My mathematical research is focused on geometric topology and contact geometry, especially the theory of braids and transverse knots.

Braids play an important role not only in the study of knots but also in mathematical areas as diverse as contact geometry, representation theory, and mapping class groups. For example, in the field of dynamical system, Ghys [G] has applied braid theory to study Lorenz flows. Moreover, braiding appears unexpectedly even in algebraic geometry, cryptography, homotopy groups of spheres, operator algebras, quantum computer, and robotics. This statement describes my recent results and future research directions in geometric aspects of braid theory.

The *braid index* is a classical invariant of knots and links and it measures complexity of knots. Alexander's theorem states that any knot, $\mathcal{K}$, can be realized as the *closure* of an n-strand braid. Moreover, $\mathcal{K}$ has infinitely many *braid representatives*. The braid index, $b_{\mathcal{K}}$, of $\mathcal{K}$ is defined to be the minimal value of n such that $\mathcal{K}$ can be represented by an n-braid. In contrast to skein-type invariants such as Jones polynomials and Alexander polynomials, there is no known algorithm to compute the braid index. Until the discovery of the Khovanov-Rozansky homology [KR1, KR2], the Morton-Franks-William's inequality [Mo, FW], was the only available tool to estimate the braid index.

I have been working to distinguish knots by studying an *infinite set of pairs*, where each pair consists of the braid index and the algebraic length of a braid (see Conjecture 1, *generalized Jones conjecture*). This view point provides a much finer invariant of knots than the braid index itself. I have applied the Khovanov-Rozansky homology as well as conventional methods to verify the conjecture for numerous classes of knots [K1, K2, K3].

In addition, studying such pairs has broad application to classification of transversal knots in the standard contact structure of S^3, which is another goal of my research. Bennequin [B] observed that, in S^3 with the standard contact structure, studying transversal knots is essentially the same as studying braids. Ding-Geiges proved that any contact 3-manifold is obtained by a contact surgery along a Legendrian link [DG]. The fact that a transversal knot is a "push off" of a Legendrian knot implies that braid theory is involved in the study of contact manifolds. Indeed, one of the transversal knot invariants, the *self-linking number*, is defined to be the braid index minus the algebraic length. I conjectured that transverse knots would be classified by topological type, the self-linking number, and the *negative flype braid move* (Conjecture 9). In collaboration with Harvey and Plamenevskaya [HKP], and extensively in [K4], I studied properties of the cyclic branched covers of (S^3, ξ_{st}) under negative flype braid moves.

Braid theory has made successful contributions to the classification of transverse knots, cf. [BM2, BW]. Reversely, I am interested in obtaining a topological invariance constrained by obstructions that have their origins in the standard contact structure of S^3. I am trying to apply the convex surface theory in contact geometry to solve Conjecture 1, a purely knot theoretical problem, .

In joint work [KP] with Pavelescu, I am also trying to classify transverse knots in a general contact manifold via Giroux correspondence [Gi].

In the past, I have studied $\mathbb{R}$-actions on the type II_1 factors and type II_1 subfactors of the von Neumann algebra, [K6, K7].

In the following sections, I will describe, in detail, the interaction of the braid theory with the Khovanov's theory, contact and symplectic geometry.

1 Geometric braids

In my Ph.D. thesis [K1], I proposed and studied the so called Generalized Jones conjecture, which describes some 'geographical' relation between the braid index and the algebraic length of a braid.

Any knot $\mathcal{K}$ in S^3 is the closure of a braid. Let $\mathcal{B}_{\mathcal{K}}$ be the infinite set of braid representatives of $\mathcal{K}$. The *braid index* $n_{\mathcal{K}}$, an invariant of a knot $\mathcal{K}$, is the minimal value of n such that $\mathcal{K}$ can be represented by an n-strand braid. Let $n(b)$ be the number of braid strands of a braid b. Let $a(b)$ be the algebraic length of b as an element of the Artin braid group.

Conjecture 1 (Generalized Jones Conjecture) [K1, MT] *Let $\mathcal{K} \subset S^3$ be a topological knot type. Let $\Phi : \mathcal{B}_{\mathcal{K}} \to \mathbb{N} \times \mathbb{Z}$ be defined by $\Phi(b) = (n(b), a(b))$. Then the entire image $\Phi(\mathcal{B}_{\mathcal{K}})$ of the map satisfies:*

$$\Phi(\mathcal{B}_{\mathcal{K}}) = \{(n_{\mathcal{K}} + x + y, a_{\mathcal{K}} + x - y) \mid x, y \in \mathbb{Z}_{\geq 0}\},$$

that is a lattice in the infinite shaded region shown in Figure 1. Therefore, the set $\Phi(\mathcal{B}_{\mathcal{K}})$ is a new *knot-invariant.*

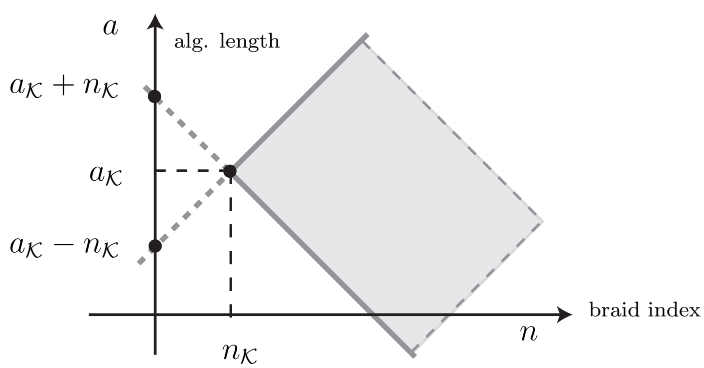

Figure 1: The region of braid representatives of $\mathcal{K}$

In particular, as Jones [J, p. 357] conjectured, the point $(b_{\mathcal{K}}, a_{\mathcal{K}})$ is a new knot-invariant. Studying the entire image $\Phi(\mathcal{B}_{\mathcal{K}})$, rather than just the corner point, would help classify transversal knots; this will be discussed further in Section 3.

Perhaps the most successful approach to Conjecture 1 is the *Morton-Franks-Williams* (*MFW*) *inequality* [Mo, FW]. The MFW inequality is sharp on the unknot, trivial unlinks, alternating fibred knots and 2-bridge knots [Mu], Lorentz knots and torus knots [FW], and knots up to 10 crossings except 9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}, [J].

Theorem 2 [K1] *If the MFW-inequality is sharp on $\mathcal{K}$ then Conjecture 1 holds for $\mathcal{K}$.*

Therefore, Conjecture 1 holds for all the knots listed above. I confirmed the exceptional five knots [K2]. In the general setting, I proved the following:

Theorem 3 [K1] *If Conjecture 1 holds for $\mathcal{K}, \mathcal{L}$ then Conjecture 1 also holds for the connect sum $\mathcal{K} \# \mathcal{L}$ and the (p, q)-cable of $\mathcal{K}$.*

However, the MFW-inequality is not an ideal tool for proving Conjecture 1:

Theorem 4 [K1] *There exist infinitely many knots on which the MFW-inequality is not sharp.*

The generalized Jones conjecture is not merely a problem in classical knot theory. In fact, it led me into broader fields such as Khovanov-Rozansky homology and contact geometry, as described in Sections 2 and 3.

2 Braids and Khovanov-Rozansky homology

Khovanov introduced a new idea, *categorification*, which replaces set-theoretic theorems by category-theoretic analogues. It has extensive influence over algebraic geometry, low dimensional topology, and representation theory. A categorification of the Jones/HOMFLY-PT polynomial is the Khovanov/Khovanov-Rozansky homology [K, KR1, KR2]. Dunfield-Gukov-Rasmussen and Wu [DGR, W] proved the *KR-MFW inequality* which is a categorification of the MFW-inequality. A parallel statement to Theorem 2 holds: If the KR-MFW inequality is sharp on $\mathcal{K}$ then Conjecture 1 holds for $\mathcal{K}$ [K3].

Since the KR-homology carries more information than the HOMFLY-PT polynomial, the KR-MFW inequality should be sharper than the MFW-inequality. In [K3], I studied this gap: Let $i = 1, 2$ be the standard generator of the braid group B_3 satisfying $1\ 2\ 1 = 2\ 1\ 2$.

Theorem 5 [K3] *Let $\mathcal{K}^\star := (1\ 2\ 2\ 1)^2\ 1\ 2^{-3}$. On $\mathcal{K}^\star$ and its mirror image the MFW-inequality is not sharp but the KR-MFW inequality is sharp.*

These are the first examples which show that the KR-homology is more effective than the HOMFLY-PT polynomial in terms of detecting the braid index.

Computation of the KR-homology is much harder than that of the HOMFLY-PT polynomial. The techniques I used are Rasmussen's skein exact sequence of the $sl(n)$-homology and his spectral sequence theorem [R].

I have also discovered a *sufficient condition* for knots which guarantees that *both* the MFW and the KR-MFW inequalities are *not* sharp. In fact, once the braid index is greater than 3, Conjecture 1 is beyond the scope of the KR-homology. There are infinitely many examples of knots satisfying my sufficient condition [K3]; these are the first known examples.

Theorem 6 [K3] *Let* $BM_{x,y,z,w} := 1^x\ 2^y\ 3^{-1}\ 2^z\ 1^w\ 2\ 3\ 2\ 2\ 3$ *be a 4-braid. There are infinitely many four tuples* (x, y, z, w) *such that both the KR-MFW and the MFW inequalites are not sharp on* $BM_{x,y,z,w}$.

Let $\mathcal{K}_k = (1\ 2\ 2\ 1)^{2k}\ 1\ 2^{-2k-1}$ and $\mathcal{L}_k = (1\ 2\ 2\ 1)^{2k+1}\ 1\ 2^{-2k+1}$ where $k \in \mathbb{N}$. Thus, $\mathcal{K}_1 = \mathcal{K}^\star$. Elrifai [E] proved that the MFW inequality is sharp for all the knots of braid index = 3, except for $\mathcal{K}_k, \mathcal{L}_k$ and their mirror images. I wish to extend the result of Theorem 5 to all $\mathcal{K}_k, \mathcal{L}_k$ and their mirror images by computing their KR-homology, i.e., the generalized Jones conjecture holds for all the 3-braid knots.

3 Braids and Contact geometry

I am also interested in application of braid theory to contact geometry especially the classification of transversal knots.

A contact 3-manifold (M, ξ) is an oriented 3-manifold M with a nowhere integrable 2-plane field ξ, called contact structure. Many contact manifolds is realized as a boundary of symplectic manifolds. A knot $K \subset M$ is called *transverse/Legendrian* if it is transverse/tangent to ξ_p for all $p \in K$. Any closed contact 3-manifold is obtained by a contact surgery along a Legendrian link in the standard contact structure (S^3, ξ_{st}) [DG]. Braids in $\mathbb{R}^3$, transversal knots, and Legendrian knots in (S^3, ξ_{std}) are closely related by the following correspondence, due to Bennequin [B] and Epstein-Fuchs-Meyer [EFM]:

$$\{\text{braids}\}/(+)\text{stabilization} = \{\text{transverse knots}\} = \{\text{Legendrian knots}\}/(\text{-})\text{stabilization}.$$

3.1 Self linking number and the generalized Jones conjecture

An invariant of transverse knots called *self-linking number* $sl(T)$ is defined in terms of a relative Euler class of the vector bundle ξ restricted to a Seifert surface of the transverse knot. Under Bennequin's identification, it has a combinatorial description:

$$sl(T) = (\text{the algebraic length } a(T)) - (\text{the number of the braid strands } n(T)). \qquad (3.1)$$

Let $\overline{sl}(\mathcal{K})$ denote the *maximal* self-linking number for all the transversal knots of topological type $\mathcal{K}$. Conjecture 1 can be interpreted as follows:

Conjecture 7 [K1] *The maximal self-linking number* $\overline{sl}(\mathcal{K})$ *is equal to* $a_{\mathcal{K}} - n_{\mathcal{K}}$.

Again, the knots listed in Section 1 are evidences for Conjecture 7. Recently, Etnyre-van Horn-Morris [EV] found a sufficient condition of Conjecture 7 for quasi-positive knots, which are intersections of S^3 and complex analytic plane curves in $\mathbb{C}^2$. Inspired by [EV], the quasi-positive knots is a next reasonable class I would like to examine for Conjecture 7.

More general approach to Conjectures 1 and 7 currently I am trying is comparison of the characteristic foliations in contact geometry and the braid foliations.

3.2 Classification and negative flype braid move

Apart from Conjectures 1 and 7, the classification of transversal knots is a significant research interest of mine.

A topological knot type $\mathcal{K}$ has infinitely many transversal knot representatives. If transversal knots for $\mathcal{K}$ are completely classified by the self-linking number, $\mathcal{K}$ is called *transversally simple.* Many transversally non-simple knots have been discovered using various techniques: Birman-Menasco [BM2] used the braid foliation, Etnyre-Honda [EH2] (cf. [MM]) used the convex surface theory, and Ng-Ozsváth-Thurston [NOT], Lisca-Ozsvaáth-Stipsicz-Szabó [LOSS], and Khandhawit-Ng [KN] used the Heegaard Floer theory.

In joint work with Harvey and Plamenevskaya, we observed the following interesting fact:

Proposition 8 [HKP] *All the transversally non-simple knots in* [BM2, EH2, NOT, LOSS, KN] *admit negative flype braid moves* (*Figure 2*).

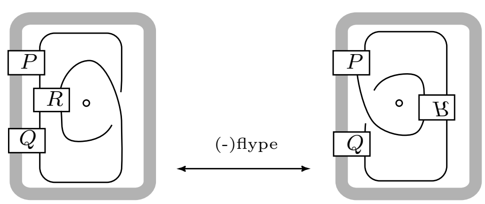

Figure 2: A negative flype. The gray bands are multiple strands. Boxes P, Q, R are braidings.

My goal is to clarify the reason behind Proposition 8, that is, to understand when and why a negative flype changes transverse knot classes.

Conjecture 9 (**Classification of transversal knots**) *Transversal knots* T_1, T_2 *in* (S^3, ξ_{sym}) *are transversally isotopic if and only if*

1. T_1 *and* T_2 *have the same* topological type *and the same* self linking number, *and*
2. T_1 *and* T_2 *are* not *related by a* (*non trivial*) *negative flype, up to positive stabilization and braid conjugation.*

In [HKP], we studied the behavior of branched coverings under negative flype and extended the result of Plamenevskaya [Pl].

Theorem 10 [HKP] *The double branched covers of* (S^3, ξ_{st}) *about transversal knots related by a negative flype are contactomorphic. The p-fold cyclic branched covers about a Birman-Menasco pair* [BM2] *are contactomorphic.*

The main idea is to find a contact surgery descriptions of the branched covering manifolds, then apply Legendrian Reidemeister moves. We also found the following properties of branched covers with regard to homotopy invariants, Stein fillability, and overtwisted-ness:

Theorem 11 [HKP] *Let* T *be a transversal knot in* (S^3, ξ_{std}) *and fix* $p \geq 2$. *Let* s_T *be the* Spin^c *structure induced by the contact structure* $\xi_p(T)$ *of the p-fold covering . Then the Chern class* $c_1(s_T) = 0$ *and the* three-dimensional invariant $d_3(\xi_p(T))$, *introduced by Gompf* [Go]*, is completely determined by the topological type of* T *and its self-linking number.*

Theorem 12 [HKP] *The p-fold cyclic cover about a transverse knot* T *is Stein fillable* (*resp. overtwisted*) *if* T *is represented by a quasipositive braid* (*resp. is destabilizable*).

Later, I further extended Theorem 10 into the following:

Theorem 13 [K4] *The p-fold cyclic branched covers about transversal knots related by a negative flype are contactomorphic.*

There is a different construction of transversally non-simple knots using the *connect sum*, hinted at Etnyre-Honda [EH1] and Vertesi [V].

Theorem 14 [K5] *Let* $\mathcal{K}_1, \mathcal{K}_2$ *be prime knots in* S^3. *Let* T_1, T_1' (*resp.* T_2, T_2') *be transverse knots in* (S^3, ξ_{st}) *of topological type* $\mathcal{K}_1$ (*resp.* $\mathcal{K}_2$). (*We allow the possibility that* $\mathcal{K}_1 = \mathcal{K}_2$ *and* $T_1 = T_2$.) *Suppose that* (1) T_1, T_1' *have the same self-linking number but are not transversely isotopic, and* (2) T_2, T_2' *are transversely isotopic and cannot be transversely destabilized. Then the connect sums* $T_1 \# T_2$ *and* $T_1' \# T_2'$ *are not transversely isotopic. Thus,* $\mathcal{K}_1 \# \mathcal{K}_2$ *is transversally non-simple.*

3.3 Open book decomposition and braids

My current research in progress [KP] is to apply braid theory to classification of transverse knots in arbitrary contact manifolds.

One of the central results about the topology of contact 3-manifolds is the Giroux correspondence [Gi]:

$$\left\{ \begin{array}{l} \text{contact structure } \xi \text{ on } M^3 \\ \text{up to isotopy} \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{open book decomposition } (\Sigma, \phi) \\ \text{of } M^3 \text{ up to positive stabilization} \end{array} \right\},$$

where Σ is a surface and ϕ is a diffeomorphism of Σ fixing $\partial\Sigma$ pointwise. Thurston-Winkelnkemper [TW] gave a construction of a contact structure $\xi_{(\Sigma,\phi)}$ from (Σ, ϕ). A natural open book decomposition of S^3 is (D^2, id_{D^2}) and it corresponds to ξ_{st}. Bennequin's identification [B] of transversal knots in (S^3, ξ_{st}) with braids in (D^2, id_{D^2}) has been extended to general manifolds by Pavelescu [Pa]:

$$\{ \text{ transversal knots in } (M_{(\Sigma,\phi)}, \xi_{(\Sigma,\phi)}) \} \longleftrightarrow \{ \text{ braids in } (\Sigma, \phi) \}$$

In a work in progress with Pavelescu [KP], we study braids in general open book (Σ, ϕ) with an intention to classify transversal knots in the corresponding contact manifolds. Many differences exist between the classical braids in (D^2, id_{D^2}) and braids in (Σ, ϕ). For example, the braid index $n(b) = 1$ is not equivalent to $b =$ (the unknot). Also, a parallel statement to the Jones conjecture does not hold in general (Σ, ϕ) [KP]. Our project is to generalize the combinatorial formula (3.1) of the self-linking number and describe it in terms of a 'braid word', which is an element of the mapping class of the page surface of the open book. Currently, we have obtained a description of the self-linking number for planer open books. We are also interested in understanding the behavior of transverse knots under 'negative flype braid moves' in general open book decompositions.

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