An induction for bimodules arising from subfactors

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Abstract

We first give necessary and sufficient conditions for a new induction, which we define, for completely positive (CP)-maps between type II_1 factors, in terms of bimodules of type II_1 factors. Our induction is characterized by a condition weaker than that for a half braiding and contains α-induction as a particular case.

Based on the above result, we define an inclusion of pointed bimodules and find an extension procedure of an inclusion of pointed bimodules, which is compatible with the Jones basic construction.

1 Introduction

Significance of bimodules in theory of von Neumann algebras was first emphasized by A. Connes in an analogy to representations of compact groups, and well studied in [11]. In the paragroup theory, introduced by A. Ocneanu [9] for the purpose of classifying subfactors, bimodules play an important role. Moreover, systems of bimodules give rise to 3-dimensional topological invariants, which is also a contribution of Ocneanu, and they are extensively studied in the field of tensor categories.

It is a discovery of Longo and Rehren, that one can induce a certain endomorphism of the smaller algebra to the larger of a net of subfactors. Xu [14] later found the dual form of their induction and applied it especially to the subfactors associated with the conformal inclusions. His work was so stimulating that it has been extensively studied by a number of people. For his construction, braiding property of a system of endomorphisms is essential. Izumi defined a notion of a *half braiding* of an endomorphism with respect to a finite system of endomorphisms [4] and proved that the existence of a half braiding is a necessary and sufficient condition for an induction of sectors associated with the Longo-Rehren inclusion. Categorically, a system of sectors of a type III factor with the intertwiner spaces corresponds to a system of bimodules arising from a type II_1 subfactor with the intertwiner spaces. In the first part of this paper, we aim to generalize both of their inductions in the language of bimodules or CP-maps of type II_1 factors. We succeed in giving a necessary and sufficient condition for an induction of a bimodule or a CP-map.

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Based on this study, we define an inclusion of pointed bimodules. Suppose we have two Jones towers [5] of type II_1 factors $N \subset M \subset M_1 \subset \cdots$, $P \subset Q \subset Q_1 \subset \cdots$, an N-P bimodule ${}_NK_P$ and an M-Q bimodule ${}_MH_Q$. Assume $K \subset H$ and this inclusion is compatible with the left N and the right P actions. We define an inclusion ${}_NK_P \subset {}_MH_Q$ if they satisfy the conditions of our induction (see Definition 3.4 for the precise notation).

Our next concern is to construct an M_1-Q_1 bimodule ${}_{M_1}L_{Q_1}$ from ${}_NK_P \subset {}_MH_Q$ such that ${}_MH_Q \subset {}_{M_1}L_{Q_1}$. We take the relative tensor products for both side of ${}_NK_P$ and obtain ${}_{M_1}L_{Q_1}$ and we can make a tower of bimodules. This type of iteration for constructing a tower does not work within the framework of α-induced endomorphisms of type III factors, because in general, an α-induced system does not have a braiding.

2 Correspondence between CP-maps and bimodules

In this section, we summerize the identification of CP-maps and bimodules. Although this was first introduced by A. Connes in his unpublished manuscript and nowadays is a well-known fact, we give a review of relationships of CP-maps and bimodules because they play essential roles in this paper. (See [11] for a general theory of bimodules.)

Let M, Q be type II_1 factors. Let us fix a notation for a pointed bimodule. We denote a pair of an M-Q bimodule ${}_MH_Q$ and an M-Q *cyclic* and the right Q-*bounded* vector ξ by $({}_MH_Q, \xi)$. Here the M-Q cyclic condition means that $M\xi Q$ is a dense subset of H and the right Q-boundedness means that there exists a positive constant C_ξ such that $\|\xi \cdot q\| \leq C_\xi \|\hat{q}\|$ for any $q \in Q$. We say pointed bimodules $({}_MH_Q, \xi)$ and $({}_M\tilde{H}_Q, \tilde{\xi})$ are *isomorphic* when there exists a unitary $u \in \mathrm{Hom}({}_MH_Q, {}_M\tilde{H}_Q)$ such that $\tilde{\xi} = u\xi$. Here we denote the set of bounded linear operators from H to $\tilde{H}$ compatible with the left M and the right Q-actions by $\mathrm{Hom}({}_MH_Q, {}_M\tilde{H}_Q)$.

(1) Let us explain how to use the Stinespring dilation theorem to obtain $({}_MH_Q, \xi)$ from a given (not necessarily unital) CP-map $\varphi : M \to Q$. Consider an algebraic tensor product of M and Q which we denote by $M \otimes_{\mathrm{alg}} Q$. Define an inner product

$$\langle x \otimes a \mid y \otimes b \rangle := \mathrm{tr}_Q(\varphi(y^*x)ab^*), \tag{2.1}$$

where tr_Q is the unique trace for Q. We take the quotient by the kernel of the inner product and take a completion of $M \otimes_{\mathrm{alg}} Q$, then we get a Hilbert space which is naturally regarded as an M-Q bimodule. We denote it by ${}_MM \otimes_\varphi Q_Q$.

Let $\varphi(x) = V^*\pi(x)V$ be the Stinespring dilation decomposition where V is the map from $L^2(Q)$ to $M \otimes_\varphi Q$ with $V : \hat{q} \mapsto 1 \otimes_\varphi q$ and π is the representation of M on $M \otimes_\varphi Q$ with $\pi(x)(a \otimes_\varphi b) = (xa) \otimes_\varphi b$. It is easy to see that $V\hat{1} = 1 \otimes_\varphi 1$ is an M-Q cyclic and right Q-bounded vector. We have obtained a pair $({}_MM \otimes_\varphi Q_Q, 1 \otimes_\varphi 1)$ from the original CP-map φ.

(2) On the other hand, we make a CP-map from a given pair $({}_MH_Q, \xi)$ of M-Q bimodule and M-Q cyclic and right Q-bounded vector. Let V be a map from

$Q \subset L^2(Q)$ to H satisfying $V : \hat{q} \mapsto \xi \cdot q$ for $q \in Q$. We may define the map V on the entire space $L^2(Q)$ because ξ is right Q-bounded. Denote the representation of M on H by $\pi : M \to B(H)$. For $x \in M$, note that

$$\varphi(x) := V^* \pi(x) V$$

sits not only in $B(L^2(Q))$ but also in Q, since $\varphi(x)$ commutes with the right Q multiplication on $L^2(Q)$, that is,

$$\varphi(x) \in (JQJ)' \cap B(L^2(Q)) = Q'' \cap B(L^2(Q)) = Q,$$

where J is the conjugation operator of Q. It is easy to prove that φ is a CP-map. Thus we construct a CP-map φ from $({}_M H_Q, \xi)$.

Now we discuss (1) and (2) more in detail, namely, that the correspondence is one-to-one, and it is compatible with the adjoint operation of CP-maps ($\varphi \mapsto \varphi^*$) and the conjugation (${}_M H_Q \mapsto {}_Q \overline{H}_M$) (see [11, Page 22]).

Definition 2.1 *Let* $\varphi : M \to Q$ *be a CP-map with a (both left* M *and right* Q*) bounded vector* $1 \otimes_\varphi 1 \in {}_M M \otimes_\varphi Q_Q$. *Put* $\Phi_q(x) := \mathrm{tr}_Q(\varphi(x) q)$ *with* $0 \leq q \in Q$. *Since*

$$\Phi_q(x^* x) = \mathrm{tr}_Q(\varphi(x^* x) q) \leq \|q\|_\infty \langle x(1 \otimes_\varphi 1) \mid x(1 \otimes_\varphi 1)\rangle_{M \otimes_\varphi Q} \leq C \|q\|_\infty \mathrm{tr}_M(x^* x)$$

for some $C > 0$ *(here we use left boundedness of* $1 \otimes_\varphi 1$*), there exists a unique element* $0 \leq \varphi^*(q) \in M$ *satisfying* $\mathrm{tr}_Q(\varphi(x) q) = \mathrm{tr}_M(x \varphi^*(q))$ *by the Radon-Nikodym type theorem. (The uniqueness follows from the faithfulness of* tr_M*.) We may extend the domain of* φ^* *to* Q, *and obtain a CP-map* $\varphi^* : Q \to M$ *which we call the* adjoint *of* φ.

Theorem 2.2 *We have a one-to-one correspondence between a CP-map* $\varphi : M \to Q$ *with a bounded vector* $1 \otimes_\varphi 1 \in {}_M M \otimes_\varphi Q_Q$, *and a pair of an* M*-*Q *bimodule and an* M*-*Q *cyclic and bounded vector, via the above procedures (1), (2). Moreover, they are compatible with the adjoint operation of CP-maps and the conjugate operation of bimodules.*

$$\begin{array}{ccc} ({}_M H_Q, \xi) & \overset{(1),(2)}{\longleftrightarrow} & \varphi : M \to Q \\ \updownarrow & & \updownarrow \\ ({}_Q \overline{H}_M, \overline{\xi}) & \overset{(1),(2)}{\longleftrightarrow} & \varphi^* : Q \to M \end{array}$$

Proof One-to-one correspondence holds by the descriptions of (1) and (2). (Here, we only need to assume the right boundedness of the vector.)

Suppose we have a CP-map $\varphi : M \to Q$. Construct $(Q \otimes_{\varphi^*} M, 1 \otimes_{\varphi^*} 1)$ from φ^*, then it is conjugate to $(M \otimes_\varphi Q, 1 \otimes_\varphi 1)$ with

$$Q \otimes_{\varphi^*} M \ni q \otimes_{\varphi^*} m \mapsto m^* \otimes_\varphi q^* \in M \otimes_\varphi Q.$$

Construct a CP-map $\tilde{\varphi}$ from $({}_Q\overline{H}_M, \overline{\xi})$. Then, we have

$$\begin{aligned}\mathrm{tr}_M(\tilde{\varphi}(q)m) &= \langle q\cdot\overline{\xi} \mid \overline{\xi}\cdot m^*\rangle = \langle \overline{\xi}\cdot m \mid q^*\cdot\overline{\xi}\rangle = \langle \overline{m^*\cdot\xi} \mid \overline{\xi\cdot q}\rangle \\ &= \langle \xi\cdot q \mid m^*\cdot\xi\rangle = \mathrm{tr}_Q(\varphi(m)q),\end{aligned}$$

where $m\in M, q\in Q$ and φ is a CP-map obtained from $({}_MH_Q, \xi)$. The above computation shows that $\tilde{\varphi} = \varphi^*$. □

Claim 2.3 *Suppose that a CP-map $\varphi : M \to Q$ corresponds to a pair $({}_MH_Q, \xi)$. Then we have the following:*

1. *When $\varphi : M \to Q$ is unital if and only if the pointed vector ξ is Q-tracial, i.e., $\mathrm{tr}_Q(q) = \langle \xi q \mid \xi\rangle$ for $q \in Q$;*
2. *The CP-map preserves the trace of M, i.e., $\mathrm{tr}_Q \circ \varphi = \mathrm{tr}_M$, if and only if ξ is M-tracial, that is, $\mathrm{tr}_M(x) = \langle x\xi \mid \xi\rangle$ for any $x \in M$.*

Proof 1. (⇒) Let $\varphi(x) = V^*\pi(x)V$ be the Stinespring decomposition as in (1). Since $\xi = 1\otimes_\varphi 1 = V\hat{1}$, we have

$$\langle \xi q \mid \xi\rangle = \langle V\hat{q} \mid V\hat{1}\rangle = \langle V^*V\hat{q} \mid \hat{1}\rangle = \langle \varphi(1)\hat{q} \mid \hat{1}\rangle = \mathrm{tr}(q).$$

(⇐) For any $q\in Q$, we have

$$\mathrm{tr}(\varphi(1)q) = \mathrm{tr}(V^*\pi(1)Vq) = \langle V\hat{q} \mid V\hat{1}\rangle = \langle \xi\cdot q \mid \xi\rangle = \mathrm{tr}(q),$$

thus $\varphi(1) = 1$.

2. (⇒) Since $\langle x(1\otimes_\varphi 1) \mid 1\otimes_\varphi 1\rangle = \mathrm{tr}_Q(\varphi(x)) = \mathrm{tr}_M(x)$, the vector $1\otimes_\varphi 1$ is M-tracial.

(⇐) It follows from $\mathrm{tr}_Q(\varphi(x)) = \langle V^*\pi(x)V\hat{1} \mid \hat{1}\rangle_{L^2(Q)} = \langle x\xi \mid \xi\rangle_H = \mathrm{tr}_M(x)$. □

Definition 2.4 *Let $({}_NX_P, \xi)$, $({}_NY_P, \eta)$ be pointed bimodules. If $s\xi\oplus t\eta \in {}_N(X\oplus Y)_P$ is N-P cyclic, for positive numbers s, t, we define a* direct sum *by*

$$({}_N(X\oplus Y)_P, s\xi\oplus t\eta)\,. \tag{2.2}$$

Let $\varphi, \psi : N \to P$ be CP-maps corresponding to $({}_NX_P, \xi)$, $({}_NY_P, \eta)$, respectively. Then (2.2) corresponds to $s^2\varphi + t^2\psi$.

Without loss of generality, we may assume the positivity of s and t, since we could change the pointed bimodules within their isomorphism classes.

Remark 2.5 Unfortunately, this correspondence of CP-maps and pointed bimodules is not compatible with composition of CP-maps and the relative tensor product of bimodules. Consider the following simple example. Let $N\subset M$ be a type II_1 subfactor, $\varphi : N \to M$ the inclusion map and $\psi : M \to N$ the unique trace-preserving conditional expectation. The CP-maps φ, ψ correspond to $({}_NL^2(M)_M, \hat{1}), ({}_ML^2(M)_N, \hat{1})$ respectively. We have $({}_NL^2(M)_N, \hat{1})$ by taking their relative tensor product over M. The composition of the CP-maps $\psi\cdot\varphi$ is the identity map of N and it corresponds to $({}_NL^2(N)_N, \hat{1})$ which is not equal to $({}_NL^2(M)_N, \hat{1})$, unless $N = M$.

3 Inclusions of pointed bimodules

Before we define an inclusion of pointed bimodules, we give the following theorem.

Theorem 3.1 *Let $N \subset M, P \subset Q$ be type II_1 subfactors with finite indices. Suppose we have an N-P bimodule ${}_NX_P$ of finite type, i.e., $[{}_NX_P] := \dim_N X \dim X_P < \infty$. The following are equivalent:*

1. *There exists an M-Q bimodule ${}_MY_Q$ with $[{}_MY_Q] < \infty$, such that $X \subset Y$ as Hilbert spaces compatible with the right P and the left N-actions, and the following hold. We have equalities of Jones indices and right dimensions of bimodules:*
$$[M : N] = [Q : P], \quad \dim_N X = \dim_M Y. \tag{3.1}$$
(From (3.1), we obtain $\dim X_P = \dim Y_Q$.) There exists an N-P cyclic and bounded vector $\xi \in X$, such that $\eta \in Y$, the image of ξ by the embedding $X \hookrightarrow Y$, is an M-Q cyclic and bounded. Moreover, if we denote the corresponding CP-maps to $({}_NX_P, \xi)$ and $({}_MY_Q, \eta)$ by $\varphi_\xi : N \to P$ and $\psi_\eta : M \to Q$, respectively, we have
$$\psi_\eta|_N = \varphi_\xi, \quad \psi_\eta^*|_P = \varphi_\xi^*;$$
2. *There exists an M-Q bimodule ${}_MY_Q$ such that*
$$\tau : {}_NX \otimes_P Q_Q \simeq {}_NM \otimes_M Y_Q, \tag{3.2}$$
and
$$\sigma : {}_MM \otimes_N X_P \simeq {}_MY \otimes_Q Q_P \tag{3.3}$$
hold, and there exist $\xi \in X$, an N-P cyclic and bounded vector, and $\eta \in Y$ satisfying
$$\tau(\xi \otimes_P 1) = 1 \otimes_M \eta, \tag{3.4}$$
$$\sigma(1 \otimes_N \xi) = \eta \otimes_Q 1; \tag{3.5}$$
3. *We have an isomorphism of bimodules:*
$$\Phi : {}_NM \otimes_N X_P \simeq {}_NX \otimes_P Q_P, \tag{3.6}$$
satisfying the following two conditions: (3a), (3b).
   (a) *There exists an N-P cyclic and bounded vector $\xi \in X$ such that*
   $$\Phi(1 \otimes_N \xi) = \xi \otimes_P 1.$$
   (b) *Let $s \in \mathrm{Hom}({}_MM \otimes_N M_M, {}_MM_M)$ be the conditional expectation from M_1 to M with the identification of ${}_MM \otimes_N M_M$ and ${}_MM_{1M}$. Define $t \in \mathrm{Hom}({}_QQ \otimes_P Q_Q, {}_QQ_Q)$ in the same way. Then, for any $\zeta \in {}_NM \otimes_N X_P$, we have*
   $$(\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)(\zeta) = (\Phi^{-1} \otimes_P \mathrm{id}_Q)(\mathrm{id}_X \otimes_P t^*)\Phi(\zeta).$$

It is well known that the right boundedness of a vector is equivalent to the left boundedness in case a bimodule is of finite type (see [13, Lemma 23] for example).

Remark 3.2 Thanks to cyclicity of $\xi \in {}_N X_P$, the conditions *2, 3* of Theorem 3.1 can be rephrased in the stronger forms as follows.

2'. The same isomorphisms as in (3.2), (3.3) *hold so that* any N-P *cyclic and bounded vector* $\xi \in N$*, there exists* $\eta \in Y$ *such that* (3.4) , (3.5) *hold.*

3'. The isomorphism (3.6) *holds so that for* any N-P *cyclic and bounded vector* $\xi \in X$*, we have* (3a)*, and, we have* (3b)*.*

Remark 3.3 When $N = P$ and $M = Q$, we note that the third condition is related to the half braiding [4, Definition 4.2] in the following sense. In (3.6), ${}_N X_N$ commutes only with ${}_N M_N$, on the other hand, in [4], a sector has to commute with every endomorphism of the system. For the pentagon type equality (3b), we only deal with the specific homomorphism $s(= t)$, so that (3b) is weaker than the braiding fusion equation of [4].

So called α-induction (defined by Longo and Rehren [7], further studied in [14], later in [1], and named in [1]) is a particular case of *3* $\Rightarrow$ *1.*

Moreover Kosaki's result, on restriction of an automorphism of M to that of N associated with vertices of the principal graph of $N \subset M$ [6], is generalized in the equivalence of *1* and *2* of Theorem 3.1. More precisely, when a CP-map α is an automorphism, the adjoint α^* and the inverse α^{-1} coincide with each other, and one can remove (3.3), (3.5) of Theorem 3.1.

By checking the second condition of Theorem 3.1, it is easy to see that the direct sum of pointed bimodules (see Definition 2.4) is compatible with the inducution of pointed bimodules.

Proof of Theorem 3.1 (1 $\Rightarrow$ 2) Since $\varphi_\xi = \psi_\eta$ on N, we have

$$\begin{aligned} {}_N X \otimes_P Q_Q &\simeq {}_N(N \otimes_{\varphi_\xi} P) \otimes_P Q_Q \simeq {}_N N \otimes_{\varphi_\xi} Q_Q \\ &\subset {}_N M \otimes_{\psi_\eta} Q_Q \\ &\simeq {}_N M \otimes_M (M \otimes_{\psi_\eta} Q)_Q \simeq {}_N M \otimes_M Y_Q. \end{aligned}$$

By (3.1), we have $\dim_N(X \otimes_P Q) = \dim_N(M \otimes_M Y)$, then $\tau : {}_N X \otimes_P Q_Q \simeq {}_N M \otimes_M Y_Q$. Applying the same procedure to $\varphi_\xi^*, \psi_\eta^*$, we obtain ${}_M M \otimes_N X_P \simeq {}_M Y \otimes_Q Q_P$.

Since $\xi \in {}_N X_P$ corresponds to $1 \otimes_{\varphi_\xi} 1 \in {}_N N \otimes_{\varphi_\xi} P_P$, with the above computations, we have $\tau(\xi \otimes_P 1) = 1 \otimes_M \eta$. The other equation $\sigma(1 \otimes_N \xi) = \eta \otimes_Q 1$ holds in the same way.

(2 $\Rightarrow$ 1) Equalities (3.1) follow from the isomorphisms (3.2), (3.3). Let $V : L^2(P) \ni \hat{p} \mapsto \xi p \in X$ and let $\pi : N \to B(X)$ be the representation of N. Define a CP-map $\varphi_\xi : N \ni x \mapsto V^* \pi(x) V \in P$. We regard φ_ξ as a map from N to Q and denote it by $\tilde{\varphi}_\xi$. Let $\tilde{V} : L^2(Q) \ni \hat{q} \mapsto (\xi \otimes_P 1)q \in X \otimes_P Q$ and $\tilde{\pi} : N \to B(X \otimes_P Q)$ be the representation of N with $\tilde{\pi} = \pi \otimes_P \mathrm{id}_Q$. Then we have $\tilde{\varphi}_\xi(x) = \tilde{V}^* \tilde{\pi}(x) \tilde{V}$. Let

$W : L^2(Q) \ni \hat{q} \mapsto \eta q \in Y$, and let $\sigma : M \to B(Y)$ be the representation of M and define $\psi_\eta(y) := W^*\sigma(y)W$. For $n \in N, q_1, q_2 \in Q$, we have

$$\begin{aligned}
\langle \tilde{\varphi}_\xi(n)\hat{q}_1 \mid \hat{q}_2\rangle_{L^2(Q)} &= \langle n(\xi \otimes_P 1)q_1 \mid (\xi \otimes_P 1)q_2\rangle_{X\otimes_P Q} \\
&= \langle n(1 \otimes_M \eta)q_1 \mid (1 \otimes_M \eta)q_2\rangle_{M\otimes_M Y} \\
&= \langle \psi_\eta(n)\hat{q}_1 \mid \hat{q}_2\rangle_{L^2(Q)}
\end{aligned}$$

thanks to (3.2), that is, $\psi_\eta|_N = \tilde{\varphi}_\xi = \varphi_\xi$. The isomorphism (3.3) yields $\psi_\eta^*|_P = \varphi_\xi^*$.

($2 \Rightarrow 3$) We obtain (3.6) from (3.2) , (3.3). Let $\tilde{\sigma} : {}_M M \otimes_N X_P \to {}_M Y_P$ be a composition of σ and the natural isomorphism ${}_M Y \otimes_Q Q_P \ni \zeta \otimes_Q q \mapsto \zeta q \in {}_M Y_P$. The isomorphism $\Phi(\zeta) := \tau^{-1}(1 \otimes_M \tilde{\sigma}(\zeta))$ with $\zeta \in {}_N M \otimes_N X_N$ gives ${}_N M \otimes_N X_P \simeq {}_N X \otimes_P Q_P$. It is easy to see that $\Phi(1 \otimes_N \xi) = \xi \otimes_P 1$.

In the following, we are going to show (3b). Let $\{m_j\}, \{n_k\}$ be the Pimsner-Popa basis of $N \subset M, P \subset Q$ respectively. We embed the left hand side of (3b) $(\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)(\zeta)$ into ${}_N M \otimes_M M \otimes_N X \otimes_P Q_P$ and apply $\mathrm{id}_M \otimes_M \sigma \otimes_P \mathrm{id}_Q$, then we obtain

$$\sum_j 1 \otimes_M (\sigma \otimes_P \mathrm{id}_Q)(\mathrm{id}_M \otimes_N \tau^{-1})(m_j \otimes_N m_j^* \otimes_M \tilde{\sigma}(\zeta)) \in {}_N M \otimes_M Y \otimes_Q Q \otimes_P Q_P. \quad (3.7)$$

On the other hand, we embed $(\mathrm{id}_X \otimes_P t^*)\Phi(\zeta) = \sum_k \tau^{-1}(1 \otimes_M \tilde{\sigma}(\zeta))n_k \otimes_P n_k^*$ into ${}_N X \otimes_P Q \otimes_Q Q \otimes_P Q_P$ and apply $\tau \otimes_Q \mathrm{id}_Q \otimes_P \mathrm{id}_Q$ to get

$$\sum_k 1 \otimes_M \tilde{\sigma}(\zeta) \otimes_Q n_k \otimes_P n_k^* \in {}_N M \otimes_M Y \otimes_Q Q \otimes_P Q_P. \quad (3.8)$$

From (3.9) in the below, (3.7) and (3.8) coincide with each other. Since $\Phi \otimes_P \mathrm{id}_Q = (\tau^{-1} \otimes_Q \mathrm{id}_Q \otimes_P \mathrm{id}_Q)(\mathrm{id}_M \otimes_M \sigma \otimes_P \mathrm{id}_Q)$, we have proved (3b).

Applying the Frobenius reciprocity [2, Prop. 9.70] to τ and σ, we can embed ${}_N X_P$ into ${}_N Y_P$ with

$$\hat{\tau} : {}_N X_P \ni a\xi b \mapsto \tau(a\xi b \otimes_P 1) \otimes_Q 1 = a(1 \otimes_M \eta)b \otimes_Q 1 \in {}_N M \otimes_M Y \otimes_Q Q_P,$$

$$\hat{\sigma} : {}_N X_P \ni a\xi b \mapsto 1 \otimes_M \sigma(1 \otimes_N a\xi b) = 1 \otimes_M a(\eta \otimes_Q 1)b \in {}_N M \otimes_M Y \otimes_Q Q_P,$$

where $a \in N, b \in P$. (Remark that $\{1_Q\}$ is a base of Q_Q and $\{1_M\}$ is a base of ${}_M M$.) Therefore, $\hat{\tau} = \hat{\sigma}$. We naturally identify ${}_N M \otimes_M Y \otimes_Q Q_P$ with ${}_N Y_P$. By the embedding $\hat{\tau} = \hat{\sigma}$, we may regard ${}_N X_P$ as a subspace of ${}_N Y_P$. Let $\rho : {}_N M \otimes_M Y \otimes_Q Q_P \to {}_N X_P$ be the orthogonal projection. Then we have $\tau^{-1}(\zeta_1) = \sum_k \rho(\zeta_1 \otimes_Q n_k) \otimes_P n_k^*$ and $\sigma^{-1}(\zeta_2) = \sum_j m_j \otimes_N \rho(m_j^* \otimes_M \zeta_2)$, where $\zeta_1 \in {}_N M \otimes_M Y_Q$, $\zeta_2 \in {}_M Y \otimes_Q Q_P$. The isomorphism $(\sigma \otimes_P \mathrm{id}_Q)(\mathrm{id}_M \otimes_N \tau^{-1})$ yields ${}_M M_1 \otimes_M Y_Q \simeq {}_M Y \otimes_Q Q_{1Q}$, and we have

$$(\sigma \otimes_P \mathrm{id}_Q)(\mathrm{id}_M \otimes_N \tau^{-1})(1 \otimes_M \zeta) = \zeta \otimes_Q 1 \quad (3.9)$$

for $\zeta \in Y$, because

$$\begin{array}{ccc}
{}_M M \otimes_N M \otimes_M Y_Q & \overset{\mathrm{id}_M \otimes_N \tau^{-1}}{\simeq} & {}_M M \otimes_N X \otimes_P Q_Q \\
\sum_j m_j \otimes_N m_j^* \otimes_M \zeta & \mapsto & \sum_{j,k} m_j \otimes_N \rho(m_j^* \otimes_M \zeta \otimes_Q n_k) \otimes_P n_k^*,
\end{array}$$

and

$$\begin{array}{ccc} {}_MY \otimes_Q Q \otimes_P Q_Q & \overset{\sigma^{-1}\otimes_P \mathrm{id}_Q}{\simeq} & {}_MM \otimes_N X \otimes_P Q_Q \\ \sum_k \zeta \otimes_Q n_k \otimes_P n_k^* & \mapsto & \sum_{j,k} m_j \otimes_N \rho(m_j^* \otimes_M \zeta \otimes_Q n_k) \otimes_P n_k^*. \end{array}$$

($3 \Rightarrow 2$) We denote the Hilbert subspace

$$(\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)(M \otimes_N X) = (\Phi^{-1} \otimes_P \mathrm{id}_Q)(\mathrm{id}_X \otimes_P t^*)(X \otimes_P Q)$$

of $M \otimes_N X \otimes_P Q$ by Y. This Y is closed under the left M and the right Q actions in the following way. Let $\alpha := (\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)$ and $\beta := (\Phi^{-1} \otimes_P \mathrm{id}_Q)(\mathrm{id}_X \otimes_P t^*)$. For any $\zeta \in Y$, there exist $\zeta_1 \in M \otimes_N X$ and $\zeta_2 \in X \otimes_P Q$ such that $\zeta = \alpha(\zeta_1) = \beta(\zeta_2)$. Since s is a homomorphism of M-M bimodules, we have

$$\begin{aligned} a\zeta &= a\alpha(\zeta_1) = a(\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)(\zeta_1) \\ &= (\mathrm{id}_M \otimes_N \Phi)a(s^* \otimes_N \mathrm{id}_X)(\zeta_1) \\ &= (\mathrm{id}_M \otimes_N \Phi)(s^* \otimes_N \mathrm{id}_X)(a\zeta_1) = \alpha(a\zeta_1) \end{aligned}$$

for $a \in M$. We have $\zeta b = \beta(\zeta_2)b = \beta(\zeta_2 b)$ for $b \in Q$ in the same way. Therefore, we may regard Y as an M-Q bimodule. From the construction of ${}_MY_Q$, we have

$$\begin{aligned} {}_NM \otimes_M Y_Q &= {}_NM \otimes_M [(\Phi^{-1} \otimes_P \mathrm{id})(\mathrm{id} \otimes_P t^*)(X \otimes_P Q)]_Q \\ &\simeq {}_N[(\Phi^{-1} \otimes_P \mathrm{id})(\mathrm{id} \otimes_P t^*)(X \otimes_P Q)]_Q \simeq {}_NX \otimes_P Q_Q. \end{aligned}$$

The isomorphism ${}_MY \otimes_Q Q_P \simeq {}_MM \otimes_N X_P$ follows in the same way.

Put $\eta := (\Phi^{-1} \otimes_P \mathrm{id})(\mathrm{id} \otimes_N s^*)(1 \otimes_N \xi) = (\Phi^{-1} \otimes_P \mathrm{id})(\mathrm{id} \otimes_P t^*)(\xi \otimes_P 1) \in Y$, then it satisfies $\tau(\xi \otimes_P 1) = 1 \otimes_M \eta$, and $\sigma(1 \otimes_N \xi) = \eta \otimes_Q 1$. □

Definition 3.4 *Under the same notations as above, we define an inclusion of pointed bimodules:* $({}_NK_P, \xi) \subset ({}_MH_Q, \eta)$ *when they satisfy 1 or 2 of Theorem 3.1.*

Let us explain some examples.

Example 3.5 Let

$$\begin{array}{ccc} M & \supset & Q \\ \cup & & \cup \\ N & \supset & P \end{array}$$

be a *non-degenerate* commuting square [12, P. 172] of type II_1 factors with $[M : P] < \infty$, then we have

$$({}_NN_P, \hat{1}) \subset ({}_MM_Q, \hat{1}).$$

This example means that an inclusion of bimodules is a generalized non-degenerate commuting square.

Example 3.6 Let $N \subset M$ be the GHJ subfactor [3, Sec. 4.5] with index $3+\sqrt{3}$. The following are principal and dual principal graphs of $N \subset M$.

$*$ ${}_N X_N$

${}_M Y_{1M}$ ${}_M Y_{2M}$ ${}_M Y_{3M}$ ${}_M Y_{4M}$
$*$

Each vertex of the graphs corresponds to an irreducible bimodule arising from $N \subset M$ (see [2, Chap. 10] for example). We can choose cyclic and bounded vectors $\xi \in {}_N X_N, \xi_j \in {}_M Y_{jM}$ such that

$$\begin{aligned} ({}_N X_N, \xi) &\subset ({}_M (Y_1 \oplus Y_3)_M, \eta_1 \oplus \eta_3) \\ ({}_N X_N, \xi) &\subset ({}_M (Y_1 \oplus Y_4)_M, \eta_1 \oplus \eta_4) \\ ({}_N X_N, \xi) &\subset ({}_M Y_{2M}, \xi_2). \end{aligned}$$

The first two inclusions follow from α-inductions. We get the third one because $X \simeq \overline{X}$ and $Y_2 \simeq \overline{Y}_2$ as bimodules with a map $\hat{x} \mapsto \overline{\hat{x^*}}$.

4 An application of the main theorem to the asymptotic inclusions

Let $A := M \vee (M' \cap M_\infty) \subset M_\infty =: B$ be the asymptotic inclusion obtained from a hyperfinite type II_1 subfactor $N \subset M$ with finite Jones index and finite depth [9]. By Popa's generating property [12], we may naturally identify A with $M \otimes_{\mathbf{C}} M^{\mathrm{op}}$. Let $\Delta_{N \subset M} = \{{}_M X_{jM}\}$ (resp. $\Delta_{A \subset B}$) be the finite closed system of irreducible M-M (resp. B-B) bimodules arising from $N \subset M$ (resp. $A \subset B$). Assume ${}_M X_{1M} = {}_M M_M$. By $\sum(\Delta)$, denote the set of bimodules obtained as finite direct sums of bimodules in Δ. It has been proved by Masuda [8] that the asymptotic inclusion and the Longo-Rehren inclusion [7] are essentially the same objects. Longo-Rehren's construction is described in the sector language, on the other hand, the asymptotic inclusion is a type II_1 subfactor and naturally studied in terms of bimodules. Translating the statement of [4, Lemma 4.5] into our bimodule terminology, for any ${}_B Y_B \in \sum(\Delta_{A \subset B})$, there exists ${}_M X_M \in \sum(\Delta_{N \subset M})$ such that

$$\begin{aligned} \tau : {}_A({}_M X_M \otimes_{\mathbf{C}} {}_{M^{\mathrm{op}}} M^{\mathrm{op}}{}_{M^{\mathrm{op}}}) \otimes_A B_B &\simeq {}_A B \otimes_B Y_B, \\ \sigma : {}_B B \otimes_A ({}_M X_M \otimes_{\mathbf{C}} {}_{M^{\mathrm{op}}} M^{\mathrm{op}}{}_{M^{\mathrm{op}}})_A &\simeq {}_B Y \otimes_B B_A, \end{aligned}$$

i.e.,

$${}_A(X \otimes_{\mathbf{C}} M^{\mathrm{op}}) \otimes_A B_A \simeq {}_A B \otimes_A (X \otimes_{\mathbf{C}} M^{\mathrm{op}})_A.$$

Since

$$ {}_A B_A \simeq {}_A(\bigoplus_{X_j \in \Delta_{N\subset M}} {}_M X_{jM} \otimes_{\mathbf{C}} {}_{M^{\mathrm{op}}} {X_j^{\mathrm{op}}}_{M^{\mathrm{op}}})_A \tag{4.1} $$

(by [10], see also [2, Sec. 12.6]), we have

$$ \bigoplus_j {}_A[(X \otimes_M X_j) \otimes_{\mathbf{C}} X_j^{\mathrm{op}}]_A \simeq \bigoplus_j {}_A[(X_j \otimes_M X) \otimes_{\mathbf{C}} X_j^{\mathrm{op}}]_A. $$

Since $\mathrm{Hom}({}_M {X_j}_M, {}_M X_{kM}) \simeq \delta_{j,k}\mathbf{C}$, comparing the both sides component-wise with respect to $\otimes_{\mathbf{C}}$, we have

$$ \Phi_X(j) : {}_M X \otimes_M {X_j}_M \simeq {}_M X_j \otimes_M X_M, \tag{4.2} $$

for any ${}_M {X_j}_M \in \Delta_{N\subset M}$, which corresponds to [4, Definition 4.2., (i) (1)].

Take an M-M cyclic and bounded vector $\xi \in X$. (In general, ${}_M X_M \in \sum(\Delta_{N\subset M})$ is not irreducible. Assume that ${}_M X_M$ is irreducibly decomposed as ${}_M X'{}_M \oplus {}_M X'{}_M \oplus {}_M X''{}_M$. Let $\xi'_1, \xi'_2 \in X'$ and $\xi'' \in X''$ be non-zero vectors and ξ'_1, ξ'_2 are linearly independent. Because the set of bounded vectors is dense subset of the original Hilbert space, we can take ξ'_1, ξ'_2, ξ'' so that $\xi'_1 \oplus \xi'_2 \oplus \xi''$ is M-M cyclic and bounded for ${}_M X'{}_M \oplus {}_M X'{}_M \oplus {}_M X''{}_M$.)

Put $\eta_\tau, \eta_\sigma \in Y$ with $\tau((\xi \otimes_{\mathbf{C}} 1) \otimes_A 1_B) =: 1_B \otimes_B \eta_\tau$ and $\sigma(1_B \otimes_A (\xi \otimes_{\mathbf{C}} 1)) =: \eta_\sigma \otimes_B 1_B$. We want to see

$$ (\mathrm{id}_B \otimes_B \sigma^{-1})(\tau \otimes_B \mathrm{id}_B)((\xi \otimes_{\mathbf{C}} 1) \otimes_A 1 \otimes_B 1) = 1 \otimes_B 1 \otimes_A (\xi \otimes_{\mathbf{C}} 1), $$

in order that $\eta_\tau = \eta_\sigma$ holds. Since ${}_M X_{1M} = {}_M M_M$, i.e., ${}_A A_A = {}_A(X_1 \otimes_{\mathbf{C}} X_1^{\mathrm{op}})_A$, it is enough to show that $\Phi_X(1)(\xi) = \xi$. In case of the above example, there exist a unitary

$$ A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{C}) \simeq \mathrm{End}({}_M(X' \oplus X')_M) $$

and $a'' \in \mathbf{C}$ with $|a''| = 1$ such that

$$ \Phi_X(1)(\xi'_1 \oplus \xi'_2 \oplus \xi'') = (a\xi'_1 + b\xi'_2) \oplus (c\xi'_1 + d\xi'_2) \oplus a''\xi''. $$

We can adjust τ, σ, so that $A = 1$ and $a'' = 1$. Apply the same arguments for a general ${}_M X_M$.

Let $\{\rho_j^{k,l}(e)\}$ be an orthonormal basis of $\mathrm{Hom}({}_M {X_j}_M, {}_M X_k \otimes_M X_{lM})$. By the formula of S of [8, p.256], and since the asymptotic inclusion has the trivial relative commutant, i.e.,

$$ \dim[\mathrm{Hom}({}_B B \otimes_A B_B, {}_B B_B)] = \dim[\mathrm{End}({}_B B_A)] = 1, $$

$s^* = t^*$ of (3b) is expressed as

$$ \lambda^{1/2} \sum_{\substack{j,k,l \\ e}} \sqrt{\frac{d(j)}{d(k)d(l)}} \rho_j^{k,l}(e) \otimes_{\mathbf{C}} \rho_j^{k,l}(e)^{\mathrm{op}}, $$

where $\lambda := [B:A]$ and $d(j) := [X_j]^{1/2}$. Thanks to *2* $\Rightarrow$ *3* of Theorem 3.1, we have

$$\bigoplus_{\substack{j,k,l\\ e}} \Big([\Phi_X(k) \otimes_{\mathbf{C}} \mathrm{id}_{X_k}^{\mathrm{op}}] \otimes_A [\mathrm{id}_{X_l} \otimes_{\mathbf{C}} \mathrm{id}_{X_l}^{\mathrm{op}}]\Big) \Big([\mathrm{id}_X \otimes_{\mathbf{C}} \mathrm{id}_M^{\mathrm{op}}] \otimes_A [\rho_j^{k,l}(e) \otimes_{\mathbf{C}} \rho_j^{k,l}(e)^{\mathrm{op}}]\Big)$$

$$= \bigoplus_{\substack{j,k,l\\ e}} \Big([\mathrm{id}_{X_k} \otimes_{\mathbf{C}} \mathrm{id}_{X_k}^{\mathrm{op}}] \otimes_A [\Phi_X(e)^{-1} \otimes_{\mathbf{C}} \mathrm{id}_{X_l}^{\mathrm{op}}]\Big)$$
$$\Big([\rho_j^{k,l}(e) \otimes_{\mathbf{C}} \rho_j^{k,l}(e)^{\mathrm{op}}] \otimes_A [\mathrm{id}_X \otimes_{\mathbf{C}} \mathrm{id}_M^{\mathrm{op}}]\Big) \Big(\Phi_X(j) \otimes_{\mathbf{C}} \mathrm{id}_{X_j}^{\mathrm{op}}\Big).$$

Since

$$\rho_j^{k,l}(e)^* \rho_{j'}^{k',l'}(e') = \delta_{j,j'}\delta_{k,k'}\delta_{l,l'}\delta_{e,e'},$$

comparing the right components with respect to $\otimes_{\mathbf{C}}$, we obtain

$$(\Phi_X(k) \otimes_M \mathrm{id}_{X_l})(\mathrm{id}_X \otimes_M \rho_j^{k,l}(e)) = (\mathrm{id}_{X_k} \otimes_M \Phi_X(e)^{-1})(\rho_j^{k,l}(e) \otimes_M \mathrm{id}_X)\Phi_X(j)$$

for each j, k, l, e. In particular, for every $\rho \in \mathrm{Hom}({}_M X_{j\,M}, {}_M X_k \otimes_M X_{l\,M})$, we have

$$(\Phi_X(k) \otimes_M \mathrm{id}_{X_l})(\mathrm{id}_X \otimes_M \rho) = (\mathrm{id}_{X_k} \otimes_M \Phi_X(e)^{-1})(\rho \otimes_M \mathrm{id}_X)\Phi_X(j) \qquad (4.3)$$

which corresponds to the braiding fusion equation of [4, Definition 4.2 (i) (2)]. From (4.2) and (4.3), we have shown that this $\{\Phi_X(j)\}_{X_j \in \Delta_{N\subset M}}$ is a half braiding.

5 Construction of a tower of pointed bimodules.

Let us first fix some notations. Assume that $({}_N H_{0\,P}, \xi) \subset ({}_M H_Q, \xi)$ (see Def 3.4) and φ_0 (resp. φ) corresponds to $({}_N H_{0\,P}, \xi)$ (resp. $({}_M H_Q, \xi)$). We denote the type II_1 factor obtained by the Jones basic construction for the original subfactor $N \subset M$ (resp. $P \subset Q$) by M_1 (resp. Q_1).

5.1 An extension of CP-maps.

Here we construct a CP-map φ_1 from M_1 to Q_1 which is an extension of $\varphi : M \to Q$ and makes the following diagram commutative

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \varphi_1\ } & Q_1 \\ \tilde{\mathcal{E}}\downarrow & & \downarrow\tilde{\mathcal{F}} \\ M & \xrightarrow{\ \varphi\ } & Q \end{array}$$

i.e., $\tilde{\mathcal{F}} \cdot \varphi_1 = \varphi \cdot \tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}}$ (resp. $\tilde{\mathcal{F}}$) is the unique trace-preserving conditional expectation from M_1 to M (resp. from Q_1 to Q).

We identify ${}_M H_Q$ (resp. ${}_N H_{0P}$) with ${}_M M \otimes_\varphi Q_Q$ (resp. ${}_N N \otimes_{\varphi_0} P_P$). From Definition 3.4, we have $\dim H_Q = \dim H_{0P}$. Put $\alpha := \dim H_Q$ and $n := [\alpha] \geq 1$, the biggest integer less than or equal to α. Then the following hold:

$$(M \otimes_\varphi Q)_Q \simeq \bigoplus^{\alpha} L^2(Q)_Q \quad \text{as right } Q\text{-bimodules},$$

$$(N \otimes_{\varphi_0} P)_P \simeq \bigoplus^{\alpha} L^2(P)_P \quad \text{as right } P\text{-bimodules},$$

where $\overset{\alpha}{\oplus} L^2(Q)$ means

$$\bigoplus^{\alpha} L^2(Q) := \underbrace{L^2(Q) \oplus \cdots \oplus L^2(Q)}_{n \text{ times}} \oplus pL^2(Q)$$

$$\bigoplus^{\alpha} L^2(P) := \underbrace{L^2(P) \oplus \cdots \oplus L^2(P)}_{n \text{ times}} \oplus pL^2(P)$$

with a projection $p \in P$ of its trace $\mathrm{tr}(p) = \alpha - n$. We may and do assume that these isomorphisms are compatible with the embedding of $N \otimes_{\varphi_0} P_P \subset M \otimes_\varphi Q_Q$ as right P-modules. We denote the corresponding vector of $N \otimes_{\varphi_0} P$ to

$$(0, \cdots, 0, \underset{\hat{j}}{\hat{1}}, 0, \cdots, 0) \in \bigoplus^{\alpha} L^2(P)$$

by $\xi_j \in N \otimes_{\varphi_0} P$ for $j = 1, \cdots, n$. We define ξ_{n+1} so as to correspond to $(0, \cdots, 0, \hat{p}) \in \overset{\alpha}{\oplus} L^2(P)$. We may assume that $\xi_1 = 1 \otimes_\varphi 1$, which means that we identify $1 \otimes_\varphi q \in M \otimes_\varphi Q$ with $(\hat{q}, 0, \cdots, 0) \in \overset{\alpha}{\oplus} L^2(Q)$.

Suppose a vector $\eta \in M \otimes_\varphi Q$ corresponds to $(\hat{q}_1, \cdots, \hat{q}_n) \in \overset{\alpha}{\oplus} L^2(Q)$. Then we have

$$\eta = \sum_{i=1}^{n+1} \xi_i \cdot q_i.$$

Define $u_j \in \mathrm{End}(M \otimes_\varphi Q_Q)$ with

$$u_j(\eta) = \xi_j \cdot q_1, \quad u_j^*(\eta) = \xi_1 \cdot q_j = 1 \otimes_\varphi q_j.$$

In general, we take a sequence $\eta_m = \sum_i \xi_i q_{m,i}$ in the right Q-bounded subspace of $M \otimes_\varphi Q_Q$ such that $\lim_m \|\eta - \eta_m\| = 0$, then we can define u_j. We may naturally regard these u_j's as elements of $\mathrm{End}(N \otimes_{\varphi_0} P_P)$. If we express u_j as an element of

$B(\overset{\alpha}{\bigoplus} L^2(Q)) \simeq \tilde{p}(M_{n+1}(\mathbf{C}) \otimes Q)\tilde{p}$, we have

$$u_j \longleftrightarrow {}_{j\rightarrow}\begin{pmatrix} 0 & & \\ \vdots & & \\ 1 & \text{\Large 0} & \\ \vdots & & \\ 0 & & \end{pmatrix} \in \tilde{p}(M_{n+1}(\mathbf{C}) \otimes Q)\tilde{p}$$

where

$$\tilde{p} = \begin{pmatrix} 1 & & & \text{\Large 0} \\ & \ddots & & \\ & & 1 & \\ \text{\Large 0} & & & p \end{pmatrix} \in M_{n+1}(\mathbf{C}) \otimes Q.$$

Let $\varphi(x) = V^*\pi(x)V$ with $x \in M$ be the Stinespring dilation decomposition, namely, $\pi : M \to B(M \otimes_\varphi Q)$ is the representation of M and V satisfies $V : L^2(Q) \ni \hat{q} \to 1 \otimes_\varphi q \in M \otimes_\varphi Q$.

Now we are interested in how our π, V and vectors in $M \otimes_\varphi Q$ are expressed in the matricial notations. The following correspondences hold:

$$\pi(x) \longleftrightarrow {}_{k\rightarrow}\begin{pmatrix} \ddots & \overset{j}{\downarrow} & \cdot^{\cdot^{\cdot}} \\ & V^*u_k^*\pi(x)u_jV & \\ \cdot^{\cdot^{\cdot}} & & \ddots \end{pmatrix} \in \tilde{p}(M_{n+1}(\mathbf{C}) \otimes Q)\tilde{p},$$

$$V \longleftrightarrow {}_{j\rightarrow}\begin{pmatrix} V^*u_1^*V \\ \vdots \\ V^*u_j^*V \\ \vdots \\ V^*u_{n+1}^*V \end{pmatrix} \in \bigoplus^{n+1} Q$$

and

$$x \otimes_\varphi y \longleftrightarrow {}_{j\rightarrow}\begin{pmatrix} V^*u_1^*\pi(x)V\hat{y} \\ \vdots \\ V^*u_j^*\pi(x)V\hat{y} \\ \vdots \\ V^*u_{n+1}^*\pi(x)V\hat{y} \end{pmatrix} \in \bigoplus^{n+1} L^2(Q).$$

Definition 5.1 *We extend this π to a $*$-homomorphism $\pi_1 : M_1 \to B(M \otimes_\varphi Q)$ by setting*

$$\pi_1(e) := \sum_{i=1}^{n+1} u_i V f V^* {u_i}^*,$$

where e, f are the Jones projections of $N \subset M, P \subset Q$ respectively.

With the matricial notations, $\pi_1(e)$ is denoted by

$$\pi_1(e) \longleftrightarrow \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix} \in M_{n+1}(\mathbf{C}) \otimes B(L^2(Q)).$$

To show well-definedness of π_1 we need some claims.

Claim 5.2 *The vector $\xi_k \in N \otimes_{\varphi_0} P$ is right P-bounded, i.e., right Q-bounded.*

Proof For each j and for any $p \in P$, we have

$$\|\xi_k \cdot p\|_{N \otimes_\varphi P} = \|(0, \cdots, \hat{p}, \cdots, 0)\|_{\bigoplus^{\alpha} L^2(P)} = \|\hat{p}\|_{L^2(P)}.$$

□

Claim 5.3 *For any $x \in M$ and $k = 1, \cdots, n+1$, the vector $x \cdot \xi_k$ is right Q-bounded and we have*

$$x \cdot \xi_k = \sum_{j=1}^{n+1} \xi_j \langle \xi_j \mid x \cdot \xi_k \rangle_Q^\circ.$$

Proof It is easy to see the right Q-boundedness of $x \cdot \xi_k$ from the following. For an arbitrary $q \in Q$, we have

$$\|(x \cdot \xi_k) \cdot q\| = \|x \cdot u_k(1 \otimes q)\| \le \|x\|_\infty \|1 \otimes q\| = \|x\|_\infty \|q\|_2.$$

According to [2, Definition 9.7] and Claim 5.2, we can uniquely define the right Q-valued inner product of ξ_j and $x \cdot \xi_k$ and denote it by $\langle \xi_j \mid x \cdot \xi_k \rangle_Q^\circ$. Suppose $x \cdot \xi_k = \sum_{j=1}^{n+1} \xi_j \cdot x_j$ is the right Q decomposition with the right Q-basis, then we have

$$\langle \xi_j \mid x \cdot \xi_k \rangle_Q^\circ = \langle \xi_j \mid \sum_l \xi_l \cdot x_l \rangle_Q^\circ = x_j.$$

The orthogonality implies $\langle \xi_i \cdot y \mid \xi_j \cdot z \rangle_Q^\circ = \delta_{i,j} y^* z$.

□

Claim 5.4 *For any $j, k = 1, \cdots, n+1$, and $x \in M$, we have*

$$\mathcal{F}_P(V^* u_j^* \pi(x) u_k V) = V^* u_j^* \pi(\mathcal{E}_N(x)) u_k V.$$

This claim means that the commutativity $\mathcal{F}_P \cdot \varphi = \varphi \cdot \mathcal{E}_N$ also holds in the matricial level, namely, the following diagram is commutative.

$$\begin{array}{ccc} M \ni x & \longrightarrow & \begin{array}{c} \scriptstyle k \\ \scriptstyle \downarrow \end{array} \\ & & {\scriptstyle j\rightarrow}\begin{pmatrix} \ddots & & \cdots \\ & V^* u_j^* \pi(x) u_k V & \\ \cdots & & \ddots \end{pmatrix} \\ \downarrow & & \downarrow \\ & & \begin{array}{c} \scriptstyle k \\ \scriptstyle \downarrow \end{array} \\ N \ni \mathcal{E}_N(x) & \longrightarrow & {\scriptstyle j\rightarrow}\begin{pmatrix} \ddots & & \cdots \\ & V^* u_j^* \pi(\mathcal{E}_N(x)) u_k V & \\ \cdots & & \ddots \end{pmatrix} \end{array}$$

Proof It is enough to show

$$fV^*u_j^*\pi(x)u_kVf\hat{p} = V^*u_j^*\pi(\mathcal{E}_N(x))u_kV\hat{p} \quad \text{for } p \in P.$$

Take a sequence $\xi_{i,n} \in N \otimes_{\varphi_0}^{\mathrm{alg}} P$ such that $\lim_{n\to\infty} \|\xi_i - \xi_{i,n}\| = 0$. Since the right Q-valued inner product $\langle \xi_{j,m} \mid x \cdot \xi_{k,n} \rangle_Q^\circ$ converges to $\langle \xi_j \mid x \cdot \xi_k \rangle_Q^\circ$ in the weak operator topology, we have

$$\begin{aligned} fV^*u_j^*\pi(x)u_kVf\hat{p} &= fV^*u_j^*(x \cdot \xi_k \cdot p) = fV^*u_j^*(\sum_{l=1}^{n+1} \xi_l \cdot \langle \xi_l \mid x\xi_k \rangle_Q^\circ \, p) \\ &= fV^*(\xi_1 \cdot \langle \xi_j \mid x\xi_k \rangle_Q^\circ \, p) = \mathcal{F}_P(\langle \xi_j \mid x\xi_k \rangle_Q^\circ)\hat{p} \\ &= \text{w-}\lim_{m,n} \mathcal{F}_P(\langle \xi_{j,m} \mid x\xi_{k,n} \rangle_Q^\circ)\hat{p}, \end{aligned}$$

and

$$\begin{aligned} V^*u_j^*\pi(\mathcal{E}_N(x))u_kV\hat{p} &= V^*u_j^*(\mathcal{E}_N(x) \cdot \xi_k \cdot p) \\ &= \langle \xi_j \mid \mathcal{E}_N(x)\xi_k \rangle_P^\circ \hat{p} = \text{w-}\lim_{m,n} \langle \xi_{j,m} \mid \mathcal{E}_N(x)\xi_{k,n} \rangle_P^\circ \hat{p} \end{aligned}$$

thanks to Claim 5.3. Recall the equality

$$\langle a \otimes_\varphi b \mid c \otimes_\varphi d \rangle_Q^\circ = b^*\varphi(a^*c)d \tag{5.1}$$

holds on $M \otimes_\varphi^{\mathrm{alg}} Q$ and $\xi_{i,n} \in N \otimes_{\varphi_0}^{\mathrm{alg}} P$. Since we obtain $\varphi_0 \cdot \mathcal{E}_N = \mathcal{F}_P \cdot \varphi$ by (3.3), we have $\mathcal{F}_P(\langle \xi_{j,m} \mid x\xi_{k,n} \rangle_Q^\circ) = \langle \xi_{j,m} \mid \mathcal{E}_N(x)\xi_{k,n} \rangle_P^\circ$. □

Proposition 5.5 *The $*$-homomorphism π_1 is well-defined.*

Proof It is enough to show the following equality: $\pi_1(exe) = \pi_1(\mathcal{E}_N(x)e)$ for $x \in M$. From Claim 5.4, we have

$$\begin{aligned} \pi_1(exe) &= \pi_1(e)\pi_1(x)\pi_1(e) = \sum_{j,k}^{n+1} u_jVfV^*u_j^*\pi(x)u_kVfV^*u_k^* \\ &= \sum_{j,k}^{n+1} u_jVV^*u_j^*\pi(\mathcal{E}_N(x))u_kVfV^*u_k^* = \pi(\mathcal{E}_N(x))\pi_1(e) \\ &= \pi_1(\mathcal{E}_N(x)e). \end{aligned}$$

□

Now we are ready to define a CP-map from M_1 to Q_1 which is an extension of φ.

Theorem 5.6 *Let $V : L^2(Q) \to M \otimes_\varphi Q$ be the linear map as above. For $x \in M_1$, let*

$$\varphi_1(x) := V^*\pi_1(x)V.$$

Then φ_1 is a CP-map from M_1 to Q_1 and have a property such as $\mathcal{F}_Q \cdot \varphi_1 = \varphi \cdot \mathcal{E}_M$, which means, $\varphi_1^* = \varphi$ on Q. Moreover, if φ is unital, so is φ_1.

Proof Since $\varphi_1(x)$ is in JPJ, $\varphi_1(x)$ is an element of Q_1.

It is known that an arbitrary element of M_1 is finite sum of elements such as aeb where $a, b \in M$ [2, Proposition 9.28], so that we only have to check $\mathcal{F}_Q \cdot \varphi_1(aeb) = \varphi \cdot \mathcal{E}_M(aeb)$. We have

$$\begin{aligned}
\mathcal{F}_Q \cdot \varphi_1(aeb) &= \mathcal{F}_Q(V^*\pi_1(a)\pi_1(e)\pi_1(b)V) \\
&= \sum_j^{n+1} \mathcal{F}_Q(V^*\pi(a)u_j V f V^* u_j^* \pi(b)V) \\
&= \sum_j^{n+1} (V^*\pi(a)u_j V)\mathcal{F}_Q(f)(V^* u_j^* \pi(b)V) \\
&= \sum_j^{n+1} [Q:P]^{-1} V^*\pi(a)u_j V V^* u_j^* \pi(b)V = [Q:P]^{-1}V^*\pi(a)\pi(b)V \\
&= [Q:P]^{-1}\varphi(ab) = \varphi \cdot \mathcal{E}_M(aeb).
\end{aligned}$$

□

Example 5.7 (cf. Example 3.5.) Let

$$\begin{array}{ccc}
M & \supset & Q \\
\cup & & \cup \\
N & \supset & P
\end{array}$$

be a non-degenerate commuting square of type II_1 factors with $[M : P] < \infty$. Let $\varphi : M \to Q$ and $\varphi_0 : N \to P$ be the unique trace-preserving conditional expectations which correspond to $({}_M M_Q, \hat{1})$ and $({}_N N_P, \hat{1})$ respectively. Applying Theorem 5.6, we also obtain $\varphi_1 : M_1 \to Q_1$ the unique trace-preserving conditional expectation.

Proof First we remark that Jones projections of $N \subset M$ and $P \subset Q$ can be identified naturally, and denote it by f. Let V, π, π_1 be as above. For $x, y \in Q$, we have

$$\begin{aligned}
\pi_1(xfy)V &= V(xfy) \qquad \text{on} \qquad L^2(Q), \\
V^*\pi_1(xfy) &= (xfy)V^* \qquad \text{on} \qquad {}_M M \otimes_\varphi Q_Q \simeq {}_M M_Q.
\end{aligned}$$

Thus for $a \in M_1$ and $p, q \in Q_1$,

$$\varphi_1(paq) = V^*\pi_1(p)\pi_1(a)\pi_1(q)V = pV^*\pi_1(a)Vq = p\varphi_1(a)q$$

holds. Since $\varphi_1(f) = f$, the trace-preserving condition follows easily. □

5.2 An extension of an inclusion of pointed bimodules

We translate back the extension procedure of CP-maps into the bimodule language.

Definition 5.8 *Suppose we have an inclusion of bimodules $({}_N H_{0P}, \xi) \subset ({}_M H_Q, \xi)$. Let $\{m_j\} \subset M$ (resp. $\{n_k\} \subset Q$) be a Pimsner-Popa basis of $N \subset M$ (resp. $P \subset Q$), and $\Phi : H \to H_0$ be the orthogonal projection, compatible with the left N and the right P-action. By using the relative tensor products, we construct an M_1-Q_1 bimodule $({}_{M_1} H_{1Q_1}, \xi_1)$ as follows.*

$$ {}_{M_1} H_{1Q_1} := {}_{M_1} M \otimes_N H_0 \otimes_P Q_{Q_1} $$

$$ \xi_1 := [Q:P]^{-1/2} \sum_{j,k} m_j \otimes_N \Phi(m_j^* \xi n_k) \otimes_P n_k^* $$

For $aeb \in M_1, pfq \in Q_1, x, z \in M, y, w \in Q$ and $\xi, \eta \in H_0$, bimodule structure and an inner product of H_1 are well-defined as

$$ \begin{aligned} (aeb) \cdot x \otimes_N \xi \otimes_P y \cdot (pfq) &:= a\mathcal{E}_N(bx) \otimes_N \xi \otimes_P \mathcal{F}_P(yp)q, \\ \langle x \otimes_N \xi \otimes_P y \mid z \otimes_N \eta \otimes_P w \rangle_{H_1} &:= \langle \mathcal{E}_N(z^*x)\xi\mathcal{F}_P(yw^*) \mid \eta \rangle_{H_0}. \end{aligned} $$

Since

$$ \dim_{M_1} H_1 = \dim_{M_1} M \dim_N H_0 \dim_P Q = \dim_N H_0 = \dim_M H, $$

we have $({}_M H_Q, \xi) \subset ({}_{M_1} H_{1Q_1}, \xi_1)$ (see Def 3.4) with the following map:

$$ {}_M H_Q \ni \eta \mapsto [Q:P]^{-1/2} \sum_{j,k} m_j \otimes_N \Phi(m_j^* \eta n_k) \otimes_P n_k^* \in {}_{M_1} H_{1Q_1}. $$

Remark that this map is norm-preserving because

$$ \begin{aligned} \| \sum_{j,k} m_j \otimes_N \Phi(m_j^* \eta n_k) \otimes_P n_k^* \|^2 &= \sum_{j,k} \|\Phi(m_j^* \eta n_k)\|^2 = \sum_{j,k} \|\pi_1(e)(m_j^* \eta n_k)\|^2 \\ &= \sum_{j,k} \|\pi_1(e)\pi_1(m_j^*)(\eta n_k)\|^2 \\ &= \sum_{j,k} \langle \pi_1(m_j e m_j^*)\eta n_k \mid \eta n_k \rangle \\ &= \sum_{k} \langle \eta n_k \mid \eta n_k \rangle = [Q:P]\|\eta\|^2 \end{aligned} $$

and we have $\xi = \xi_1$ by the map.

Theorem 5.9 *With the same notations as in Subsection 5.1, we have*

$$ {}_{M_1} M_1 \otimes_{\varphi_1} Q_{1Q_1} \simeq {}_{M_1} M \otimes_N (N \otimes_{\varphi_0} P) \otimes_P Q_{Q_1} $$

and

$$ {}_{Q_1} Q_1 \otimes_{\varphi_1^*} M_{1M_1} \simeq {}_{Q_1} Q \otimes_P (P \otimes_{\varphi_0^*} N) \otimes M_{M_1}. $$

Proof On the dense part, consider the following map:

$$\begin{array}{ccc} {}_{M_1}M_1 \otimes_{\varphi_1} Q_{1Q_1} & \to & {}_{M_1}M \otimes_N (N \otimes_{\varphi_0} P) \otimes_P Q_{Q_1} \\ \Cup & & \Cup \\ aeb \otimes_{\varphi_1} xfy & \mapsto & [Q:P]^{-1/2} a \otimes_N \Phi(b \otimes_{\varphi_0} x) \otimes_P y, \end{array}$$

which gives an isomorphism. The second isomorphism follows in the same way. □

Example 5.10 (cf. Example 3.5, Example 5.7.) Let N, M, P, Q be as in Example 3.5. If we extend $({}_N N_P, \hat{1}) \subset ({}_M M_Q, \hat{1})$ by Definition 5.8 and Theorem 5.9, we have

$${}_{M_1}M \otimes_N N \otimes_P Q_{Q_1} \simeq {}_{M_1}M \otimes_N (N \otimes_{\varphi_0} P) \otimes_P Q_{Q_1} \simeq {}_{M_1}M_1 \otimes_{\varphi_1} Q_{1Q_1} \simeq {}_{M_1}M_{1Q_1}$$

by the map

$$\hat{a} \otimes_N \hat{b} \otimes_P \hat{c} \mapsto [Q:P]^{1/2} a\widehat{fbf}c$$

with the common Jones projection f of $N \subset M$ and $P \subset Q$.

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