1+t & -t & & & & & & & & \\
 & & 1 & \ddots & -t & & & & & & & \\
 & & & 1 & -1+t & -t & & & & & & \\
 & & & & & 1 & & & & & & \\ \hline
1 & & & & & & -1+t & -t & & & & \\
 & & & & & & 1 & \ddots & -t & & & \\
 & & & & & & & 1 & -1+t & & & \\ \hline
 & 1 & & & & & & & & -1+t & -t & \\
 & & & & & & & & & 1 & \ddots & -t \\
 & & & & & & & & & & 1 & -1+t
\end{array}\right]$$

$= (-1+t)\Delta_{x-1,y,z}(t) + t\Delta_{x-2,y,z}(t).$

If $\Delta_{i,y,z}(t) = (-1)^i(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots)$ for $i = x-1$ and $x-2$, then

$$\begin{aligned}
\Delta_{x,y,z}(t) &= (-1+t)(-1)^{x-1}(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots) + t(-1)^{x-2}(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots) \\
&= (-1)^x(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots).
\end{aligned}$$

In fact, $\Delta_{x,y,z}(t) = (-1)^{x+y+z}(1 - 5t + \cdots)$ for all $x, y, z \in \{2, 3\}$. By induction, $\Delta_{x,y,z}(t) = (-1)^{x+y+z}(1 - 5t + \cdots)$ for all $x, y, z \geq 2$.

Other cases follow by similar arguments. □

Finally we are ready to prove the theorem.

Proof of Theorem 2.3.2. By Lemmas 2.3.4, 2.3.6, 2.3.8, our knot $\mathcal{K}_{2m}$ where $(m \geq 1)$ cannot be a 3-braid. Thus by Lemma 2.3.3, Theorem 2.3.2 follows. □

Chapter 3

Uniqueness of the algebraic crossing number at minimal braid index

3.1 Sharpness of the MFW inequality and conjectures

It has been conjectured (see [11] p.357 for example) that the exponent sum in a minimal braid representation is a knot invariant.

Conjecture 3.1.1 (Main Conjecture): *Let $\mathcal{K}$ be a knot type of braid index $b_{\mathcal{K}}$. If K^1 and K^2 are braid representatives of $\mathcal{K}$ with $b_{K^1} = b_{K^2} = b_{\mathcal{K}}$ then their algebraic crossing numbers have $c_{K^1} = c_{K^2}$*

We deform it into a stronger form:

Conjecture 3.1.2 *Let $\mathcal{B}_{\mathcal{K}}$ be the set of braid representatives of $\mathcal{K}$. Let $\Phi : \mathcal{B}_{\mathcal{K}} \to \mathbb{N} \times \mathbb{Z}$ be a map such that $\Phi(K) := (b_K, c_K)$ for $K \in \mathcal{B}_{\mathcal{K}}$. Then there exists a* *unique* $c_{\mathcal{K}} \in \mathbb{Z}$ *with*

$$\Phi(\mathcal{B}_{\mathcal{K}}) = \{(b_{\mathcal{K}} + x + y, c_{\mathcal{K}} + x - y) \mid x, y \in \mathbb{N}\}, \tag{3.1}$$

the infinite quadrant region in Figure 3.1.

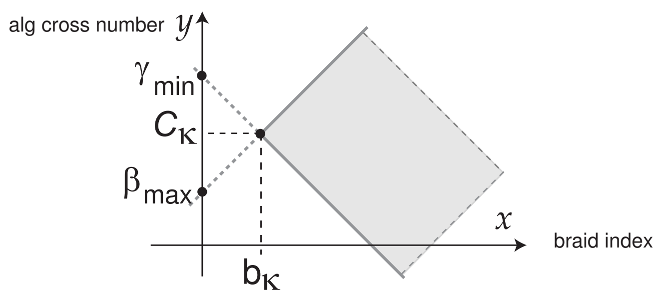

Figure 3.1: The region of braid representatives of $\mathcal{K}$

The inclusion "$\supset$" is trivial by the following argument: Let $K_\star \in B_{\mathcal{K}}$ be a minimal braid representative with $\Phi(K_\star) = (b_{\mathcal{K}}, c_{\mathcal{K}})$. Suppose $K \in B_{\mathcal{K}}$ is obtained from $K_\star$ after applying $(+)$-stabilization x-times and then $(-)$-stabilization y-times. Then $(b_{\mathcal{K}} + x + y, c_{\mathcal{K}} + x - y) = \Phi(K) \in \Phi(\mathcal{B}_{\mathcal{K}})$.

The MFW inequality (1.2) says that $c_K \geq -b_K + (d_+ + 1),\ c_K \leq b_K + (d_- - 1)$ for any $K \in \mathcal{B}_{\mathcal{K}}$. Thus

$$\Phi(\mathcal{B}_{\mathcal{K}}) \subset \{(x, y) \mid b_{\mathcal{K}} \leq x,\ -x + (d_+ + 1) \leq y \leq x + (d_- - 1)\}. \tag{3.2}$$

Before we provide examples of the conjectures we present:

Theorem 3.1.3 *Sharpness of the MFW inequality implies the truth of Conjectures 3.1.1 and 3.1.2. In particular;*

$$b_{\mathcal{K}} = \frac{d_+ - d_-}{2} - 1, \quad c_{\mathcal{K}} = \frac{d_+ + d_-}{2}.$$

Proof of Theorem3.1.3. Let $K_\star \in \mathcal{B}_\mathcal{K}$ be a minimal braid representative. Since the MFW inequality (1.2) is sharp on $\mathcal{K}$, we have $c_{K_\star} - b_\mathcal{K} + 1 = d_-$, and $d_+ = b_\mathcal{K} + c_{K_\star} - 1$, i.e., $c_{K_\star} = (d_+ + d_-)/2$ which is independent of the choice of $K_\star$. Thus we denote $c_{K_\star} =: c_\mathcal{K}$. In this case, the right side of (3.2) coincides with the right side of (3.1) and we have the other inclusion "$\subset$" of (3.1). □

Example 3.1.4 Both of the conjectures are true for unlinks, torus links, closed positive braids with a full twist (for example, the Lorenz links) [8], 2-bridge links and alternating fibred links [16], where the MFW inequality is sharp and one can apply Theorem 3.1.3.

Also Conjecture 3.1.1 applies to links with braid index ≤ 3 [4]. However, this case has been settled by a completely different way, the classification of 3-braids. Namely, any link of braid index 3 admits a unique conjugacy class of 3-braid representatives or has at most two conjugacy classes of 3-braid representatives related to each other by a flype move, which does not change the algebraic crossing number of the link.

Every transversal knot TK in S^3 with the standard contact structure is transversally isotopic to a transversal closed braid K [1]. The *Bennequin number* β is an invariant of transversal knots. By the identification of TK and K, we have

$$\beta(K) = c_K - b_K.$$

If Conjecture 3.1.2 is true for $\mathcal{K}$, then the *maximal* Bennequin number $\beta_{\max}(\mathcal{K})$ for the knot type $\mathcal{K}$ is realized on $\mathcal{B}_\mathcal{K} \ni K$'s with plotted vertices $\Phi(K)$ on the upper half boundary of the quadrant region of Figure 3.1. Let

$$\gamma(K) := c_K + b_K.$$

Then γ satisfies $\gamma_{\min}(\mathcal{K}) = -\beta_{\max}(\overline{\mathcal{K}})$ where $\overline{\mathcal{K}}$ is the mirror image of $\mathcal{K}$. See Figure 3.1. Thus, investigation of $\beta_{\max}(\mathcal{K})$ is related to Conjecture 3.1.2.

Another approach to Conjecture 3.1.2 is the following:

We say that $\mathcal{K}$ is *exchange reducible*, if for any braid representative K of $\mathcal{K}$, there exists a sequence of braid representatives $K = K_0 \longrightarrow K_1 \longrightarrow \cdots \longrightarrow K_m$, such that "$\longrightarrow$" is either exchange move, $\pm$-destabilization or braid isotopy and K_m is a minimal braid representative of $\mathcal{K}$.

Theorem 3.1.5 *If* (1) *a knot or link is exchange reducible and* (2) *Conjecture 3.1.1 is true for that knot or link, then Conjecture 3.1.2 is also true for that knot or link.*

The unlinks and iterated torus links are such examples.

Proof of Theorem 3.1.5. The first statement of the theorem is obtained by the definition of exchange reducibility.

($\star$) To the best of our knowledge, the unlinks (Theorem 1 of [6]) and iterated torus knots (Theorem 1 of [14]) are the only *known* examples having exchange-reducibility. (It's an interesting open question whether there are others.)

($\star\star$) Conjecture 3.1.1 is true for the unlinks and torus links, see Example 3.1.4.

($\star\star\star$) Theorem 3.2.7 below and ($\star\star$) imply the truth of Conjecture 3.1.1 for iterated torus knots.

By ($\star$), ($\star\star$), ($\star\star\star$), the second statement of the theorem follows. $\square$

3.2 Cabling and the conjecture

In this section, we study behavior of the deficit of the MFW inequality under cabling and prove Theorem 3.2.1. As a consequence, we observe that the Conjecture

3.1.1 is true for many of the knots and links that appeared in Chapter 2, where the MFW-inequality is not sharp, i.e., we cannot apply Theorem 3.1.3.

We also prove, in Theorems 3.2.7 and 3.2.9, that the truth of Conjecture 3.1.1 is "inherited" through cabling operations.

Let us fix some notation. Let $\mathcal{K}$ be a knot type. Denote the (p, q)-cable of $\mathcal{K}$ by $\mathcal{K}_{p,q}$. Let K be a braid representative with $b_K = b_{\mathcal{K}}$ and with algebraic crossing number c_K. Put

$$k := q - p \cdot c_K$$

and let $K_{p,q}$ denote the p-parallel copies of K with a k/p-twist (see Figure 3.2). We can assume that $K_{p,q}$ is on the boundary of a tubular neighborhood N of K

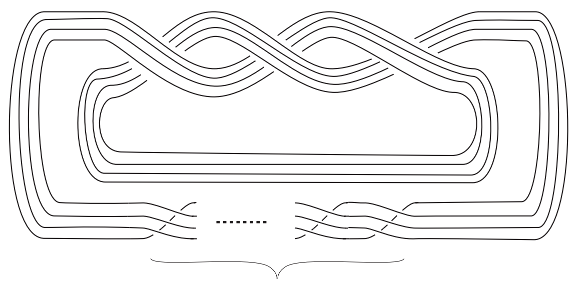

Figure 3.2: $(4, q)$-cable $(q = 4 \cdot 3 + k)$ of the right hand trefoil

(thus K is the core of solid torus N). Then

$$q = \mathrm{lk}(K_{p,q}, K)$$

where 'lk' is the linking number. Figure 3.2 shows that the algebraic crossing number of $K_{p,q}$ is

$$c_{K_{p,q}} = p^2 c_K + k(p-1) = q(p-1) + p \cdot c_K. \tag{3.3}$$

Thanks to [21], we know that the braid index of $\mathcal{K}_{p,q}$ satisfies

$$b_{\mathcal{K}_{p,q}} = p \cdot b_{\mathcal{K}}. \tag{3.4}$$

Theorem 3.2.1 *Suppose K^1 and K^2 are braid representatives of $\mathcal{K}$ with $b_{K^1} = b_{K^2} = b_{\mathcal{K}}$ and with distinct algebraic crossing numbers $c_{K^1} < c_{K^2}$, (i.e., Conjecture 3.1.1 does not apply to $\mathcal{K}$). Then the deficit $D_{\mathcal{K}_{p,q}}$ of the MFW inequality for (p,q)-cable $\mathcal{K}_{p,q}$ is;*

$$D_{\mathcal{K}_{p,q}} \geq \frac{p}{2}(c_{K^2} - c_{K^1}) \geq p. \tag{3.5}$$

Proof of Theorem 3.2.1. Thanks to (3.4), and by the construction of $(K^1)_{p,q}$ and $(K^2)_{p,q}$, they are both minimal braid representatives of $\mathcal{K}_{p,q}$ i.e., $b_{{K^1}_{p,q}} = b_{{K^2}_{p,q}} = b_{\mathcal{K}_{p,q}} = p \cdot b_{\mathcal{K}}$.

Let k_1, k_2 be integers satisfying $q = pc_{K^1} + k_1 = pc_{K^2} + k_2$. By (3.3) we have

$$c_{{K^1}_{p,q}} = p^2 c_{K^1} + k_1(p-1), \quad c_{{K^2}_{p,q}} = p^2 c_{K^2} + k_2(p-1).$$

Therefore, $c_{{K^2}_{p,q}} - c_{{K^1}_{p,q}} = p(c_{K^2} - c_{K^1})$. By (1.2) we have

$$c_{{K^2}_{p,q}} - b_{\mathcal{K}_{p,q}} + 1 \leq d_- \leq d_+ \leq c_{{K^1}_{p,q}} + b_{\mathcal{K}_{p,q}} - 1,$$

and by Definition 2.1.1,

$$D_{\mathcal{K}_{p,q}} \geq \frac{1}{2}(c_{{K^2}_{p,q}} - c_{{K^1}_{p,q}}) = \frac{p}{2}(c_{K^2} - c_{K^1}). \tag{3.6}$$

This is the first inequality of (3.5).

Notice that K^1 and K^2 are related each other by a sequence of Markov moves [2]. Let $K^1 = B_1 \to B_2 \to \cdots \to B_n = K^2$ be a Markov tower. Each arrow corresponds to either braid isotopy, stabilization or destabilization moves.

Let (x_i, y_i) be the braid index and the algebraic crossing number of B_i. Then $(x_{i+1}, y_{i+1}) - (x_i, y_i) = (0,0), (\pm 1, \pm 1)$ or $(\mp 1, \pm 1)$ depending on the move corresponding to the arrow between B_{i+1} and B_i. Since $x_1 = x_n = b_{\mathcal{K}}$ the difference $c_{K^1} - c_{K^2} = y_1 - y_n \neq 0$ must be an even integer. Therefore, we get the second inequality of (3.5). □

Corollary 3.2.2 *Conjecture 3.1.1 is true for all* $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$.

Proof. Knotscape computes that the deficit of 2-cable $\mathcal{K}_{2,2c_K+1}$ is 1 for each knot.

$\mathcal{K}$	$b_{\mathcal{K}}$	$D_{\mathcal{K}}$	K	c_K	$D_{\mathcal{K}_{2,2c_K+1}}$
9_{42}	4	1	$aaacBAAcB$	1	1
9_{49}	4	1	$aabbcbAbbcB$	7	1
10_{132}	4	2	$AbcaaaBBBcb$	3	1
10_{150}	4	1	$aabbcbABccB$	5	1
10_{156}	4	1	$aaacBAAcbAb$	3	1

Comparing with (3.5), each $\mathcal{K}$ must have unique algebraic crossing number. □

Remark 3.2.3 One can also observe by Knotscape that the $(2, 2c_{\mathcal{K}_n} + 1)$-cable of $\mathcal{K}_n = BM_{-1,-2,n,2}$ has deficit $= 1$ if $|n|$ is small. i.e., Conjecture 3.1.1 is true for $\mathcal{K}_n$ if $|n|$ is small.

Corollary 3.2.2 implies:

Corollary 3.2.4 *Conjecture 3.1.1 is true for the prime links* $(9_{42})^n$ *(see Figure 2.2)*.

Proof. We know that 9_{42} has unique algebraic crossing number $= 1$ by Corollary 3.2.2. Since each link component of $(9_{42})^n$ is 9_{42}, we have proved this corollary. □

With regard to the deficit of cabled links, we conjecture that:

Conjecture 3.2.5 *For any q, the limit $\lim_{p\to\infty} D_{\mathcal{K}_{p,q}}$ of deficits exists.*

Remark 3.2.6 If Conjecture 3.2.5 is true, then (3.5) of Theorem 3.2.1 implies the truth of Conjecture 3.1.1.

We present another property of cabling:

Theorem 3.2.7 *Let $\mathcal{K}$ be a non-trivial knot type. If Conjecture 3.1.1 is true for $\mathcal{K}$ then it is also true for $\mathcal{K}_{p,q}$ when $p \geq 2$.*

In particular, if $c_{\mathcal{K}}$ and $c_{\mathcal{K}_{p,q}}$ denote the unique algebraic crossing numbers of $\mathcal{K}$ and $\mathcal{K}_{p,q}$ respectively in their minimal braid representatives then we have

$$c_{\mathcal{K}_{p,q}} = (p-1)q + p \cdot c_{\mathcal{K}}.$$

Remark 3.2.8 Suppose $\mathcal{K}$ is the right hand trefoil. The MFW inequality is sharp on $\mathcal{K}$. Since its cable $\mathcal{K}_{2,7}$ has deficit $D_{\mathcal{K}_{2,7}} = 1$, we cannot apply Theorem 3.1.3. However Theorem 3.2.7 guarantees the truth of the conjecture for $\mathcal{K}_{2,7}$.

The following proof is inspired by the work of Williams [21], whose main result can be seen in formula (3.4). Note that his result holds not only for cable knots but also for generalized cable links. For the sake of completeness we repeat part of his discussion.

Proof of Theorem 3.2.7. Assume Conjecture 3.1.1 is true for $\mathcal{K}$ and denote the unique algebraic crossing number at minimal braid index by $c_{\mathcal{K}}$.

Let K be a braid representative of $\mathcal{K}$. Suppose K' is a braid representative of $\mathcal{K}_{p,q}$ on the boundary of a small tubular (solid torus) neighborhood N of K. We may regard the z-axis as the braid axis. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism of compact support so that ($\star$) $\phi(K') \subset \partial\phi(N)$ has exactly $p \cdot b_{\mathcal{K}}$ maxima and $p \cdot b_{\mathcal{K}}$ minima (both non-degenerate critical points) and no other critical points and ($\star\star$)

the "height" function $h : \partial\phi(N) \simeq T^2 \to \mathbb{R}$ is a Morse function. In particular, $\phi(K')$ has a braid position with braid index $p \cdot b_{\mathcal{K}}$.

By ($\star\star$), a generic intersection of the horizontal plane with $\partial\phi(N) \simeq T^2$ consists of disjoint simple closed curves. Furthermore, these simple closed curves are either meridians of T^2 or trivial in T^2 since $\mathcal{K}$ is knotted (Remark 1 of [21]).

Remark 2 in [21] says that there is a plane π (parallel to the (xz)-plane) intersecting transversely with T^2 in a meridian.

Let J be an innermost one among such meridians. Then J bounds a disk $d \subset \pi \cap \phi(N)$ which separates $\phi(K')$ into arcs $\{C_i\}$. Close each C_i with aid of some arc $D_i \subset d$ and set $\hat{K}_i := C_i \cup D_i$. See Figure 3.3.

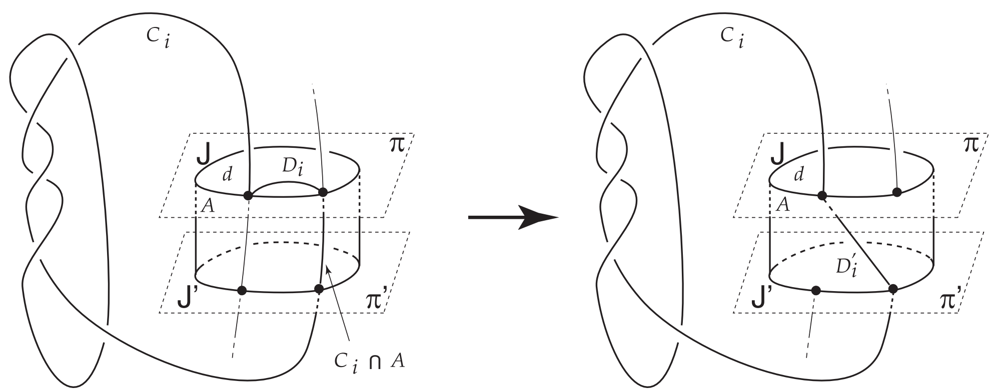

Figure 3.3: Construction of K_i from $\hat{K}_i$

Thanks to Remark 3 in [21], p of $\hat{K}_i$'s are non-trivial (i.e., do not bound any disk in $\phi(N)$) since the linking number of J and $\phi(K')$ pushed a little bit into the interior of $\phi(N)$ is p.

Discard trivial $\hat{K}_i$'s.

Our $\hat{K}_i$'s are not in a braid position. As in [21], we make them have a braid position: Choose another plane π' just below π and call the annulus between the two planes A (see Figure 3.3). We may assume that the other boundary curve

$J' \subset \partial A$ is parallel to J.

As in the passage of Figure 3.3 replace the arc $D_i \cup (C_i \cap A)$ (the left sketch) with $D_i' \subset A$ (the right sketch) and construct p-parallels;

$$K_i := (C_i - (C_i \cap A)) \cup D_i' \subset \partial\phi(N) \quad \text{for } i = 1, \cdots, p, \tag{3.7}$$

which is in a braid position. Also the K_i's are disjoint from each other and each is isotopic to the core of the solid torus $\phi(K) \simeq \mathcal{K}$, thus $b_{\mathcal{K}} \leq b_{K_i}$. Then we have

$$\begin{aligned} p \cdot b_{\mathcal{K}} &\leq \sum_{i=1}^{p} \{b_{K_i} = \text{number of max of } K_i\} \\ &\leq \{\text{number of max of } \phi(K')\} = p \cdot b_{\mathcal{K}} \end{aligned}$$

where the last equality holds by ($\star\star$) above. This implies that (†) there are no trivial $\hat{K}_i$'s (we didn't have to discard anything), (††) each knot has $b_{\mathcal{K}} = b_{K_i}$. Let $n, 0 \leq m < p$ be integers such that

$$q = p(c_{\mathcal{K}} + n) + m.$$

By (†), the p-component link $L := K_1 \cup \cdots \cup K_p$ is obtained from $\mathbf{K}' := \phi(K')$ by using the meridian disk d to create a cutout and adding an m/p-twist along the annulus A, then gluing the end-points. See Figure 3.4. In other words, L is the

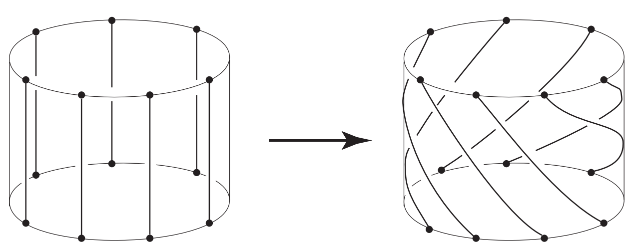

Figure 3.4: From $A \cap \mathbf{K}'$ to $A \cap L$, where $p = 7, m = 2$

$(p, p(c_{\mathcal{K}} + n))$-cable of K. From (††) we have $c_{K_i} = c_K = c_{\mathcal{K}}$. Therefore, L has the algebraic crossing number

$$\begin{aligned} c_L &= \sum_{i=1}^{p} c_{K_i} + \sum_{i \neq j} \mathrm{lk}(K_i, K_j) = p \cdot c_{\mathcal{K}} + p(p-1)(c_{\mathcal{K}} + n) \\ &= p^2 c_{\mathcal{K}} + p(p-1)n \end{aligned}$$

and $c_{\mathbf{K}'} - c_L = m(p-1)$. Thus,

$$\begin{aligned} c_{\mathbf{K}'} &= p^2 c_{\mathcal{K}} + p(p-1)n + m(p-1) \\ &= p^2 c_{\mathcal{K}} + (p-1)(pn+m) \\ &= p^2 c_{\mathcal{K}} + (q - p \cdot c_{\mathcal{K}}) \\ &= (p-1)q + p \cdot c_{\mathcal{K}}, \end{aligned}$$

which is independent of the choice of $\mathbf{K}' \in \mathcal{B}_{\mathcal{K}_{p,q}}$. Compare with (3.3). This concludes the uniqueness of the algebraic crossing number of $\mathcal{K}_{p,q}$ at minimal braid index.

□

A similar result to Theorem 3.2.7 holds for links.

Theorem 3.2.9 *Let $\mathcal{L} = \mathcal{K}^{(1)} \cup \cdots \cup \mathcal{K}^{(l)}$ be an l-component link of braid index $= b_{\mathcal{L}}$. Assume that each $\mathcal{K}^{(j)}$ is a non-trivial knot. Let $\mathcal{L}' := \mathcal{K}^{(1)}_{p,q_1} \cup \cdots \cup \mathcal{K}^{(l)}_{p,q_l}$ be the p-cable of $\mathcal{L}$ such that $q_j = \mathrm{lk}(\mathcal{K}^{(j)}, \mathcal{K}^{(j)}_{p,q_j})$ for $j = 1, \cdots, l$.*

If $\mathcal{L}$ and every component $\mathcal{K}^{(j)}$ have unique algebraic crossing numbers $c_{\mathcal{L}}, c_{\mathcal{K}^{(j)}}$ in minimal braid representations, then so does $\mathcal{L}'$.

Furthermore, let k_j satisfy $q_j = p \cdot c_{\mathcal{K}^{(j)}} + k_j$ then

$$c_{\mathcal{L}'} = p^2 c_{\mathcal{L}} + (p-1)(k_1 + \cdots + k_l). \tag{3.8}$$

Remark 3.2.10 The assumption for $\mathcal{K}^{(j)}$ in the second paragraph of Theorem 3.2.9 is essential by the following reason: If $b_{\mathcal{L}} = \sum_{j=1}^{l} b_{\mathcal{K}^{(j)}}$, then the existence of unique algebraic crossing number of $\mathcal{L}$ in minimal braid representation implies that each $\mathcal{K}^{(j)}$ also has unique algebraic crossing number. However, this is not true in general. For instance, assume that $\mathcal{L} = \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}$ is a 2-component link and has two braid representatives related to each other by a flype move as in Figure3.5. The thick gray arcs are parallel braid strands of $\mathcal{L}$. Braidings occur inside the

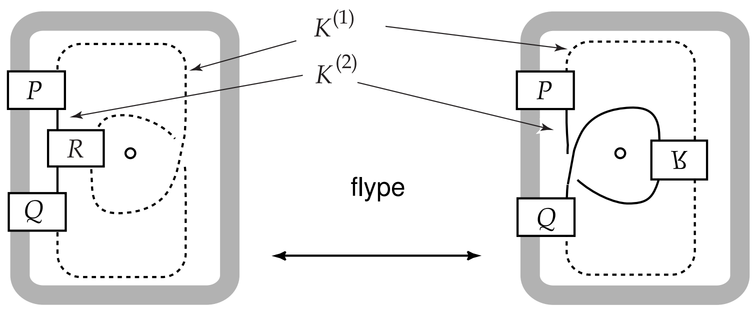

Figure 3.5: A flype move

boxes P, Q, R. In particular, box R contains even number of half twists of $\mathcal{K}^{(1)}$ (dashed arc) and $\mathcal{K}^{(2)}$ (black arc). The flype move preserves the number of braid strands and the algebraic crossing number of the link, but it changes the algebraic crossing numbers of link components $\mathcal{K}^{(1)}, \mathcal{K}^{(2)}$. Namely, in the passage from the left sketch to the right sketch, the algebraic crossing number of $\mathcal{K}^{(1)}$ decreases by 1 and the one for $\mathcal{K}^{(2)}$ increases by 1.

The above observation for a flype move leads to:

Proposition 3.2.11 *A flype move in general cannot be a composition of exchange moves and braid isotopy.*

It has been known that sometimes a flype move is a combination of exchange moves and braid isotopy, and sometimes it is realized by nothing but braid isotopy (see [4] for example).

Proof of Proposition 3.2.11. Any exchange move and braid isotopy do not change the algebraic crossing number of each link component. However, as we saw in Remark 3.2.10, some flype move does. Thus, we get Proposition 3.2.11. $\square$

Proof of Theorem 3.2.9. Suppose $L = K^{(1)} \cup \cdots \cup K^{(l)}$ is a minimal braid representative of $\mathcal{L} = \mathcal{K}^{(1)} \cup \cdots \cup \mathcal{K}^{(l)}$, i.e., $b_L = b_{\mathcal{L}}$. Let $c_{x,y} := 2 \cdot \mathrm{lk}(\mathcal{K}^{(x)}, \mathcal{K}^{(y)})$ for $x \neq y$. Then

$$c_{\mathcal{L}} = c_L = \sum_{1 \leq x < y \leq l} c_{x,y} + \sum_{j=1}^{l} c_{\mathcal{K}^{(j)}}. \tag{3.9}$$

Let $k_j, n_j, 0 \leq m_j < p$ be integers with

$$q_j = p \cdot c_{\mathcal{K}^{(j)}} + k_j = p(c_{\mathcal{K}^{(j)}} + n_j) + m_j \quad \text{for } j = 1, \cdots, l. \tag{3.10}$$

Williams proved that the braid index of $\mathcal{L}' = \mathcal{K}^{(1)}_{p,q_1} \cup \cdots \cup \mathcal{K}^{(l)}_{p,q_l}$ is $p \cdot b_{\mathcal{L}}$ [21]. Let L' be a minimal braid representative of $\mathcal{L}'$. Let N_j be a tubular neighborhood of $K^{(j)}$. Let ϕ be a compact support diffeo morphism of $\mathbb{R}^3$ such that $\phi(L') =: K'^{(1)} \cup \cdots \cup K'^{(l)} \subset \partial\phi(N_1 \cup \cdots N_l)$ has a minimal braid position.

As we did in (3.7), for each $j = 1, \cdots, l$, construct p parallels $K^{(j)}_1, \cdots, K^{(j)}_p \subset \partial\phi(N_j)$ from each $K'^{(j)}$ by cutout by an (inner most) meridian disk $d_j \subset \phi(N_j)$ and adding an m_j/p-twist along annulus $A_j \subset \partial\phi(N_j)$ then gluing. Thus,

$$\mathrm{lk}(K^{(j)}_i, K^{(h)}) = \begin{cases} c_{\mathcal{K}^{(j)}} + n_j & \text{when } j = h, \\ \mathrm{lk}(K^{(j)}, K^{(h)}) = \frac{1}{2} c_{j,h} & \text{otherwise.} \end{cases} \tag{3.11}$$

Let

$$L_i := K^{(1)}_i \cup \cdots \cup = K^{(l)}_i \quad \text{for } i = 1, \cdots, p.$$

Thanks to [21] we know that $L_i \simeq L$ and $b_{L_i} = b_L = b_{\mathcal{L}}$. By assumption of Theorem 3.2.9, it follows $c_{L_i} = c_L = c_{\mathcal{L}}$. The $(p \cdot l)$-component link $L_1 \cup \cdots \cup L_p$ has the algebraic crossing number;

$$\begin{aligned}
c_{L_1\cup\cdots\cup L_p} &= \sum_{i=1}^{p} c_{L_i} + \sum_{x\neq y} \mathrm{lk}(L_x, L_y) \\
&= p \cdot c_{\mathcal{L}} + \sum_{i=1}^{p} (p-1)\mathrm{lk}(L_i, L) \\
&= p \cdot c_{\mathcal{L}} + \sum_{i=1}^{p} (p-1)\{\sum_{j=1}^{l} \mathrm{lk}(K_i^{(j)}, K^{(j)}) + \sum_{x\neq y} \mathrm{lk}(K_i^{(x)}, K^{(y)})\} \\
&\overset{(3.11)}{=} p \cdot c_{\mathcal{L}} + p(p-1)\{\sum_{j=1}^{l} (c_{\mathcal{K}^{(j)}} + n_j) + \sum_{x<y} c_{x,y}\} \\
&\overset{(3.9)}{=} p \cdot c_{\mathcal{L}} + p(p-1)(c_{\mathcal{L}} + \sum_{j=1}^{l} n_j) \\
&= p^2 c_{\mathcal{L}} + p(p-1)(\sum_{j=1}^{l} n_j).
\end{aligned}$$

Since only the difference between $L_1\cup\cdots\cup L_p$ and L' occur on the annuli $A_1, \cdots, A_l$, we have

$$\begin{aligned}
c_{L'} &= c_{L_1\cup\cdots\cup L_p} + (p-1)\sum_{j=1}^{l} m_j \\
&= p^2 c_{\mathcal{L}} + (p-1)\sum_{j=1}^{l} (pn_j + m_j) \\
&\overset{(3.10)}{=} p^2 c_{\mathcal{L}} + (p-1)\sum_{j=1}^{l} k_j,
\end{aligned}$$

which is independent of the choice of braid representative L'. □

With regard to Conjecture 3.1.2 we have:

Theorem 3.2.12 *Let $\mathcal{L} = \mathcal{K}^{(1)} \cup \cdots \cup \mathcal{K}^{(l)}$ be an l-component link satisfying all* *the assumptions in Theorem 3.2.9. If Conjecture 3.1.2 is true for* $\mathcal{L}$ *then it is also true for its p-cable* $\mathcal{L}' := \mathcal{K}^{(1)}_{p,q_1} \cup \cdots \cup \mathcal{K}^{(l)}_{p,q_l}$.

Proof of Theorem 3.2.12. Let L (resp. L') be a braid representative of $\mathcal{L}$ (resp. $\mathcal{L}'$). Take tubular neighborhoods $N = N_1 \cup \cdots \cup N_l$ of L (each N_j is a solid torus) and let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a compact support diffeo morphism such that $\phi(L) =: K^{(1)} \cup \cdots \cup K^{(l)}$, $\phi(L') =: K'^{(1)} \cup \cdots \cup K'^{(l)}$ have braid positions. They are not necessarily minimal braid representatives and in general $b_{\phi(L')} \neq p \cdot b_{\phi(L).}$ We may assume that $K'^{(j)} \subset \partial\phi(N_j) \simeq T^2$.

Let plane $\pi_j = \{(x, y, z) | y = y_0\}$, innermost meridian loop $J_j \subset \pi_j \cap \partial\phi(N_j)$, and meridian disk $d_j \subset \pi_j \cap \phi(N_j)$ be as in the proof of Theorem 3.2.7. We may assume that the braid axis is *not* contained in π_j i.e., $y_0 \neq 0$.

We deform $K'^{(j)}$ in the following way: Suppose sub-arcs $u \subset K'^{(j)}$ and $v \subset J_j$ bound a disk $\mathcal{D} \subset \partial\phi(N_j) \simeq T^2$. If $\mathcal{D}$ is innermost, then replace u with v. Repeat this until $K'^{(j)}$ and J_j do not bound any disk in T^2. Add up all the linking numbers of $\partial\mathcal{D}$'s with the z-axis and denote it by $x^{(j)} \geq 0$.

Next, from the deformed $K'^{(j)}$ above, construct p-parallels $K'^{(j)}_1, \cdots, K'^{(j)}_p$ as in (3.7). Since the plane π_j does not contain the z-axis, the m_j/p-twist along a thin annulus does not change the number of braid strands.

Suppose that $b_{K^{(j)}} = b_{\mathcal{K}^{(j)}} + y^{(j)}$ and $b_{K'^{(j)}_i} = b_{\mathcal{K}^{(j)}} + y^{(j)} + z^{(j)}_i$ with $y^{(j)}$, $y^{(j)} + z^{(j)}_i \geq 0$. Let $L_i := K^{(1)}_i \cup \cdots \cup = K^{(l)}_i$ then

$$b_{L_i} = \sum_{j=1}^{l} b_{K^{(j)}_i} = b_{\mathcal{L}} + \sum_{j=1}^{l} (y^{(j)} + z^{(j)}_i), \tag{3.12}$$

$$b_{\phi(L')} = \sum_{i=1}^{p} b_{L_i} + \sum_{j=1}^{l} x^{(j)} \overset{(3.12)}{=} p \cdot b_{\mathcal{L}} + \sum_{j=1}^{l} (x^{(j)} + p \cdot y^{(j)} + \sum_{i=1}^{p} z^{(j)}_i). \tag{3.13}$$

Since $L_i \simeq \mathcal{L}$, our assumption of this theorem and (3.12) give us

$$c_{\mathcal{L}} - \sum_{j=1}^{l}(y^{(j)} + z_i^{(j)}) \leq c_{L_i} \leq c_{\mathcal{L}} + \sum_{j=1}^{l}(y^{(j)} + z_i^{(j)}). \tag{3.14}$$

As in the proof of Theorem 3.2.9, let $k_j, n_j, 0 \leq m_j < p$ satisfy $q_j = p \cdot c_{\mathcal{K}^{(j)}} + k_j = p(c_{\mathcal{K}^{(j)}} + n_j) + m_j$. Then we have

$$\begin{aligned} c_{L_1 \cup \cdots \cup L_p} &= \sum_{i=1}^{p} c_{L_i} + \sum_{x \neq y} \operatorname{lk}(L_x, L_y) \\ &\overset{(3.14)}{\leq} p(c_{\mathcal{L}} + \sum_{j=1}^{l} y^{(j)}) + \sum_{i=1}^{p}\sum_{j=1}^{l} z_i^{(j)} + p(p-1)(c_{\mathcal{L}} + \sum_{j=1}^{l} n_j) \\ &= p^2 c_{\mathcal{L}} + p(p-1)(\sum_{j=1}^{l} n_j) + \sum_{j=1}^{l}(p \cdot y^{(j)} + \sum_{i=1}^{p} z_i^{(j)}), \end{aligned}$$

and

$$\begin{aligned} c_{\phi(L')} &\leq c_{L_1 \cup \cdots \cup L_p} + \sum_{j=1}^{l} m_j + \sum_{j=1}^{l} x^{(j)} \\ &\leq p^2 c_{\mathcal{L}} + (p-1)\sum_{j=1}^{l} k_j + \sum_{j=1}^{l}(x^{(j)} + p \cdot y^{(j)} + \sum_{i=1}^{p} z_i^{(j)}). \end{aligned} \tag{3.15}$$

Similarly,

$$p^2 c_{\mathcal{L}} + (p-1)\sum_{j=1}^{l} k_j - \sum_{j=1}^{l}(x^{(j)} + p \cdot y^{(j)} + \sum_{i=1}^{p} z_i^{(j)}) \leq c_{\phi(L')}. \tag{3.16}$$

We conclude the theorem by (3.8), (3.13), (3.15) and (3.16). □

3.3 Connect sum and the conjecture

We will prove the following:

Theorem 3.3.1 *If Conjecture 3.1.1 is true for knot types $\mathcal{K}^1$ and $\mathcal{K}^2$ then it is also true for the connect sum $\mathcal{K}^1 \sharp \mathcal{K}^2$.*

In particular, denoting the unique algebraic crossing numbers of $\mathcal{K}^i$ in minimal braid representatives by $c_{\mathcal{K}^i}$ we have

$$c_{\mathcal{K}^1 \sharp \mathcal{K}^2} = c_{\mathcal{K}^1} + c_{\mathcal{K}^2}.$$

Before we prove Theorem 3.3.1 let us recall two important known results:

Lemma 3.3.2 ([12] Theorem 2.12.) *Up to ordering of summands, there is a unique expression for a knot type $\mathcal{K}$ as a finite connect sum of prime knots.*

Lemma 3.3.3 (**The composite braid theorem,** [5].) *Let $\mathcal{K}$ be a composite link, and let K be an arbitrary closed n-braid representative of $\mathcal{K}$. Then there is an obvious composite n-braid representative $K^\bullet$ of $\mathcal{K}$ (see Figure 3.6) and a finite sequence of closed n-braids:*

$$K = K_0 \to K_1 \to \cdots \to K_m = K^\bullet$$

such that K_{i+1} is obtained from K_i by either braid isotopy or an exchange move.

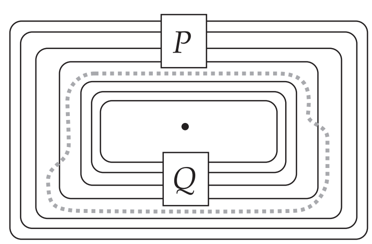

Figure 3.6: An obvious composite braid

Proof of Theorem 3.3.1. Since an exchange move does not change the algebraic crossing number, Lemmas 3.3.2 and 3.3.3 imply the truth of Theorem 3.3.1. □

As a corollary of Theorem 3.3.1 we have:

Theorem 3.3.4 *If Conjecture 3.1.2 is true for* $\mathcal{K}^1, \mathcal{K}^2$ *then it is also true for* $\mathcal{K}^1 \sharp \mathcal{K}^2$.

Proof of Theorem 3.3.4. Let K be a braid representative of $\mathcal{K}^1 \sharp \mathcal{K}^2$. By Lemma 3.3.3, after applying exchange moves and braid isotopy to K one can get a composite braid representative $K^\bullet = K^1 \sharp K^2$. Suppose $b_{K^i} = b_{\mathcal{K}^i} + x_i$ with $x_i \geq 0$. Then

$$\begin{aligned} b_K &= b_{K^\bullet} = b_{K^1} + b_{K^2} - 1 = (b_{\mathcal{K}^1} + b_{\mathcal{K}^2} - 1) + (x_1 + x_2) \\ &= b_{\mathcal{K}^1 \sharp \mathcal{K}^2} + (x_1 + x_2). \end{aligned} \tag{3.17}$$

Our assumption gives $c_{\mathcal{K}^i} - x_i \leq c_{K^i} \leq c_{\mathcal{K}^i} + x_i$. Since $c_K = c_{K^\bullet} = c_{K^1} + c_{K^2}$, we have $(c_{\mathcal{K}^1} + c_{\mathcal{K}^2}) - (x_1 + x_2) \leq c_K \leq (c_{\mathcal{K}^1} + c_{\mathcal{K}^2}) + (x_1 + x_2)$. Thanks to Theorem 3.3.1,

$$c_{\mathcal{K}^1 \sharp \mathcal{K}^2} - (x_1 + x_2) \leq c_K \leq c_{\mathcal{K}^1 \sharp \mathcal{K}^2} + (x_1 + x_2). \tag{3.18}$$

The truth of Conjecture 3.1.2 follows by (3.17), (3.18). □

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