Connect sum and transversely non simple knots

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(*Received*)

Abstract

We prove that transversal non-simplicity is preserved under taking connect sum, generalizing Veŕtesi's result [**11**].

1. *Introduction*

The goal of this paper is to prove:

Theorem 1·1. *Let K_1, K_2 be prime knot types in S^3. Let T_1, T_1' (resp. T_2, T_2') be transverse knots in (S^3, ξ_{sym}) of topological type K_1 (resp. K_2). Suppose that*

(i) *T_1, T_1' have the same self linking number but are not transversely isotopic, and*

(ii) *T_2, T_2' are transversely isotopic and cannot be transversely destabilized.*

Then the connect sums $T_1 \# T_2$ and $T_1' \# T_2'$ are not transversely isotopic.

(*We allow the possibility that K_1, K_2 have the same topological type and T_1, T_2 are transversely isotpic.*)

Remark 1·2. *If T_2 is transversely destabilizable and T_1, T_1' are related to each other by a negative flype move* (*see* [**5**] *for definition*), *then $T_1 \# T_2$ is transversely isotopic to $T_1' \# T_2'$.*

The idea behind Theorem 1·1 was a result of Veŕtesi, who proved a specialized version of it in her paper [**11**]. Her result holds only when the transversally non-simple knots in question can be distinguished by invariants in Heegaard Floer homology theory studied in [**9**]. We make no such restrictions. In fact, Birman-Menasco proved the existence of infinitely many transversely non simple knots [**4**] that the Heegaard Floer homology invariants do not distinguish [**9**].

We will give two proofs. The first proof is given in Section 2. It uses the theory of transversal closed braids, and is based upon ideas in [**1**], [**2**], [**5**], [**10**], [**12**]. The second proof is given in Section 3. It is inspired by a suggestion of John Etnyre that our theorem ought to follow from a theorem of Etnyre-Honda [**7**], and uses techniques based upon the well-known idea that every transversal knot type can be represented by a transversal pushoff of some Legendrian knot.

2. *Proof of Theorem* 1·1

Throughout this paper, $T, T_{i=1,2}$ denote transversal knots in (S^3, ξ_{sym}) the symmetric contact structure of S^3. Regard S^3 as a one point compactification of $\mathbb{R}^3$ equipped with the cylindrical coordinates (r, θ, z). Thanks to Bennequin [**1**] we identify transversal knots in (S^3, ξ_{sym}) with closed braids in $\mathbb{R}^3$ about the z-axis.

DEFINITION 2·1. *Suppose T_1, T_2 have braid presentations*

$$T_1 = b\ \sigma_{n-1} b'\ \sigma_{n-1}^{-1}, \quad T_1 = b\ \sigma_{n-1}^{-1} b'\ \sigma_{n-1}$$

where n is the braid index and b, b' are some braid words in $\sigma_1, \cdots, \sigma_{n-2}$, the standard generators of the braid group B_n. See Figure 1. *Then we say T_1 and T_2 are related to each other by an* exchange move.

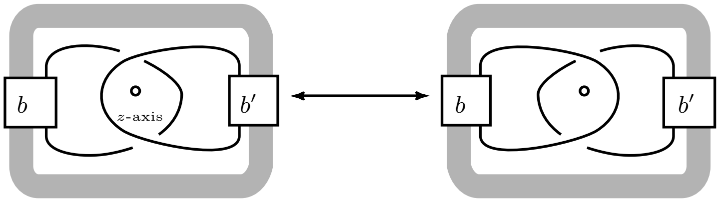

Fig. 1. An exchange move between T_1 (left) and T_2 (right). Thick gray bands are $(n-1)$ parallel strands and b, b' are some braidings.

As shown in [**5**, Lemma 1], an exchange move is a composition of a positive braid stabilization and a positive braid destabilization. Thus, an exchange move is a transversal isotopy. We use the following notations:

- $T_1 = T_2$ if T_1, T_2 are braid isotopic (conjugate);
- $T_1 \overset{e}{=} T_2$ if T_1, T_2 are exchange equivalent;
- $T_1 \sim T_2$ if T_1, T_2 are transversely isotopic;
- $S^+(T)$ for a transverse knot obtained by a number of positive braid stabilizations of T. It is known that $S^+(T) \sim T$.

Notice that $T_1 = T_2 \Rightarrow T_1 \overset{e}{=} T_2 \Rightarrow T_1 \sim T_2$.

DEFINITION 2·2. *We define the* braid connect sum $T_1 \# T_2$ *of T_1 and T_2 as in Figure* 2.

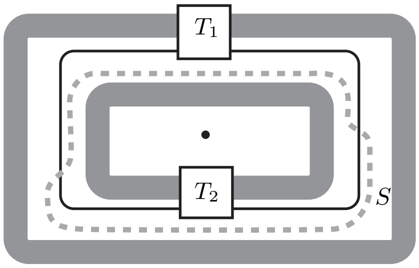

Fig. 2. Connect sum $T_1 \# T_2$ and dividing sphere S (dashed). Thick bands are multi strands.

This definition of connect sum is well defined thanks to Birman-Menasco [**2**]:

THEOREM 2·3. [**2**, Composite braid theorem] *Any composite braid can be reduced to the form in Figure* 2 *by exchange moves and braid isotopy.*

DEFINITION 2·4. *A* transverse stabilization *of T is a negative braid stabilization, that is, addition of an negative trivial kink about the z-axis. We call the inverse operation* transverse destabilization.

Now we are ready to prove our main theorem.

Proof of Theorem 1·1 Suppose, on the contrary, that $T_1 \# T_2 \sim T_1' \# T_2'$. Thanks to Orevkov-Shevchishin [**10**] and Wrinkle [**12**], after a number of positive braid stabilizations we get $S^+(T_1 \# T_2) = S^+(T_1' \# T_2')$. Due to Birman-Wrinkle [**5**, Lemma 2], one can slide a trivial stabilization loop to any place around the braid by exchange moves and braid isotopy. Therefore,

$$T_1 \# S^+(T_2) \overset{e}{=} S^+(T_1 \# T_2) = S^+(T_1' \# T_2') \overset{e}{=} T_1' \# S^+(T_2'). \tag{2·1}$$

To simplify notation, since $S^+(T) \sim T$, we will denote $S^+(T_2)$ by T_2 and $S^+(T_2')$ by T_2'. Let $f : S^3 \to S^3$ be a diffeomorphism corresponding to the composition of the exchange moves and braid isotopy in (2·1) so that

$$f(T_1 \# T_2) = T_1' \# T_2'.$$

We may think that the restriction of f to T_1 does not change θ-coordinate i.e.,

$$f|_{T_1}(r, \theta, z) = (r', \theta, z'). \tag{2·2}$$

In the following, we will deduce $T_1 \sim T_1'$, which contradicts our assumption.

Let $S \subset S^3$ (resp. S') be a 2-sphere separating T_2 (resp. T_2') and T_1 (resp. T_1') as in Figure 2. Let p, q (resp. p', q') denote the intersection points of $S \cap T_1 \# T_2$ (resp. $S' \cap T_1' \# T_2'$). Let $\tilde{T_1}, \tilde{T_2}$ (resp. $\tilde{T_1'}, \tilde{T_2'}$) be two arcs obtained by cutting $T_1 \# T_2$ (resp. $T_1' \# T_2'$) at p and q (resp. p', q'). Suppose $\partial T_2 = \{-p\} \cup \{q\}$ (resp. $\partial T_2' = \{-p'\} \cup \{q'\}$) with respect to the positive orientation of the braid $T_1 \# T_2$. We have

$$T_1' \# T_2' = \tilde{T_1'} \cup \tilde{T_2'} = f(\tilde{T_1} \cup \tilde{T_2}) = f(\tilde{T_1}) \cup f(\tilde{T_2}).$$

Let

$$A = f(\tilde{T_2}) \cap \tilde{T_1'}, \quad B = f(\tilde{T_2}) \cap \tilde{T_2'}, \quad C = \tilde{T_2'} \setminus B, \quad D = \tilde{T_1'} \setminus A,$$

where arcs A, B, C or D may be empty and may have more than one component.

By small perturbation, $f(S) \cap S'$ consists of a number of disjoint circles and $f(S) \cup S'$ divides S^3 into a number of 3-balls. Since $T_1' \# T_2'$ intersects $f(S)$ (resp. S') only at two points $f(p), f(q)$ (resp. p', q'), some of the balls do not intersect $T_1' \# T_2'$. We deform $f(S)$ to remove such empty balls, starting with the innermost one without moving $T_1' \# T_2'$. See Figure 3. We use the same notation $f(S)$ for the changed $f(S)$.

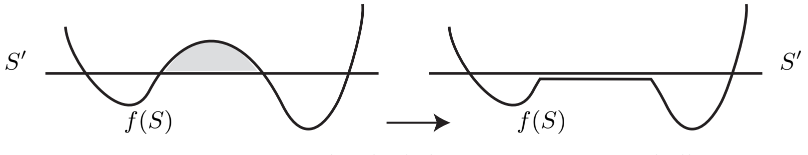

Fig. 3. Removing the shaded innermost empty 3-ball.

Eventually we have the following five cases. Note that by (2·2), we have $\theta_{f(p)} = \theta_p$ and $\theta_{f(q)} = \theta_q$. See Figure 4.

- $f(S) \cap S'$ consists of one circle and S^3 is divided by $f(S) \cup S'$ into four 3-balls. Cyclic order of $\theta_p, \theta_q, \theta_{p'}, \theta_{q'}$ for each case is;

  Case (1) $\theta_p < \theta_{q'} < \theta_q < \theta_{p'}$; Case (1′) $\theta_p < \theta_{p'} < \theta_q < \theta_{q'}$.

- $f(S)$ and S' are disjoint, and S^3 is divided by $f(S) \cup S'$ into two 3-balls and $S^2 \times (0,1)$.

$$\text{Case (2) } \theta_p < \theta_{q'} < \theta_{p'} < \theta_q; \quad \text{Case (3) } \theta_p < \theta_{p'} < \theta_{q'} < \theta_q;$$

$$\text{Case (4) } \theta_p < \theta_q < \theta_{q'} < \theta_{p'}; \quad \text{Case (5) } \theta_p < \theta_q < \theta_{p'} < \theta_{q'}.$$

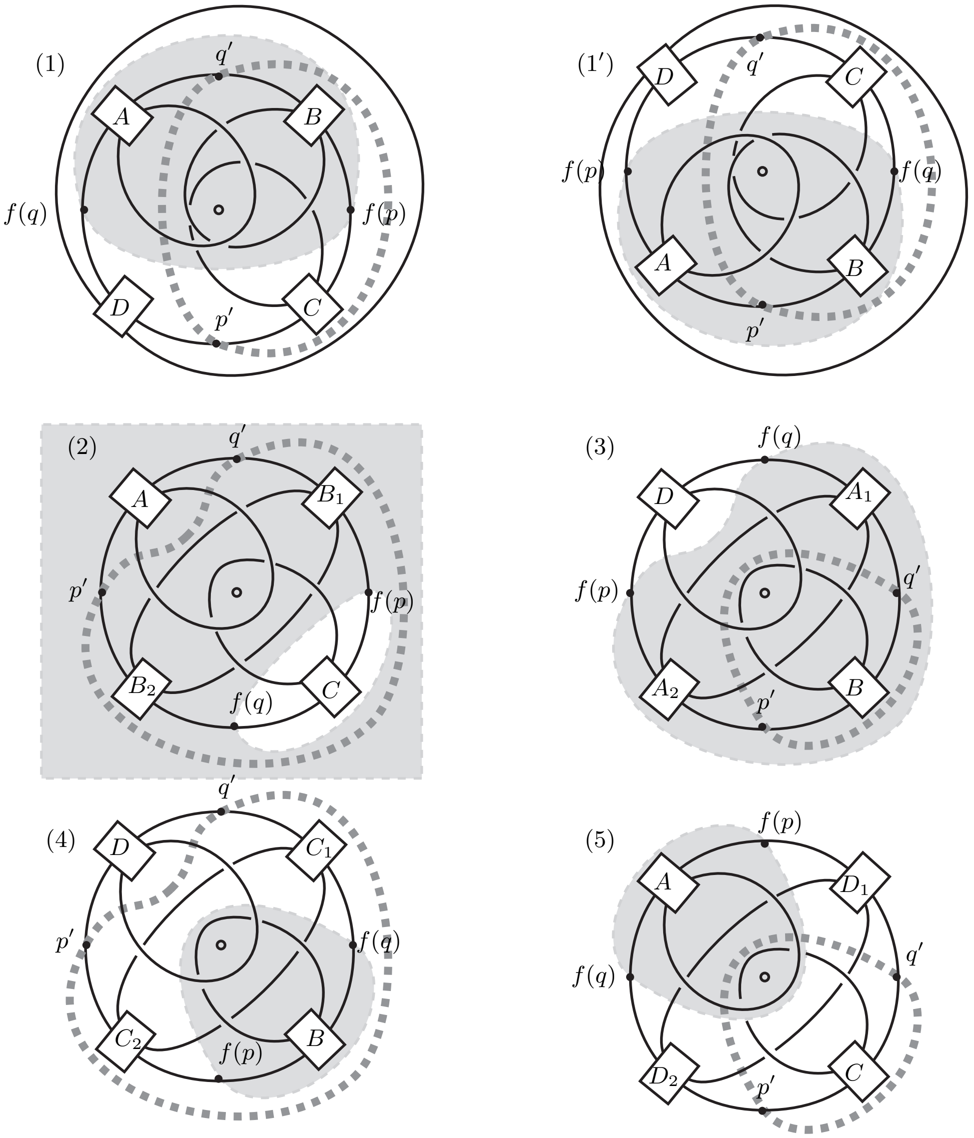

Fig. 4. Transverse knot $T_1'\#T_2'$ where $A_1 \cup A_2 = A$, $B_1 \cup B_2 = B$, $C_1 \cup C_2 = C$ and $D_1 \cup D_2 = D$. Sphere $S' = \partial\Delta'$ is dashed. The 3-ball $f(\Delta)$ containing $f(\tilde{T_2})$ is shaded. Braid strands may be weighted.

Let $\overline{A \cup D}$ (resp. $\overline{B \cup C}$) be a closed braid obtained by filling the braid blocks B, C (resp. A, D) with trivial braid strands of braid index 1. They are determined uniquely up to braid isotopy.

CLAIM 2·5. *We have* $\overline{A \cup D} = T_1'$ *(resp.* $\overline{B \cup C} = T_2'$*).*

Proof. This is clear from Definition 2·2 of the connect sum. □

Let $\overline{C \cup D}$ (resp. $\overline{A \cup B}$) be a closed braid obtained by filling the braid blocks A, B (resp. C, D) with trivial braid strands of braid index 1. Note that $\overline{C \cup D}$ (resp. $\overline{A \cup B}$) is unique up to exchange moves, since after an exchange move f the sphere $f(S)$ is pierced by the braid axes more than twice in general and there may be several ways to take the braid closure.

Claim 2·6. *We can extend* $f(\tilde{T}_1) = C \cup D$ (*resp.* $f(\tilde{T}_2) = A \cup B$) *to the closed braid* $\overline{C \cup D}$ (*resp.* $\overline{A \cup B}$) *and the extension satisfies* $\overline{C \cup D} \overset{e}{=} T_1$ (*resp.* $\overline{A \cup B} \overset{e}{=} T_2$).

Proof. Let $\Delta \subset S^3 \setminus S$ be the 3-ball containing $\tilde{T}_2$. Join the end points p, q of $\tilde{T}_1$ by an arc $\alpha \subset \Delta$ so that $T_1 = \tilde{T}_1 \cup \alpha$.

Suppose $f = f_k \circ \cdots \circ f_0$ where f_i $(i = 0, 1, \cdots, k)$ is an exchange move. We may assume, if necessary, as in Figure 5 by using some braid isotopy with property (2·2), that p, q are fixed by f_i. That is, each of the two exchange arcs is a sub-arc of either $\tilde{T}_1$ or $\tilde{T}_2$.

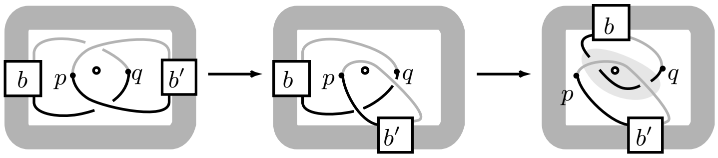

Fig. 5. By braid isotopy, points p and q can be outside the shaded exchange domain. Arc $\tilde{T}_1$ is colored gray and $\tilde{T}_2$ is black.

Here we recall some of Birman-Menasco's foundational work in [2]. Let $H_{\theta_0} \subset \mathbb{R}^3$ be the half-plane $\{(r, \theta_0, z) | 0 < r, z \in \mathbb{R}\}$. For all but a finite number of $\theta \in [0, 2\pi)$ the intersection $S \cap H_\theta$ is a disjoint union of simple closed curves and properly embedded arcs, in which case H_θ is called *non-singular*. Thanks to [2, Lemma 1 and p.135] we may assume that there are no simple closed curves. When H_θ is non-singular, we call an arc $\beta \subset S \cap H_\theta$ *essential* if the both components of H_θ split along β are pierced by our transverse knot.

An exchange move of a composite braid with separating sphere S is done by three steps. See Figure 6. First, without moving the braid we change the shape of S to make a "room" for the coming exchange move, which can be done in the exchange domain (the shaded 3-ball in the right sketch of Figure 5) away from p and q. Second, we move the braid by fixing S. Third, move S by isotopy in order to remove all the inessential arcs from the inner-most one (as in [2, p.120]) if they occur in the above procedure.

Based on this, for each exchange move f_i we define how the joining arc α changes. First, change S as f_i does by fixing $\tilde{T}_1$ but moving α in the 3-ball Δ by some exchange move if necessary, and $p, q \in S$ are fixed. Second, (a) if an exchange move f_i involves sub-arcs of $\tilde{T}_1$ then change $\tilde{T}_1 \cup \alpha$ to $f_i(\tilde{T}_1) \cup \alpha$, (b) otherwise, do not change $\tilde{T}_1 \cup \alpha$ at all. Thus, $\tilde{T}_1 \cup \alpha$ and its result after the changes are related to each other by an exchange move up to braid isotopy. Third, remove the inessential arcs as f_i does. Since the arc α is of braid index $= 1$, even after the third step there may exist inessential arcs.

Repeating this construction for each f_i, we obtain a closed braid $\overline{f(\tilde{T}_1)}$ extending $f(\tilde{T}_1) = C \cup D$ and $\overline{f(\tilde{T}_1)} \overset{e}{=} T_1$.

Furthermore, $\overline{f(\tilde{T}_1)}$ is exchange equivalent to the closed braid $A \cup B \cup C \cup D$ filling

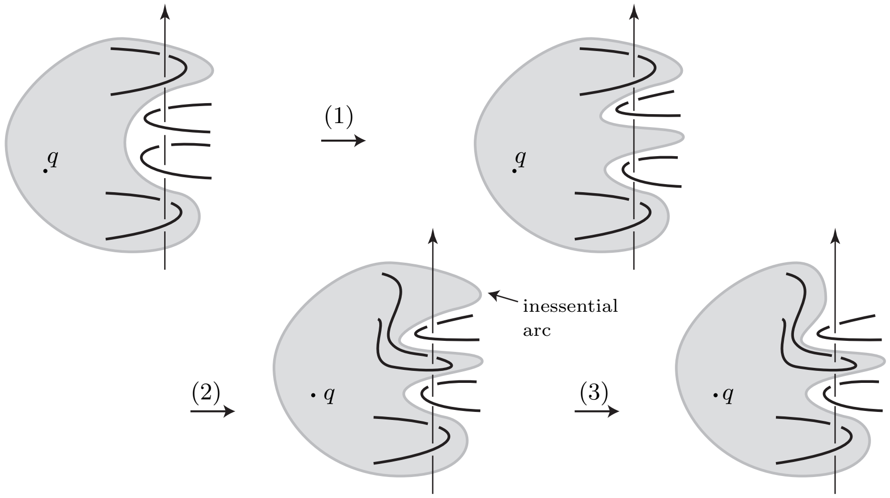

Fig. 6. The three steps of an exchange move. The 3-ball Δ is shaded. Points p (not in the sketch) and q are fixed.

the braid blocks A, B (or $A_i, B_j,\ i, j = 1, 2$) by trivial arcs of braid index $= 1$, i.e., $\overline{f(\tilde{T}_1)} \overset{e}{=} \overline{C \cup D}$. $\square$

We continue the proof of Theorem 1·1.

Construct closure $\overline{A}$ by filling the braid boxes B, C, D by trivial braid arcs of braid index $= 1$. Closure $\overline{A}$ is unique up to exchange move. Similarly, construct closures $\overline{B}$, $\overline{C}$, $\overline{D}$. By this construction and Theorem 2·3, in all the five cases we have $T_1' \# T_2' = A \cup B \cup C \cup D = \overline{A} \# \overline{B} \# \overline{C} \# \overline{D}$.

By Claims 2·5, 2·6 and Theorem 2·3 we have

$$\begin{aligned} T_1 &\overset{e}{=} \overline{C \cup D} \overset{e}{=} \overline{C} \# \overline{D}, \\ T_2 &\overset{e}{=} \overline{A \cup B} \overset{e}{=} \overline{A} \# \overline{B}, \\ T_1' &= \overline{A \cup D} \overset{e}{=} \overline{A} \# \overline{D}, \\ T_2' &= \overline{B \cup C} \overset{e}{=} \overline{B} \# \overline{C}. \end{aligned}$$

(Case 1) Recall K_2 is the topological type of $T_2 \sim T_2'$ and it is a prime knot. Since $T_2 \overset{e}{=} \overline{A} \# \overline{B}$ and $T_2' \overset{e}{=} \overline{B} \# \overline{C}$ we have two cases to study.

(Case 1.1) Suppose that topologically $\overline{B}$ is K_2 and $\overline{A}, \overline{C}$ are the unknot. Since T_2 and T_2' are not transversely destabilizable and the unknot is exchange reducible [**3**, Theorem 1], it follows that $\overline{A}, \overline{C}$ are transversely isotopic to the 1-strand braid representative of the unknot. Thus

$$T_1 \overset{e}{=} \overline{C} \# \overline{D} \sim \overline{D} \sim \overline{A} \# \overline{D} \overset{e}{=} T_1'.$$

(Case 1.2) Suppose that topologically $\overline{B}$ is the unknot and $\overline{A}, \overline{C}$ are K_2. Since $T_2 \sim T_2'$ cannot be transversely destabilized, $\overline{B}$ is transversely isotopic to the 1-strand braid and $\overline{A} \sim \overline{A} \# \overline{B} \overset{e}{=} T_2 \sim T_2' \overset{e}{=} \overline{B} \# \overline{C} \sim \overline{C}$. Thus

$$T_1 \overset{e}{=} \overline{C} \# \overline{D} \sim \overline{A} \# \overline{D} \overset{e}{=} T_1'.$$

Similar arguments hold for (Case $1'$).

(Case 2) Since K_2 is a prime knot, we have two cases to study.

(Case 2.1) Suppose that topologically $\overline{B}$ is K_2 and $\overline{A}, \overline{C}$ are the unknot. Since T_2 cannot be transversely destabilized, $\overline{A}, \overline{C}$ are transversely isotopic to the 1-strand braid. Thus

$$T_1 \overset{e}{=} \overline{C} \sim \overline{A} = T_1'.$$

(Case 2.2) Suppose that topologically $\overline{B}$ is the unknot and $\overline{C}$ is K_2. Since $T_2' \overset{e}{=} \overline{B}\#\overline{C}$ cannot be transversely destabilized, $\overline{B}$ is transversely isotopic to the 1-strand braid. Therefore,

$$T_1 \overset{e}{=} \overline{C} \sim \overline{C}\#\overline{B} \overset{e}{=} T_2' \sim T_2 \overset{e}{=} \overline{A}\#\overline{B} \sim \overline{A} = T_1'.$$

(Case 3) Since $\overline{A}\#\overline{B} \overset{e}{=} T_2 \sim T_2' = \overline{B}$ cannot be transversely destabilized, $\overline{A}$ is transversely isotopic to the 1-strand braid. Therefore,

$$T_1 \overset{e}{=} \overline{D} \sim \overline{D}\#\overline{A} \overset{e}{=} T_1'.$$

(Case 4) Since $\overline{B}\#\overline{C} \overset{e}{=} T_2' \sim T_2 \overset{e}{=} \overline{B}$ cannot be transversely destabilized, $\overline{C}$ is transversely isotopic to the 1-strand braid. Therefore,

$$T_1 \overset{e}{=} \overline{C}\#\overline{D} \sim \overline{D} = T_1'.$$

(Case 5) We have $\overline{A} \overset{e}{=} T_2 \sim T_2' = \overline{C}$ thus $T_1 \overset{e}{=} \overline{C}\#\overline{D} \overset{e}{=} \overline{A}\#\overline{D} \overset{e}{=} T_1'$.

In all the cases, we obtain $T_1 \sim T_1'$ which contradicts our assumption that $T_1 \nsim T_1'$. □

3. *Appendix*

In this section, we give an alternative proof of Theorem 1·1 as a corollary of Etnyre-Honda's classification of connected sum Legendrian knots [**7**, Theorem 3.4]. Since our ambient manifold is S^3 we use its $\mathbb{R}^3$-version taken from [**8**].

Let $\mathcal{K} \subset \mathbb{R}^3$ be a topological knot type and $\mathcal{L}(\mathcal{K})$ be the set of Legendrian representatives of $\mathcal{K}$. We denote by $\mathcal{S}_{\pm}(L)$ the $\pm$-stabilization of the Legendrian knot L.

Theorem 3·1. [**7**, Theorem 3.4] [**8**, Theorem 5.11] *Let $\mathcal{K} = \mathcal{K}_1\#\cdots\#\mathcal{K}_n$ be a topological connected sum knot type in $\mathbb{R}^3$. The map*

$$\frac{\mathcal{L}(\mathcal{K}_1) \times \cdots \times \mathcal{L}(\mathcal{K}_n)}{\approx} \to \mathcal{L}(\mathcal{K}_1\#\cdots\#\mathcal{K}_n)$$

is a bijection where the equivalence relation $\approx$ is generated by

(i) $(\ldots, \mathcal{S}_{\pm}(L_i), \ldots, L_j, \ldots) \approx (\ldots, L_i, \ldots, \mathcal{S}_{\pm}(L_j), \ldots)$

(ii) $(L_1, \ldots, L_n) \approx (L_{\sigma(1)}, \ldots, L_{\sigma(n)})$ *where σ is a permutation of $1, \ldots, n$ such that $\mathcal{K}_i = \mathcal{K}_{\sigma(i)}$.*

We also recall a theorem by Epstein-Fuchs-Meyer [**6**]: Let $L \subset (S^3, \xi_{std})$ be a Legendrian knot and $T_{\pm}(L)$ be its positive and negative transverse push offs.

Theorem 3·2. [**6**, Theorem 2.1] *Legendrian knots $L_1, L_2 \subset (S^3, \xi_{std})$ are negatively stably isotopic (i.e., $\mathcal{S}_-^k(L_1) = \mathcal{S}_-^l(L_2)$ for some $k, l \geq 0$) if and only if $T_{\pm}(L_1) \sim T_{\pm}(L_2)$.*

The next proposition explains the relationship between positive Legendrian stabilization and transverse stabilization: Let $S(T)$ be a transverse stabilization of T. Under the identification of T with a closed braid, $S(T)$ is an negative braid stabilization of T.

Proposition 3·3. *Let $L \subset (S^3, \xi_{std})$ be a Legendrian knot. We have $T_+(\mathcal{S}_+(L)) \sim S(T_+(L))$.*

Here is an alternative proof of Theorem 1·1.

Proof. Suppose that $T_1 \# T_2 \sim T_1' \# T_2'$. Let $L_1 \# L_2$ (resp. $L_1' \# L_2'$) be a Legendrian push-off of $T_1 \# T_2$ (resp. $T_1' \# T_2'$) so that

$$T_+(L_1 \# L_2) \sim T_1 \# T_2 \sim T_1' \# T_2' \sim T_+(L_1' \# L_2').$$

By Theorem 3·2, $\mathcal{S}_-^k(L_1 \# L_2) = \mathcal{S}_-^l(L_1' \# L_2')$ in $\mathcal{L}(\mathcal{K}_1 \# \mathcal{K}_2)$ for some $k, l \geq 0$. Since Legendrian stabilization is well defined (we can move the zig-zags anywhere) we have $L_1 \# \mathcal{S}_-^k(L_2) = L_1' \# \mathcal{S}_-^l(L_2')$. By Theorem 3·1,

$$(L_1, \mathcal{S}_-^k(L_2)) \approx (L_1', \mathcal{S}_-^l(L_2')).$$

Recall our assumption that $T_2 \sim T_2'$ cannot be tansversely destabilized. Thus Proposition 3·3 implies that:

CLAIM 3·4. *L_2 and L_2' cannot be positive Legendrian destabilizable.*

Suppose that $\mathcal{K}_1 \neq \mathcal{K}_2$. By the definition of $\approx$ in Theorem 3·1 and Claim 3·4, we have for some $m, n \in \mathbb{Z}$ and $x, y \in \mathbb{Z}_{\geq 0}$.

$$(\mathcal{S}_+^{-x}\mathcal{S}_-^m(L_1),\ \mathcal{S}_+^x\mathcal{S}_-^{k-m}(L_2)) = (\mathcal{S}_+^{-y}\mathcal{S}_-^n(L_1'),\ \mathcal{S}_+^y\mathcal{S}_-^{l-n}(L_2')).$$

By Theorem 3·2 and Proposition 3·3 we have

$$S^x(T_2) \sim T_+(\mathcal{S}_+^x\mathcal{S}_-^{k-m}(L_2)) = T_+(\mathcal{S}_+^y\mathcal{S}_-^{l-n}(L_2')) \sim S^y(T_2')$$

and obtain $x = y$. Then,

$$S^{-x}(T_1) \sim T_+(\mathcal{S}_+^{-x}\mathcal{S}_-^m(L_1)) \sim T_+(\mathcal{S}_+^{-y}\mathcal{S}_-^n(L_1')) \sim S^{-x}(T_1')$$

and we obtain $T_1 \sim T_1'$, which is a contradiction.

Suppose that $\mathcal{K}_1 = \mathcal{K}_2$. We have either

$$(\mathcal{S}_+^{-x}\mathcal{S}_-^m(L_1),\ \mathcal{S}_+^x\mathcal{S}_-^{k-m}(L_2)) = (\mathcal{S}_+^{-y}\mathcal{S}_-^n(L_1'),\ \mathcal{S}_+^y\mathcal{S}_-^{l-n}(L_2'))$$

or

$$(\mathcal{S}_+^{-x}\mathcal{S}_-^m(L_1),\ \mathcal{S}_+^x\mathcal{S}_-^{k-m}(L_2)) = (\mathcal{S}_+^y\mathcal{S}_-^{l-n}(L_2'),\ \mathcal{S}_+^{-y}\mathcal{S}_-^n(L_1'))$$

for some $m, n \in \mathbb{Z}$ and $x, y \in \mathbb{Z}_{\geq 0}$. The first case is covered in the case when $\mathcal{K}_1 \neq \mathcal{K}_2$. In the latter case, we obtain by Theorem 3·2 and Proposition 3·3,

$$S^{-x}(T_1) \sim S^y(T_2') \text{ and } S^x(T_2) \sim S^{-y}(T_1')$$

Since $T_2 \sim T_2'$ we have $S^{x+y}(T_2) \sim S^{x+y}(T_2')$. Therefore,

$$T_1 \sim S^{-x+x}(T_1) \sim S^{-y+y}(T_1') \sim T_1',$$

which is a contradiction. □

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