Research statement

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1 Introduction

My research interest lies in geometric topology. More specifically, I study knots and links from braid theory point of view. I am also interested in how braid theory interacts with Ozsváth-Szabó Heegaard Floer homology, contact geometry and Khovanov homology which are closely related to each other.

I have been working on problems related to knot invariants. Classification of knots and links is the main goal of knot theory, for which the study of various knot invariants are crucial.

Braid index is one of the classical invariants of knots and links. It is characterized via differential geometry as the holonomic curvature-torsion of an embedded circle (up scales) [6]. In general, it is hard to determine the braid index of a knot. However, some upper and lower bounds are known. As a lower bound, the Morton-Franks-Williams (MFW) inequality [15], [7] is quite effective for knots up to 10-crossings in the standard knot table. Furthermore, since this lower bound can be computed combinatorially, it is a very practical tool to determine braid index. In [11], I have found a sufficient condition for *non-sharpness* of the inequality (Theorem 2.1) and have constructed an infinite sequence of knots on which the inequality is *not* sharp (Theorem 2.2). To prove this, the enhanced Milnor number [18] plays an important role. The enhanced Milnor number also appears in the top-most term in the knot Floer homology of fibered knots. Furthermore, I have shown that the deficit of the inequality can be arbitrarily large (Theorem 2.3).

The *writhe* of a knot is another important quantity of a knot. It is invariant under the Reidemeister moves II and III. It has been conjectured that the writhe is invariant when a knot is presented as a closed braid of minimal braid index. I have formulated the conjecture into a stronger conjecture (Conjecture 3.4) and am approaching it geometrically by studying intersections of two disks. Theorem 3.6, 3.7 and 3.8 are partial results. Conjecture 3.4 is related to a problem in the study of transversal and Legendrian knots: If Conjecture 3.4 is true, then the maximal Bennequin number of a knot K is realized on a closed braid representative of K with minimal braid index.

2 Non-sharpness of Morton-Franks-Williams inequality

2.1 Results

It is known that any knot K has closed braid representatives. The *braid index* of a knot K is the minimal number of braid strands to obtain K as the closure of a braid. The MFW-inequality gives a lower bound for braid index. Jones found that the inequality is sharp on all knots up to 10-crossing except for the five knots 9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156} [10].

Then one can pose a question: What conditions on a knot lead to non-sharpness of the MFW-inequality?

I have found such a sufficient condition [11]:

Theorem 2.1 *Let B be a closed braid of minimal braid index. Choose a crossing c of B. Change the sign of* (*resp. resolve the crossing*) *c and denote the new close braid by B'* (*resp. B''*). *If both B' and B'' can be destabilized k-times* ($k \geq 1$), *then the MFW-inequality is not sharp on B.*

More precisely, the deficit *$D(B)$ of the MFW-inequality have*

$$D(B) = k + \delta(B, c),$$

where $\delta(B, c)$ is a non-negative integer related to differences of v-degrees of the Skein polynomials of B, B' and B'' and deficits of B' and B''.

Theorem 2.1 gives a new *local* criterion for judging non-sharpness of the MFW-inequality. In stead of computing skein polynomials, one need look at *only one* crossing.

As an application of Theorem 2.1, I have shown:

Theorem 2.2 [11] *The* Birman-Menasco (BM) block-strand diagram (*see Figure 1*) *can carry infinitely many prime knots and links* (*including the Jones' five knots* 9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}) *on which the MFW-inequality is* not *sharp.*

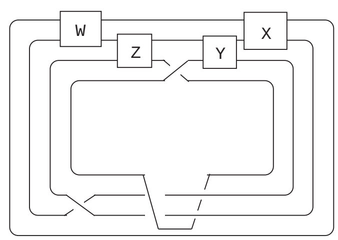

Figure 1: The Birman-Menasco diagram

In the study of braids, the braids up to braid index 3 are rather well known and classified, due to Birman, Menasco, Murasugi and others. However, braids with braid index ≥ 4 is much more difficult to work on. In fact, little is known about 4-braids except for certain well-behaved classes.

All the examples discussed in [11] have braid index ≥ 4. Indeed, the most involved part of Theorem 2.2, which I will describe in the following paragraphs, is to prove that the BM-diagram actually carries infinitely many 4-braids that *cannot* be reduced to 3-braids.

First, I choose an infinite sequence $\{B_n\}_{n=1}^{\infty}$ of braids carried by the BM-diagram. By Gabai's 'sutured manifolds' and 'C-product decompositions' [8], I conclude that every B_n is fibered. Therefore, by Giroux's theorem [9], the fibre surface of B_n is obtained from a disk by plumbing and deplumbing the Hopf-bands. Then I compute the *enhanced Milnor number* λ of B_n, defined by Neumann and Rudolph [18], which algebraically counts the number of negative Hopf-plumbings needed to construct the fiber surface of B_n. I find that every B_n has $\lambda = 1$.

Next, I list all the 3-braids with $\lambda = 1$ with help of Xu's standard forms of 3-braids [23]. A very useful feature of her standard form is that the associated Bennequin surface gives a Seifert surface of minimal genus. I find that only *four* infinite sequences of 3-braids can have $\lambda = 1$. These computations rely on Melvin and Morton's technique for 'Stalling twists' [14].

Finally, by comparing the Alexander polynomials of the knots in $\{B_n\}$ and the 3-braids in the four infinite sequences mentioned above, I conclude that B_n cannot be a 3-braid, i.e., it is a 4-braid. As a byproduct, I discover an interesting relation between Alexander polynomials of closed braids and Hopf-plumbings of Bennequin surfaces. Namely, coefficients of Alexander polynomials are stabilized after applying Hopf-plumbings sufficiently many times. The primeness of B_n follows from Thurston's hyperbolic Dehn surgery theorem [22], [19].

To the best of my knowledge, it is new to use the enhanced Milnor number for determination of braid index.

Here is another application of Theorem 2.1:

Theorem 2.3 *There is an infinite sequence of prime links on which the* deficit *of the MFW-inequality grows arbitrarily large.*

One can easily see that the deficit of the inequality can be arbitrarily large by taking the connected sum of knots on which the inequality is not sharp. This relies on the additivity of braid indices under connected sums [3]. I remark here that the construction in Theorem 2.3 is *not* the connected sum construction. Again, Thurston's hyperbolic Dehn surgery is applied to show primeness.

2.2 Future plans

So far, I have discussed non-sharpness of the MFW-inequality. I have also considered sharpness of the inequality and been working to prove the following:

Conjecture 2.4 *The MFW-inequality is sharp on knots with braid index* $= 3$.

3 The braid index and the writhe

3.1 Results

In this project, I study relations between the braid index and writhe number of a given knot. It has been conjectured that

Conjecture 3.1 *The writhe is uniquely determined at minimal braid index.*

The conjecture is justified for links with braid index ≤ 3 [2], torus links, positive closed braid with a full twist [7], 2-bridge links and alternating fibred links [16]. I try to approach this problem *without* any restriction on knots.

This problem is also related to the project discussed in Section 2.1 because if the MFW-inequality is sharp on a link L then Conjecture 3.1 is true for L. However the sharpness of the MFW-inequality is sufficient for the truth of Conjecture 3.1 it is not a necessary condition:

Theorem 3.2 *The conjecture is true for the five knots* 9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}*, on which the inequality is not sharp (see Section 2.1).*

This theorem follows as a corollary of the next theorem:

Theorem 3.3 *Let* $K_{p,q}$ *be the* (p,q)*-cable of knot (or link)* K*. Assume* K *has braid representatives* B_1, B_2 *at the minimal braid index with distinct writhes,* w_1 *and* w_2 *respectively. Then the deficit*

$D(K_{p,q})$ *of the MFW-inequality for* $K_{p,q}$ *satisfies*

$$D(K_{p,q}) \geq \frac{1}{2}p|w_1 - w_2| \geq p.$$

I computed $(2, q)$-cables of the five knots and found that all have deficits $= 1$.

The following is my strategy.

Fix a knot K. Suppose K has a braid representative with x strands and writhe $= y$. Then I plot a point at $(x, y) \in \mathbb{N} \times \mathbb{Z}$ on the standard x-y plane. In this manner, I plot points on the plane for all the braid representatives of K. Let B_0 be a braid representative of K of minimal braid index. Let x_0 be the braid index of K (i.e., also the braid index of B_0) and let y_0 be the writhe of B_0. If we stabilize B_0 positively k-times and stabilize negatively l times, them we get another braid representative of K with plotted vertex $(x_0 + k + l, y_0 + k - l)$. Thus it is natural to conjecture that:

Conjecture 3.4 [12] [***Range of braid representatives***] *Let* x_0 *be the braid index of a knot* K*. Then there exists a unique* y_0 *such that the set of all the plotted points for* K *coincides with the quadrant set* $\{(x_0 + k + l, y_0 + k - l) \mid k, l \in \mathbb{Z}_{\geq 0}\}$*. See the shaded region in Figure 2. (Symbols* $\beta_{\max}, \gamma_{\min}$ *will be discussed in Section 3.2.)*

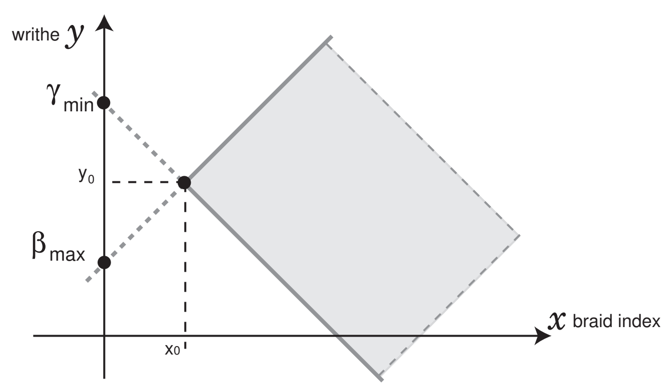

Figure 2: Range of braid representatives

Conjecture 3.4 implies the folklore conjecture.

A rough idea to prove Conjecture 3.4 is the following: First, I find commutativity rules of braid operations shown below in Conjecture 3.5. Secondly, I assume that K has a braid representative B whose plotted vertex is outside of the above quadrant region. Let B_0 be a minimal strand braid representative of K. By the Markov theorem, there is a *Markov tower* $\{B_i\}_{i=0}^n$ of braid representatives of K such that $B_n = B$ and B_{i+1} is obtained from B_i either by stabilization, destabilization or isotopy operations. Assuming that Conjecture 3.5 is true, I replace the Markov tower by another which has a braid representative with braid index $< x_0$. (This replacement does not change the lengths of Markov towers.) This would lead to a contradiction and a proof of Conjecture 3.4.

Conjecture 3.5 [12] [***Commutativity of braid operations***] *If a closed braid* B *admits operations* a *and* b *consecutively, and if* a, b *have one of the following four types, then we can find* *operations a', b' of the same type for the a, b's such that $b \cdot a = a' \cdot b'$ on B (up to braid isotopy and exchange move).*

Type	a	b
1	*Stabilization*	*Stabilization*
2	*Destabilization*	*Destabilization*
3	*Destabilization*	*Stabilization*
4	$\pm$ *Stabilization*	$\mp$ *Destabilization*

As mentioned before, if Conjecture 3.5 is true, then Conjecture 3.4 is true.

The *opposite sign* condition of a and b is necessary for type 4 by the following reason: Recall that a flype move of a closed braid that preserve the braid index is a composition of stabilization and destabilization of the *same* sign [4]. There is a closed n-braid which cannot be reduced to a $(n-1)$-braid and admits a flype move.

Applying a Lemma of Birman and Wrinkle [5], one can obtain:

Theorem 3.6 *When a, b have type 1, 2, or 3, Conjecture 3.5 is true.*

However, it is not so easy to study the fourth type, which I try to approach geometrically.

First, I define *spanning disks* of stabilization and destabilization operations. Denote the spanning disks of type 4 operations a and b by R_a and R_b respectively. I assume that $\partial R_a \cap \partial R_b = \emptyset$ and R_a, R_b intersect in a complicated manner. (If they do not intersect, then the Conjecture 3.5 is true.)

Next, I define *standard forms* of intersections $R_a \cap R_b$. A nice property of my standard form is that all the information about intersections appears just on one half plane $H_\theta = \{(r\cos\theta, r\sin\theta, z) \mid r \in \mathbb{R}_+, z \in \mathbb{R}\ \}$ for some θ.

An important intermediate result is that:

Theorem 3.7 [12] *Any intersections of R_a, R_b can be reduced to a standard form.*

Then I introduce a *complexity measure* (r, c) of the set

$$\{(R_a, R_b) \mid a, b \text{ are type 4 operations and } R_a, R_b \text{ intersect in a standard form}\}.$$

Define

$$(r, c) := (\sharp\{\text{ribbon intersections}\},\ \sharp\{\text{clasp intersections}\}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

The order of complexities is the lexicographic order. At the moment of writing, I have proved that:

Theorem 3.8 [12] *Conjecture 3.5 is true when (R_a, R_b) has complexity $(0, c)$, $(1, c)$, $(r, 0)$, $(r, 1)$ or $(2, 2)$ where $r, c \in \mathbb{Z}_{\geq 0}$.*

On the way, I have found many tricks to replace spanning disks with ones of lower complexity. As complexity increases the size of set $\{(R_a, R_b) \mid \text{complexity } (r, c)\}$ increases at least exponentially. However, if one applies the tricks, many cases of intersections can be reduced to ones with lower complexities.

3.2 Future plans

Aside from the above approach to Conjecture 3.1, I have two alternative ideas.

Related to Theorem 3.3, I am interested in how deficits of the MFW-inequality behaves under cabling. More concretely, if I can prove that

$$p > D(K_{p,q})$$

then Conjecture 3.1 would follow immediately.

The other idea is related to contact geometry: Every transversal knot TK (in S^3 with the standard contact structure) is transversally isotopic to a transversal closed braid K [1]. The *Bennequin number* β is an invariant of transversal knots. By the identification of TK and K, we have

$$\beta(K) = (\text{Writhe of } K) - (\text{braid index of } K).$$

If Conjecture 3.4 is true, then the *maximal* Bennequin number $\beta_{\max}(\mathcal{K})$ for the knot type $\mathcal{K}$ is realized on closed braid representatives with plotted vertices on the upper half boundary of the quadrant region, see Figure 2. Let

$$\gamma(K) := (\text{Writhe of } K) + (\text{braid index of } K).$$

Then γ satisfies $\gamma_{\min}(\mathcal{K}) = -\beta_{\max}(\overline{\mathcal{K}})$ where $\overline{\mathcal{K}}$ is the mirror image of $\mathcal{K}$. As in Figure 2, $\gamma_{\min}$ is realized on closed braid representatives with vertices on the lower half boundary of the quadrant region. Thus, investigation of the maximal Bennequin number is deeply related to Conjecture 3.4.

It has been known that transversal knots and Legendrian knots have many similar aspects. The Thurston-Bennequin number $tb(LK)$ of Legendrian knot LK is related to the Bennequin number of a transversal knot. Let $TB(\mathcal{K})$ denote the maximal Thurston-Bennequin number of a knot type $\mathcal{K}$. Rudolph [21] proved that

$$TB(\mathcal{K}) = \max\{n \mid \text{positive } n\text{-twisted Whitehead double of } K \text{ is strongly quasipositive }\}.$$

The right hand side is a purely topological invariant. I wonder whether a similar equality for $\beta_{\max}(\mathcal{K})$ holds. If it does, then that equality must be useful to prove Conjecture 3.4. I would like to explore this question.

4 Conclusion

My work has been focused on a lower bound for the braid index and a relation between braid index and writhe. I am also interested in interactions between the braid theory and other developing fields. For instance: The concatenation of braids is compatible with the tensor product of bimodules, which can be seen in Khovanov's works on various link invariants [13]. To detect fibreness of 3-braids, one can use the top-most term in the knot Floer homology [20]. My own research projects on braid index also leads to contact geometry. I am excited to work in this promising area.

References

[1] Bennequin, D., *Entrelacements et équations de Pfaff* Astrisque, **107-108**, (1983) 87–161.

[2] Birman, J. S. and Menasco, W. W., *Studying links via closed braids III: Classifying links which are closed* 3*-braids*, Pacific J. Math. **161**, (1993), no. 1, 25–113.

[3] Birman, J. S. and Menasco, W. W., *Studying links via closed braids. IV. Composite links and split links* Invent. Math. 102 (1990), no. 1, 115–139 Erratum: Invent. Math. 160 (2005), no. 2, 447–452.

[4] Birman, J. S. and Menasco, W. W., *Stabilization in the braid groups-I:MTWS* arXiv:math/0310279

[5] Birman, J. S. and Wrinkle, N. C., *On transversally simple knots*, J. Differential Geom. **55**, (2000), no. 2, 325–354.

[6] Ekholm, T. and Weistrand, O., *Total curvatures of holonomic links* J. Knot Theory Ramifications **9** (2000), no. 7, 893–906.

[7] Franks, J. and Williams, R. F., *Braids and the Jones Polynomial*, Trans. Amer. Math. Soc., **303**, (1987), 97–108.

[8] Gabai, D., *Detecting fibred links in* S^3 Comment. Math. Helv. **61** (1986), no. 4, 519–555.

[9] Giroux, E., *Gomtrie de contact: de la dimension trois vers les dimensions suprieures.* Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[10] Jones, V. F. R., *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math., **126**, (1987), 335–388.

[11] Kawamuro, K., *Non-sharpness of the Morton-Franks-Williams inequality*, arXiv:math.GT/0509169

[12] Kawamuro, K., *Algebraic crossing number and braid index*, in preparation.

[13] Khovanov, M., *Triply-graded link homology and Hochschild homology of Soergel bimodules* arXiv: math.GT/0510265

[14] Melvin, P. M. and Morton, H. R. *Fibred knots of genus 2 formed by plumbing Hopf bands* Proc. London Math. Soc. (2) **34** (1986), 159–168.

[15] Morton, H. R., *Seifert circles and knot polynomials*, Math. Proc. Cambridge Philos. Soc., **99**, (1986), 107–109.

[16] Murasugi, K., *On the braid index of alternations links* Trans. Amer. Math. Soc. 326 (1991), no. 1, 237–260.

[17] Murasugi, K., Przytycki, J. H., *An index of a graph with applications to knot theory* Mem. Amer. Math. Soc. 106 (1993), no. 508, x+101 pp.

[18] Neumann, W. D. and Rudolph, L., *Difference index of vectorfields and the enhanced Milnor number* Topology **29** (1990), no. 1, 83–100.

[19] Neumann, W. D. and Zagier, D., *Volumes of hyperbolic* 3*-manifolds* Topology **24** (1985), 307–332.

[20] Ni, Y., *Closed* 3*-braids are nearly fibred* arXiv:math.GT/0510243

[21] Rudolph, L. *Quasipositivity as an obstruction to sliceness* Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 1, 51–59.

[22] Thurston, W. P., *The geometry and topology of* 3*-manifolds* Mimeographed Notes, Princeton Univ. (1977).

[23] Xu, Peijun, *The genus of closed* 3*-braids*, J. Knot Theory Ramifications **1** (1992), no. 3, 303–326.