

Central sequence subfactors and double commutant properties

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Abstract

First, we construct the Jones tower and tunnel of the central sequence subfactor arising from a hyperfinite type II_1 subfactor with finite index and finite depth, and prove each algebra has the double commutant property in the ultraproduct of the enveloping II_1 factor. Next, we show the equivalence between Popa's strong amenability and the double commutant property of the central sequence factor for subfactors as above without assuming the finite depth condition.

1 Introduction

Subfactor theory was initiated by V. F. R. Jones in the 80's ([J2]). The central sequence subfactor, which is the key concept in this paper, is one of the important notions in the analytical approaches to subfactor theory. We review the historical background.

Let R be the hyperfinite II_1 factor, G a finite group and α an outer action of G on R . We fix a free ultrafilter ω on \mathbf{N} . Originally, it has been known that the classification of group actions can be understood conceptually in terms of the subfactor $R_\omega^G \subset R_\omega$, where R_ω stands for a central sequence algebra $R^\omega \cap R'$. We considered the following subfactor of S. Popa ([P3])

$$N := \left\{ \left(\begin{array}{ccc} \alpha_{g_1}(x) & & \\ & \ddots & \\ & & \alpha_{g_n}(x) \end{array} \right) \middle| x \in R \right\} \subset M := R \otimes M_n(\mathbf{C}),$$

where α_g is the image of $g \in G$ under a homomorphism (“action”) α of G into the group $\text{Aut}(R)$ of $*$ -automorphism of R . (This example was initially hinted by Jones.) Then we have $N^\omega \cap M' = R_\omega^G$ and $M_\omega = R_\omega$. This is a special case of the *central sequence subfactor* $N^\omega \cap M' \subset M^\omega \cap M'$. The central sequence subfactor has been introduced by A. Ocneanu with intention of generalizing the classification theory of group actions on factors to that of “actions” of new algebraic objects called *paragroups*

([O1], [EK]). That is, let $N \subset M$ be a hyperfinite type II_1 subfactor with finite index and finite depth, and we construct a paragroup \mathcal{G} from $N \subset M$, then the idea of the paragroup theory is to regard $N \subset M$ as “ $N \subset N \rtimes \mathcal{G}$.”

S. Popa has proved in [P4] that we can reconstruct the original subfactor from the paragroup. This is the *generating property* of Popa. It gives another proof of the uniqueness of outer actions of a finite group on the hyperfinite II_1 factor. He has also defined a notion of amenability of subfactors based on an analogy with groups and proved the equivalence of strong amenability and the generating property for hyperfinite type II_1 subfactors with finite indices. This theorem of Popa also implies several classification results about group actions on factors.

We obtain another type II_1 subfactor from $N \subset M$, which is called the *asymptotic inclusion*. It is defined by $M \vee (M' \cap M_\infty) \subset M_\infty$, where we denote the enveloping algebra of $N \subset M$ by M_∞ . Recently it has been studied in various fields such as topological quantum field theory ([EK, Section 12]), sector theory ([LR]), quantum doubles and so on.

Several properties of the central sequence subfactor have been studied by Y. Kawahigashi and Ocneanu. The following theorem of Ocneanu is especially important. ([O2], [EK, Theorem 15.32].) We discuss it later in Section 4.

Theorem (Ocneanu) The paragroups of the central sequence subfactor and of the asymptotic inclusion are mutually dual.

Looking at the proof of the theorem, we notice that $(M^\omega \cap M')' \cap M_\infty^\omega \subset (N^\omega \cap M')' \cap M_\infty^\omega$ has the same higher relative commutant as the asymptotic inclusion, and

$$\begin{aligned} N^\omega \cap M' &= ((N^\omega \cap M')' \cap M_\infty^\omega)' \cap M_\infty^\omega, \\ M^\omega \cap M' &= ((M^\omega \cap M')' \cap M_\infty^\omega)' \cap M_\infty^\omega. \end{aligned}$$

We construct the Jones towers and tunnels from

$$N^\omega \cap M' \subset M^\omega \cap M'$$

and

$$(M^\omega \cap M')' \cap M_\infty^\omega \subset (N^\omega \cap M')' \cap M_\infty^\omega$$

in M_∞^ω . We set $P_0 := N^\omega \cap M'$, $P_1 := M^\omega \cap M'$, $P_0^c := (N^\omega \cap M')' \cap M_\infty^\omega$ and $P_1^c := (M^\omega \cap M')' \cap M_\infty^\omega$, where “c” stands for the relative commutant in M_∞^ω . Then we have,

$$\begin{aligned} \cdots \subset P_{-2} \subset P_{-1} \subset P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots \subset P_\infty \subset M_\infty^\omega \\ \cdots \subset Q_{-3} \subset Q_{-2} \subset P_1^c \subset P_0^c \subset Q_1 \subset Q_2 \subset \cdots \subset Q_\infty \subset M_\infty^\omega. \end{aligned}$$

Actually, we could choose them so that they satisfy $P_{-k} = Q_k^c$ and $Q_{-k} = P_k^c$.

In the first half of this paper, we aim to show the double commutant properties, such as $P_k^{cc} = P_k$ and $Q_k^{cc} = Q_l$ for $k = 0, 1, 2, \dots, \infty$. When $k < \infty$, the conclusions follow just from estimating the Jones indices $[P_k^{cc} : P_k]$ and $[Q_k^{cc} : Q_k]$. It is not

so complicated. However, in case of $k = \infty$, we could only prove $Q_\infty^{cc} = Q_\infty$. The other one $P_\infty^{cc} = P_\infty$ is still open. The proof deeply owes to the special form of $P_1 = M^\omega \cap M'$. More precisely, when we have a sequence in P_1 , we actually have a sequence of sequences of operators. Then we would like to construct a new sequence of operators within P_1 using these sequences. Such a construction works for P_1 , but not for Q_1 .

In the second half of this paper, we eliminate the finite depth condition of $N \subset M$, and we shall prove the equivalence of the following conditions.

(1) The subfactor is strongly amenable in Popa's sense.

(2) The central sequence factor $M^\omega \cap M'$ has the double commutant property in M_∞^ω .

In the rest, we explain the outline of this proof. Since M is hyperfinite, we could identify it with $\otimes_{n=1}^\infty M_2(\mathbf{C})$, and set $A_k := \otimes_{n=1}^k M_2(\mathbf{C})$. The direction (1) \Rightarrow (2) is easy from Ocneanu's central freedom lemma. For the converse direction (2) \Rightarrow (1), we set $A := (M' \cap M_\infty)' \cap M_\infty$ and consider the following non-degenerate commuting squares.

$$\begin{array}{ccc} M & \subset & A \\ \cup & & \cup \\ A'_k \cap M & \subset & A'_k \cap A \end{array} \quad \text{with } k = 1, 2, 3, \dots$$

What we want to prove is that the next commuting square is non-degenerate.

$$\begin{array}{ccc} M^\omega & \subset & A^\omega \\ \cup & & \cup \\ M' \cap M^\omega & \subset & M' \cap A^\omega. \end{array}$$

Our idea is to “pile up” the first commuting squares and show the non-degeneracy of the second one. This proof owes much to the paper [PP] as we see in Section 5.

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2 Preliminaries

In the next section we shall study properties of the Jones tower arising from a central sequence subfactor $M' \cap N^\omega \subset M' \cap M^\omega$. We use the following notations. Let M be a hyperfinite II_1 factor and $N \subset M$ be a type II_1 subfactor with finite index and finite depth. (Here we do not assume the trivial relative commutant condition.) In the following, we denote the Jones index by $[M : N]$, i.e., $[M : N] = \dim_N L^2(M)$. (This definition makes sense whenever N is a II_1 factor. In fact, in Section 5 we define the Jones index without assuming the factoriality of M .) Then we have the Jones tower

$$N \subset M \subset M_1 \subset M_2 \subset M_3 \subset \dots \subset M_\infty.$$

Here M_∞ means the weak closure of $\bigcup_{k=0}^\infty M_k$ in the GNS-representation with respect to the trace.

In the following, we recall some results about the central sequence subfactor.

(1) If we fix a free ultrafilter ω over \mathbf{N} , we obtain an inclusion of type II_1 factors; $M' \cap N^\omega \subset M' \cap M^\omega$. We write M^ω for the ultraproduct algebra and $M' \cap M^\omega$ for the central sequence algebra. The subfactor is called the *central sequence subfactor* of $N \subset M$. It has been introduced by Ocneanu [O1], [O2]. (See [EK, Section 15.4].)

(2) It is known that $M' \cap N^\omega \subset M' \cap M^\omega$ has a trivial relative commutant. (See [EK, Lemma 15.25]. From the proof of this Lemma, one sees easily that the trivial relative commutant condition of $N \subset M$ is not needed for proving that of $M' \cap N^\omega \subset M' \cap M^\omega$!)

(3) Let $\{X_i\}_i$ be the set of (isomorphism classes of) the N - N bimodules arising from ${}_N M_M$. We set $\gamma = \sum_i [X_i]$ and call it the *global index* of $N \subset M$, where we denote by $[X_i]$ the Jones index of the bimodule X_i . As in [EK, Theorem 12.24, Lemma 15.25], we know that the index of the asymptotic inclusion of $N \subset M$, i.e., $M \vee (M' \cap M_\infty) \subset M_\infty$, is equal to the global index of $N \subset M$, and the index of the central sequence subfactor is also equal to the global index. That is,

$$[M' \cap M^\omega : M' \cap N^\omega] = [M_\infty : M \vee (M' \cap M_\infty)] = \gamma.$$

(This was first noted by Ocneanu [O1].) The asymptotic inclusion has been recently studied from many points of view, such as topological quantum field theory, group actions on factors, sector theory, and so on. When a subfactor is given by a crossed product by an outer action of a finite group G on the hyperfinite II_1 factor R , i.e., $R \subset R \rtimes G$, the global index equals to $\#G$. And the asymptotic inclusion of $R \subset R \rtimes G$ is given by $Q^{G \times G} \subset Q^G$, where Q is another hyperfinite II_1 factor, and G is embedded into $G \times G$ by $g \mapsto (g, g)$. Recalling that a paragroup is a generalization of a group, we may consider that the global index is the “order” of a paragroup and the asymptotic inclusion gives the quantum double of a paragroup in an appropriate sense. (See [EK, Section 12.8].)

(4) The principal graph of the asymptotic inclusion of $N \subset M$ is the connected component of the fusion graph of the bimodule system arising from $N \subset M$ containing the vertex corresponding to the bimodule ${}_M M_M$. Since $N \subset M$ is of finite depth, so is $M \vee (M' \cap M_\infty) \subset M_\infty$. (See [O2], [EK, page 663].)

(5) The algebra M_∞^ω is a II_1 factor. For simplicity of notations, we set $P_0 := M' \cap N^\omega$ and $P_1 := M' \cap M^\omega$. We denote $P'_k \cap M_\infty^\omega$ by P_k^c . Then, both P_0^c and P_1^c are II_1 factors and they satisfy $[P_0^c : P_1^c] = [P_1 : P_0] = \gamma$. (See [EK, Lemmas 15.26, 15.27 and 15.30].)

(6) We also recall the important facts. From the string algebra theory, the central freedom lemma of Ocneanu (we later explain it in Section 5) and Popa’s generating property, we have $P_0^{cc} = P_0$ and $P_1^{cc} = P_1$. (This is also due to Ocneanu. See [EK, Theorem 15.32].) In the next section we study this topic in more detail.

(7) If we choose a downward Jones projection $e \in P_0^c$ with $E_{P_1^c}(e) = 1/\gamma$ and set

$P_2 := \langle P_1, e \rangle$, $P_2^c = P_1^c \cap \{e\}'$, then we have both $P_0 \subset P_1 \subset P_2$ and $P_2^c \subset P_1^c \subset P_0^c$ are standard. By repeating this procedure, we can construct the Jones tower

$$P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots \subset M_\infty^\omega$$

such that

$$\cdots \subset P_3^c \subset P_2^c \subset P_1^c \subset P_0^c$$

is a tunnel. (This has been noted by Ocneanu. See [EK, Lemma 15.30].)

(8) Since $P_1^c \subset P_0^c$ are type II₁ factors and $P_0^{cc} = P_0$ and $P_1^{cc} = P_1$ as we have mentioned in (5) and (6), we also have the Jones tower

$$P_1^c \subset P_0^c \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset M_\infty^\omega$$

such that

$$\cdots \subset Q_3^c \subset Q_2^c \subset Q_1^c \subset P_0 = P_0^{cc} \subset P_1 = P_1^{cc}$$

is a tunnel. (See [EK, Lemma 15.30].)

We recall the Kosaki index and the Pimsner–Popa index.

Let $A \subset B$ be von Neumann algebras, and $P(B, A)$ be the set of all faithful normal semifinite operator-valued weights from B onto A . In [H, Theorem 5.9], Haagerup has proved the equivalence between $P(B, A) \neq \emptyset$ and $P(A', B') \neq \emptyset$. Later, Kosaki has noticed the existence of the canonical order-reversing bijection from $P(B, A)$ onto $P(A', B')$ in [K], which we denote by $E \mapsto E^{-1}$. When A, B are factors with a fixed normal conditional expectation E , Kosaki defined in [K, Definition 2.1] Index E as the scalar $E^{-1}(1)$, and proved that Index E does not depend on a Hilbert space H on which B is represented, i.e., $B \subset B(H)$. (See [K, Theorem 2.2].) When Index $E < \infty$, by setting $\tau = (E^{-1}(1))^{-1}$, he also noted that τE^{-1} is a conditional expectation from A' onto B' .

Especially when A and B are type II₁ factors and $E : B \rightarrow A$ satisfies $\text{tr}_A \circ E = \text{tr}_B$, the Jones index $[B : A] := \dim_A L^2(B)$ equals to Index E . (See [K, page 133].) In the following, we write $[B : A]_{K,E}$ for Index E to distinguish this index from the Jones index and the Pimsner–Popa index which we define below.

Definition 2.1 [PP, Section 2] [P5, Definition 1.1.1] *Let $A \overset{E}{\subset} B$ be an inclusion of von Neumann algebras with a conditional expectation E from B onto A . Then we denote*

$$(\sup\{\lambda \mid E(x) \geq \lambda x, x \in B_+\})^{-1}$$

by $[B_1 : B_2]_{PP,E}$, with the convention $0^{-1} = \infty$.

In case both B_1 and B_2 are type II₁ factors, this index coincides with the Jones index. (See [PP, Proposition 2.1].)

Definition 2.2 [P5, section 1.1] *Let $A \overset{E}{\subset} B$ be as above. If $C \overset{F}{\subset} D$ is another inclusion of von Neumann algebras with a conditional expectation F from D onto C such that $C \subset A, D \subset B$, and $F = E|_D$, then we call the square*

$$\begin{array}{ccc} A & \overset{E}{\subset} & B \\ \cup & & \cup \\ C & \overset{F}{\subset} & D \end{array}$$

a commuting square. If $\overline{\text{sp}}AD = B$, we call it a non-degenerate commuting square.

This is a generalization of a case having a trace, which has been studied in [P2], [GHJ]. When the square

$$\begin{array}{ccc} A & \overset{E}{\subset} & B \\ \cup & & \cup \\ C & \overset{F}{\subset} & D \end{array}$$

is a commuting square, we trivially have $[B : A]_{PP,E} \geq [D : C]_{PP,F}$ by the definition.

The relation between the Kosaki index and the Pimsner–Popa index is noted in [BDH, page 224] and [L, Theorem 4.1, Corollary 4.2] as follows.

Proposition 2.3 *Let $A \overset{E}{\subset} B$ be an inclusion of infinite dimensional factors with a conditional expectation E from B onto A . Then we have $[B : A]_{K,E} = [B : A]_{PP,E}$.*

3 Double commutant property

In this section we shall prove the double commutant property of the Jones tower of $N' \cap M^\omega \subset M' \cap M^\omega$. We need the following two easy lemmas.

Lemma 3.1 *Let $A \overset{E}{\subset} B$ be type II_1 subfactors of M_∞^ω and we represent these von Neumann algebras on $L^2(M_\infty^\omega)$. Let E be the unique trace-preserving conditional expectation from B onto A with $\text{Index } E < \infty$. Then we have*

$$[A' : B']_{K,\tau E^{-1}} = [B : A].$$

Proof By the definition of $\text{Index } E$ and the property of the map $E \mapsto E^{-1}$, we have the following.

$$\begin{aligned} [A' : B']_{K,\tau E^{-1}} &= \text{Index } (\tau E^{-1}) = (\tau E^{-1})^{-1}(1) \\ &= \tau^{-1}(E^{-1})^{-1}(1) = \tau^{-1} = [B : A]_{K,E} = [B : A] \end{aligned}$$

The fourth equality follows from the fact $(E^{-1})^{-1} = E$. (See [K, page 126].)

□

Lemma 3.2 *Let $A \subset B$ and $B^c \subset A^c$ be inclusions of II_1 factors contained in M_∞^ω , and represent them on $L^2(M_\infty^\omega)$. Let E be the unique trace-preserving conditional expectation from A^c onto B^c . Then the square*

$$\begin{array}{ccc} A^{c'} & \xrightarrow{\tau E^{-1}} & B^{c'} \\ \cup & & \cup \\ A^{cc} & \xrightarrow{F} & B^{cc} \end{array}$$

where $F = (\tau E^{-1})|_{B^{cc}}$ and $\tau = (E^{-1}(1))^{-1}$, is a commuting square. Furthermore, when we assume that $F(B) \subset A$ and $A' \cap B = \mathbf{C}$, $F|_B$ is the unique trace-preserving conditional expectation from B onto A .

Proof Since $M'_\infty \subset A^{c'} \subset B^{c'}$ and $M''_\infty = M_\infty$, the commuting square condition easily follows. To make it sure, let x and y be arbitrary elements of B^{cc} and $M_\infty^{\omega'}$ respectively. Since $M_\infty^{\omega'} \subset A^{c'} \subset B^{c'}$, y is in $A^{c'}$. Then we obtain

$$\tau E^{-1}(x)y = \tau E^{-1}(xy) = \tau E^{-1}(yx) = y\tau E^{-1}(x),$$

i.e.,

$$\tau E^{-1}(x) \in (M_\infty^{\omega'})' \cap A^{c'} = M_\infty^\omega \cap A^{c'} = A^{cc},$$

for any $x \in B^{cc}$. Therefore F is a conditional expectation from B^{cc} onto A^{cc} .

When $A' \cap B = \mathbf{C}$, it is known that A has the unique conditional expectation onto B . (See [S, Proposition 10.17].) Thus in this case $F|_B$ is the unique trace-preserving conditional expectation from B onto A . \square

Now we have two main theorems about the double commutant property.

Theorem 3.3 *We have $P_k^{cc} = P_k$ and $Q_k^{cc} = Q_k$ for any $k \in \mathbf{N}$.*

Proof We prove the theorem by induction on k . We have already known that $P_0^{cc} = P_0$ and $P_1^{cc} = P_1$. Suppose we have $P_k^{cc} = P_k$. We denote the unique trace-preserving conditional expectation from P_k^c to P_{k+1}^c by E . And we set τ as above. We remark that the square

$$\begin{array}{ccc} P_k^{c'} & \xrightarrow{\tau E^{-1}} & P_{k+1}^{c'} \\ \cup & & \cup \\ P_k^{cc} & \xrightarrow{F} & P_{k+1}^{cc} \end{array}$$

where $F = (\tau E^{-1})|_{P_{k+1}^{cc}}$, is a commuting square by Lemma 3.2. As we have mentioned at the beginning of this section, $P_k \subset P_{k+1}$ has a trivial relative commutant. Thus Lemma 3.2 also implies that $F|_{P_{k+1}}$ is a trace-preserving conditional expectation from P_{k+1} onto P_k . Applying Lemma 3.1 and Proposition 2.3 to $P_{k+1}^c \subset P_k^c \subset M_\infty^\omega$ and using the Pimsner–Popa inequality, we have

$$\begin{aligned} \gamma &= [P_k^c : P_{k+1}^c] = [P_k^c : P_{k+1}^c]_{K,E} = [P_{k+1}^{c'} : P_k^{c'}]_{K,\tau E^{-1}} = [P_{k+1}^{c'} : P_k^{c'}]_{PP,\tau E^{-1}} \\ &\geq [P_{k+1}^{cc} : P_k^{cc}]_{PP,F} \geq [P_{k+1} : P_k]_{PP,F} = [P_{k+1} : P_k] = \gamma. \end{aligned}$$

Thus we have $\gamma = [P_{k+1} : P_k]_{PP} = [P_{k+1}^{cc} : P_k]_{PP}$.

Since P_{k+1}^c is a II_1 factor and $P_{k+1} \subset P_{k+1}^{cc}$, we obtain

$$\begin{aligned} P_{k+1}^{cc} \cap P_{k+1}^{cc'} &\subset P_{k+1}^{cc} \cap P_{k+1}' = ((P_{k+1}^c)' \cap M_\infty^\omega) \cap P_{k+1}' \\ &= (P_{k+1}^c)' \cap (P_{k+1}' \cap M_\infty^\omega) = (P_{k+1}^c)' \cap P_{k+1}^c = \mathbf{C}. \end{aligned}$$

Hence P_{k+1}^{cc} is a II_1 factor. Then we have $[P_{k+1} : P_k]_{PP} = [P_{k+1} : P_k]$ and $[P_{k+1}^{cc} : P_k]_{PP} = [P_{k+1}^{cc} : P_k]$. Thus $[P_{k+1} : P_k] = [P_{k+1}^{cc} : P_k]$, which means $P_{k+1}^{cc} = P_{k+1}$.

In the same way, we have $[Q_1^{cc} : P_0]_{PP} = [Q_1 : P_0]_{PP}$, i.e., $Q_1^{cc} = Q_1$. We can also prove $Q_k^{cc} = Q_k$ in the same way. \square

Let P_∞ and Q_∞ be the weak closures of $\bigcup_{k=0}^\infty P_k$ and $\bigcup_{k=0}^\infty Q_k$ on $L^2(M_\infty^\omega)$ respectively. Both P_∞ and Q_∞ are II_1 factors, because P_k, Q_k and M_∞^ω are all II_1 factors. We have the following theorem.

Theorem 3.4 *We obtain $Q_\infty^{cc} = Q_\infty$.*

At first glance, this statement may seem strange. As we mentioned in Section 2, when we construct Q_k , we have an ambiguity of choosing a Jones projection at each step. This theorem claims that whatever Jones projections we may choose, the identity $Q_\infty^{cc} = Q_\infty$ holds. In Theorem 3.3, we used only estimates of the Jones index and nothing particular about ultraproducts was used. But for this theorem, it is essential that P_1 is an ultraproduct algebra.

Proof We have

$$\begin{aligned} Q_\infty^{cc} &= ((\bigvee_{l=1}^\infty Q_l)' \cap M_\infty^\omega)' \cap M_\infty^\omega \\ &= (\bigcap_{l=1}^\infty (Q_l' \cap M_\infty^\omega))' \cap M_\infty^\omega = (\bigcap_{l=1}^\infty Q_l^c)' \cap M_\infty^\omega, \end{aligned}$$

and

$$Q = \bigvee_{l=1}^\infty Q_l = \bigvee_{l=1}^\infty Q_l^{cc} = \bigvee_{l=1}^\infty (Q_l^c' \cap M_\infty^\omega).$$

It is enough for us to show the following equality.

$$(\bigcap_{l=1}^\infty Q_l^c)' \cap M_\infty^\omega = \bigvee_{l=1}^\infty (Q_l^c' \cap M_\infty^\omega)$$

It is easy to see that the right hand side is a subalgebra of the left hand side. To prove the converse inclusion, we choose

$$x = \{x_n\}_n \in (\bigcap_{l=1}^\infty Q_l^c)' \cap M_\infty^\omega.$$

If x were not in $\bigvee_{l=1}^{\infty}(Q_l^{c'} \cap M_{\infty}^{\omega})$, there would exist $\varepsilon > 0$ such that for any l ,

$$\|E_{Q_l^{c'} \cap M_{\infty}^{\omega}}(x) - x\|_2 > \varepsilon.$$

Thus, there exist unitaries $y^l = \{y_n^l\}_n \in Q_l^c$ such that $\|xy^l - y^lx\|_2 > \varepsilon/2$. Here we may assume $y_n^l \in M$ because $Q_l^c \subset P_1$. We denote the Jones projection of the subfactor $Q_{l+2}^c \subset Q_{l+1}^c$ by $f^l = \{f_n^l\}_n \in Q_l^c$. Let $\{a_l\}_{l \in \mathbf{N}}$ be an L^2 -dense subset of M . Then we have

$$\begin{aligned} f^3 &\in Q_3^c = Q_2^c \cap \{f^1\}' \\ &\quad \cap \\ f^2 &\in Q_2^c = Q_1^c \cap \{f^0\}' \\ &\quad \cap \\ f^1 &\in Q_1^c = P_0 \cap \{f^{-1}\}' \\ &\quad \cap \\ f^0 &\in P_0 = P_1 \cap \{f^{-2}\}' \\ &\quad \cap \\ f^{-1} &\in P_1 = M' \cap M^{\omega} \\ &\quad \cap \\ f^{-2} &\in P_2 = (M' \cap M^{\omega}) \vee \{f^{-2}\}' \\ &\quad \cap \\ &\quad \vdots \\ &\quad \cap \\ &\quad M_{\infty}^{\omega} \end{aligned}$$

which means

$$y^l \in Q_l^c = M^{\omega} \cap M' \cap \{f^{-2}, f^{-1}, f^0, \dots, f^{l-2}\}'.$$

Let $F_0 := \mathbf{N}$ and

$$F_k := F_{k-1} \cap [k, \infty) \cap \left\{ n \left| \begin{array}{l} \|x_n y_n^k - y_n^k x_n\|_2 > \varepsilon/2 \\ \|f_n^i y_n^k - y_n^k f_n^i\|_2 < 1/k, \text{ for } i = -2, -1, \dots, k-2 \\ \|a_i y_n^k - y_n^k a_i\|_2 < 1/k, \text{ for } i = 1, 2, \dots, k \end{array} \right. \right\}.$$

Then, each F_k is in ω and $\bigcap_{k=1}^{\infty} F_k = \emptyset$. If we set $y := \{y_n^k\}_n$ for $n \in F_k \setminus F_{k+1}$, then

$$y \in M^{\omega} \cap M' \cap \{f^{-2}, f^{-1}, f^0, \dots\}' = \bigcap_{k=1}^{\infty} Q_k^c,$$

and $\|xy - yx\|_2 > \varepsilon/2$. Thus, $x \notin (\bigcap_{k=1}^{\infty} Q_k^c)' \cap M_{\infty}^{\omega}$, which is a contradiction. \square

Unfortunately, we cannot prove $P_{\infty}^{cc} = P_{\infty}$ in the same way. In the above proof, we choose every y_n^k in M , thus $y = \{y_n^k\}_n \in M^{\omega}$. However, since P_0^c is represented as $P_0^c = \bigvee_{k=1}^{\infty} (A_{0\infty} \vee A_{0k})^{\omega}$, (here the string algebra $A_{k,l}$ does not matter, see [EK, Section 15] for more details), if we construct an element of the filter and y as above, we are not sure whether such y could be in P_0^c or not.

4 Applications to paragroups

In this section we study the double sequences of the higher relative commutants of the subfactor $M' \cap N^\omega \subset M' \cap M^\omega$. Owing to the double commutant properties, we can see the relations among the higher relative commutants clearly.

Lemma 4.1 *We have the following identities.*

- (1) $\bigvee_{k=0}^{\infty} (P_k \cap Q_l) = P_\infty \cap Q_l$
- (2) $\bigvee_{l=0}^{\infty} (P_k \cap Q_l) = P_k \cap Q_\infty$
- (3) $\bigvee_{k=0}^{\infty} (P_k \cap Q_\infty) = P_\infty \cap Q_\infty$
- (4) $\bigvee_{l=0}^{\infty} (P_\infty \cap Q_l) = P_\infty \cap Q_\infty$

Proof (1) It is clear that the left hand side is contained in the right hand side. We prove the converse inclusion. Since $Q_l^{cc} = Q_l$ and $Q_l^c \subset P_k \subset P_\infty$, the square

$$\begin{array}{ccc} P_k \cap Q_l & \subset & P_\infty \cap Q_l \\ \cap & & \cap \\ P_k & \overset{E}{\subset} & P_\infty \end{array}$$

is a commuting square. Then for any $x \in P_\infty \cap Q_l$, we have $E_{P_k \cap Q_l}(x) = E_{P_k}(x)$ and $\|E_{P_k \cap Q_l}(x)\|_\infty \leq \|x\|_\infty$. Since $\bigvee_{k=1}^{\infty} P_k = P_\infty$, we have

$$\|x - E_{P_k \cap Q_l}(x)\|_2 = \|x - E_{P_k}(x)\|_2 \rightarrow 0,$$

which means $E_{P_k \cap Q_l}(x)$ converges to x strongly. Therefore, we have $E_{\bigvee_{k=1}^{\infty} (P_k \cap Q_\infty)}(x) = x$ for any $x \in P_\infty \cap Q_\infty$.

(2), (3) Since $P_k^{cc} = P_k$ and $Q_\infty^{cc} = Q_\infty$, then the squares

$$\begin{array}{ccc} P_k \cap Q_l & \subset & Q_l \\ \cap & & \cap \\ P_k \cap Q_\infty & \subset & Q_\infty \end{array}$$

and

$$\begin{array}{ccc} P_k \cap Q_\infty & \subset & P_\infty \cap Q_\infty \\ \cap & & \cap \\ P_k & \subset & P_\infty \end{array}$$

are commuting squares. Thus we obtain equalities (2), (3) in the same way as is the proof of (1).

(4) The equalities below show (4).

$$\begin{aligned} \bigvee_{l=1}^{\infty} (Q_l \cap P_k) &= \bigvee_{l=1}^{\infty} (Q_l^c \cap P_k) = \left(\bigcap_{l=1}^{\infty} Q_l^c \right)' \cap P_k = \left(\bigcap_{l=1}^{\infty} Q_l^c \right)' \cap M_\infty^\omega \cap P_k \\ &= \left(\bigvee_{l=1}^{\infty} (Q_l^c \cap M_\infty^\omega) \right) \cap P_k = \left(\bigvee_{l=1}^{\infty} Q_l^{cc} \right) \cap P_k = Q_\infty \cap P_\infty. \end{aligned}$$

Both the second and fourth identities follow from the same arguments as in the proof of Theorem 3.4. \square

In addition to the properties mentioned in Section 2, the subfactor $N^\omega \cap M' \subset M' \cap M^\omega$ is known to have the finite depth property. To prove it, the next theorem of Ocneanu has been very useful.

Theorem 4.2 (Ocneanu) *Let $N \subset M$ be a subfactor of the hyperfinite II_1 factor with finite index and finite depth. The paragroups of the central sequence subfactor $N^\omega \cap M' \subset M' \cap M^\omega$ and of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ are mutually dual.*

This has been noted by Ocneanu in [O2]. One can see a proof in [EK, Theorem 15.32]. In general, we say that two paragroups are *dual* to each other if and only if the corresponding subfactors are dual to each other. (See [EK, page 570].) Originally the adjective “dual” comes from the following duality of groups. When G is a finite group and R a II_1 factor, we have a fixed point algebra and a crossed product by an outer action, i.e., $R^G \subset R$ and $R \subset R \rtimes G$. It is known that the R^G - R^G bimodules are indexed by \hat{G} and the R - R bimodules are indexed by G . Thus we say $R^G \subset R$ and $R \subset R \rtimes G$ are dual. Extending this duality, we say $N \subset M$ and $M \subset M_1$ are dual when $N \subset M \subset M_1$ is standard.

The finite depth condition of $N^\omega \cap M' \subset M' \cap M^\omega$ follows from this theorem and the finite depth condition of $M \vee (M' \cap M_\infty) \subset M_\infty$. Thanks to the above arguments, especially using the double commutant properties ($P_k^{cc} = P_k$ and $Q_l^{cc} = Q_l$), we simplify the proof of [EK, Theorem 15.32]. That is, in [EK] they have shown $P'_0 \cap P_k = (P_k^c)' \cap P_0^c$ by using several inclusions and two anti-isomorphisms. However, with the double commutant properties, it is quite natural for us to write the higher relative commutants by the combination of algebras, one from

$$\cdots \subset Q_3^c \subset Q_2^c \subset Q_1^c \subset P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots \subset M_\infty^\omega$$

and the other from

$$\cdots \subset P_3^c \subset P_2^c \subset P_1^c \subset P_0^c \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots \subset M_\infty^\omega.$$

Proof The double sequence of the higher relative commutants of the central sequence subfactor $N^\omega \cap M' \subset M' \cap M^\omega$ is given as in the following diagram. Here we note that $N^\omega \cap M' \subset M' \cap M^\omega$ has a trivial relative commutant, (see [EK, Lemma

15.25]), and use the conventions of Lemma 4.1.

$$\begin{array}{cccccccc}
& & & & \mathbf{C} & \subset & \mathbf{C} & \subset & P_3 \cap P_1^c & \subset \cdots \subset & P_\infty \cap P_1^c \\
& & & & \cap & & \cap & & \cap & & \cap \\
& & & & \mathbf{C} & \subset & \mathbf{C} & \subset & P_2 \cap P_0^c & \subset P_3 \cap P_0^c & \subset \cdots \subset & P_\infty \cap P_0^c \\
& & & & \cap & & \cap & & \cap & & \cap \\
& & & \mathbf{C} & \subset & \mathbf{C} & \subset & P_1 \cap Q_1 & \subset P_2 \cap Q_1 & \subset P_3 \cap Q_1 & \subset \cdots \subset & P_\infty \cap Q_1 \\
& & & \cap & & \cap & & \cap & & \cap & & \cap \\
\mathbf{C} & \subset & \mathbf{C} & \subset & P_0 \cap Q_2 & \subset & P_1 \cap Q_2 & \subset & P_2 \cap Q_2 & \subset P_3 \cap Q_2 & \subset \cdots \subset & P_\infty \cap Q_2 \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
Q_2^c \cap Q_\infty & \subset & Q_1^c \cap Q_\infty & \subset & P_0 \cap Q_\infty & \subset & P_1 \cap Q_\infty & \subset & P_2 \cap Q_\infty & \subset & P_3 \cap Q_\infty & \subset \cdots \subset & P_\infty \cap Q_\infty
\end{array}$$

On the other hand, the double sequence of the higher relative commutants of $M \vee (M' \cap M_\infty) \subset M_\infty$ is equal to that of $P_1^c \subset P_0^c$. (See [EK, Lemma 15.31].) Then the double sequence is as follows.

$$\begin{array}{cccccccc}
& & & & \mathbf{C} & \subset & \mathbf{C} & \subset & Q_2 \cap P_0 & \subset \cdots \subset & Q_\infty \cap P_0 \\
& & & & \cap & & \cap & & \cap & & \cap \\
& & & & \mathbf{C} & \subset & \mathbf{C} & \subset & Q_1 \cap P_1 & \subset Q_2 \cap P_1 & \subset \cdots \subset & Q_\infty \cap P_1 \\
& & & & \cap & & \cap & & \cap & & \cap \\
& & & \mathbf{C} & \subset & \mathbf{C} & \subset & P_0^c \cap P_2 & \subset Q_1 \cap P_2 & \subset Q_2 \cap P_2 & \subset \cdots \subset & Q_\infty \cap P_2 \\
& & & \cap & & \cap & & \cap & & \cap & & \cap \\
\mathbf{C} & \subset & \mathbf{C} & \subset & P_1^c \cap P_3 & \subset & P_0^c \cap P_3 & \subset & Q_1 \cap P_3 & \subset Q_2 \cap P_3 & \subset \cdots \subset & Q_\infty \cap P_3 \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
P_3^c \cap P_\infty & \subset & P_2^c \cap P_\infty & \subset & P_1^c \cap P_\infty & \subset & P_0^c \cap P_\infty & \subset & Q_1 \cap P_\infty & \subset & Q_2 \cap P_\infty & \subset \cdots \subset & Q_\infty \cap P_\infty
\end{array}$$

Since the asymptotic inclusion is anti-isomorphic to itself, its paragroup is opposite to itself, so the above double sequence is isomorphic to the following.

$$\begin{array}{cccccccc}
& & & & & & & \mathbf{C} & \subset & \mathbf{C} & \subset \cdots \subset & P_\infty \cap P_2^c \\
& & & & & & & \cap & & \cap & & \cap \\
& & & & & & & \mathbf{C} & \subset & \mathbf{C} & \subset P_3 \cap P_1^c & \subset \cdots \subset & P_\infty \cap P_1^c \\
& & & & & & & \cap & & \cap & & \cap \\
& & & & & & & \mathbf{C} & \subset & \mathbf{C} & \subset P_2 \cap P_0^c & \subset P_3 \cap P_0^c & \subset \cdots \subset & P_\infty \cap P_0^c \\
& & & & & & & \cap & & \cap & & \cap \\
\mathbf{C} & \subset & \mathbf{C} & \subset & P_1 \cap Q_1 & \subset & P_2 \cap Q_1 & \subset & P_3 \cap Q_1 & \subset \cdots \subset & P_\infty \cap Q_1 \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
\mathbf{C} & \subset & P_0 \cap Q_2 & \subset & P_1 \cap Q_2 & \subset & P_2 \cap Q_2 & \subset & P_3 \cap Q_2 & \subset \cdots \subset & P_\infty \cap Q_2 \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cap & & \cap & & \cap & & \cap & & \cap & & \cap \\
Q_1^c \cap Q_\infty & \subset & P_0 \cap Q_\infty & \subset & P_1 \cap Q_\infty & \subset & P_2 \cap Q_\infty & \subset & P_3 \cap Q_\infty & \subset \cdots \subset & P_\infty \cap Q_\infty
\end{array}$$

By shifting the first diagram by one line vertically, we get the third diagram, which means the paragroup of $N^\omega \cap M' \subset M' \cap M^\omega$ is dual to that of $M \vee (M' \cap M_\infty) \subset M_\infty$. \square

Theorem 4.3 *The von Neumann algebra $P_\infty \cap Q_\infty$ is a hyperfinite II_1 factor.*

Proof Since

$$P_k^c \subset \cdots \subset P_1^c \subset P_0^c \subset Q_1 \subset Q_2 \subset Q_3 \subset \cdots$$

is the Jones tower with $[P_0^c : P_1^c] = \gamma < \infty$, we have $\dim(Q_l \cap P_k) = \dim(Q_l \cap P_k^c) < \infty$. Thus Lemma 4.1 and the finite depth property of $P_1^c \subset P_0^c$ imply that $P_\infty \cap Q_\infty$ is a hyperfinite II_1 factor. \square

Corollary 4.4 *The II_1 factors P_∞ and Q_∞ are strictly smaller than M_∞^ω .*

By the definition of P_0 , it is not separable and quite a large algebra. Thus P_∞ is extremely large. Once we take an ultraproduct of M_∞ , it becomes extraordinarily large as it contains P_∞ strictly.

Proof Suppose $P_\infty = M_\infty^\omega$ on the contrary, then we have $P_\infty \cap Q_\infty = M_\infty^\omega \cap Q_\infty = Q_\infty$. Since Q_∞ contains P_0^c which is not hyperfinite, Q_∞ is not hyperfinite. By Lemma 4.3, $P_\infty \cap Q_\infty$ is hyperfinite, which is a contradiction. We remark that if we suppose $Q_\infty = M_\infty^\omega$, we similarly obtain a contradiction. \square

5 The strong amenability and the central sequence factor $M^\omega \cap M'$

In this section we shall prove the following theorem.

Theorem 5.1 *Let M be a hyperfinite II_1 factor and $N \subset M$ be a type II_1 subfactor with finite index. (Here we do not assume the trivial relative commutant condition nor finite depth condition.) The following are equivalent.*

- (1) $(M' \cap M_\infty)' \cap M_\infty = M$.
- (2) $(M'_\omega \cap M_\infty^\omega)' \cap M_\infty^\omega = M_\omega$.

We remark condition (1) is equivalent to the strong amenability of Popa. (See [P4, Theorem 4.1.2].) He has proved that the strong amenability is equivalent to his generating property. When a subfactor has a finite depth, it is easy to see that it is strongly amenable. In Section 3, we have assumed the finite depth condition of $N \subset M$, therefore condition (1) always holds.

Before the proof, we recall Ocneanu's central freedom lemma ([O1], [EK, Lemma 15.20]) as follows, since we use it here for several times.

Lemma 5.2 (Central freedom lemma, Ocneanu) *Let $L \subset P \subset Q$ be finite von Neumann algebras and L a hyperfinite factor. Then we obtain*

$$(L' \cap P^\omega)' \cap Q^\omega = L \vee (P' \cap Q)^\omega.$$

In this lemma, the hyperfiniteness condition is indispensable because we approximate L by $L_m := \otimes_{i=1}^m M_2(\mathbf{C})$ and use the finite dimensionality of L_m essentially.

Proof (Theorem 5.1 (1) \Rightarrow (2)) By using the central freedom lemma twice it is straightforward to see that (2) follows from (1), i.e.,

$$\begin{aligned} (M'_\omega \cap M_\infty^\omega)' \cap M_\infty^\omega &= ((M' \cap M^\omega)' \cap M_\infty^\omega)' \cap M_\infty^\omega = (M \vee (M' \cap M_\infty)^\omega)' \cap M_\infty^\omega \\ &= M' \cap (M' \cap M_\infty)^\omega' \cap M_\infty^\omega = M' \cap ((M' \cap M_\infty)' \cap M_\infty)^\omega \\ &= M' \cap M^\omega. \end{aligned}$$

□

We prove the converse direction in the following way.

$$((2) \Rightarrow (1)) \stackrel{(5.6)}{\Leftarrow} (5.7) \left\{ \begin{array}{l} \text{The case } [A^\omega : M^\omega] < \infty \\ (5.8) \\ \text{The case } [A^\omega : M^\omega] = \infty \left\{ \begin{array}{l} \text{The case } \dim(M' \cap A) = \infty \\ (5.3)(5.4) \\ \text{The case } \dim(M' \cap A) < \infty \\ (5.9)(5.12) \end{array} \right. \end{array} \right.$$

Though we do not know whether A is a factor or not, $[A : M]$ has a meaning in the sense of Section 2, i.e., $[A : M] := \dim_M L^2(A)$. We denote the unique trace-preserving conditional expectation from M_∞ onto M by E_M .

Proposition 5.3 *Let A be a finite von Neumann algebra with a fixed trace tr , and M its type II_1 subfactor. If, for any $\varepsilon > 0$, there exists a non-zero projection $p \in A$ such that $E_M(p) \leq \varepsilon 1_M$, then we obtain $[A : M]_{PP} = \infty$.*

In [PP, page 71], this proposition has been used since it is trivial from the definition of the Pimsner–Popa index. We include a proof here since A is not a factor now, but actually this does not cause any trouble.

Proof We suppose $[A : M]_{PP} < \infty$ on the contrary, then there exists $\varepsilon_0 > 0$ such that for any $x \in A_+$ we have $E_M(x) \geq 2\varepsilon_0 x$. From the assumption there exists a projection $0 \neq p_0 \in A$ such that $E_M(p_0) \leq \varepsilon_0 1_M$. Then we have $2\varepsilon_0 p_0 \leq E_M(p_0) \leq \varepsilon_0 1_M$. If we multiply all the three operators by p_0 both from the right and left, we obtain $2\varepsilon_0 p_0 \leq p_0 E_M(p_0) p_0 \leq \varepsilon_0 p_0$, which is a contradiction. \square

We also need the following easy lemma, so we remark it here.

Proposition 5.4 *Let A be a finite von Neumann algebra with a fixed trace tr , and M its II_1 subfactor. Then we have the following.*

(1) *When $\dim(M' \cap A) < \infty$, we have the following identity.*

$$[A : M]_{PP} = \max_{1 \leq i \leq k} \frac{[Aq_i : Mq_i]}{\text{tr}(q_i)}$$

In particular, the condition $[A : M]_{PP} = \infty$ is equivalent to the condition $[A : M] = \infty$.

(2) *When $\dim(M' \cap A) = \infty$, we obtain $[A : M]_{PP} = \infty = [A : M]$.*

Proof (1) Since $\dim(A' \cap A) \leq \dim(M' \cap A) < \infty$, there exist a finite number of minimal central projections $q_1, \dots, q_k \in A$, such that $A = Aq_1 \oplus \dots \oplus Aq_k$. If we set the isomorphism $\phi_i : Mq_i \rightarrow M$ by $\phi_i(yq_i) := y$, for any $x = x_1 + \dots + x_k \in \bigoplus_{i=1}^k Aq_i$, we have

$$\begin{aligned} E_M(x) &= \sum_{i=1}^k E_M(x_i q_i) = \sum_{i=1}^k \phi_i(E_{Mq_i}(x_i)) \text{tr}(q_i) \\ &= \sum_{i=1}^k \text{tr}(q_i) \sum_{j=1}^k \phi_i(E_{Mq_i}(x_i)) q_j \geq \sum_{i=1}^k \text{tr}(q_i) E_{Mq_i}(x_i) \\ &\geq \sum_{i=1}^k \frac{\text{tr}(q_i) x_i}{[Aq_i : Mq_i]} \geq \left(\max_{1 \leq i \leq k} \frac{[Aq_i : Mq_i]}{\text{tr}(q_i)} \right)^{-1} x. \end{aligned}$$

Thus we obtain

$$[A : M]_{PP} \leq \max_{1 \leq i \leq k} \frac{[Aq_i : Mq_i]}{\text{tr}(q_i)}$$

In case $[A : M]_{PP} = \infty$, the above inequality is enough for us, because the following identities

$$\begin{aligned} [A : M] &= \sum_{i=1}^k \dim_M L^2(q_i A q_i) \\ &= \sum_{i=1}^k \dim_{M q_i} L^2(q_i A q_i) = \sum_{i=1}^k [q_i A q_i : M q_i], \end{aligned}$$

imply $[A : M] = \infty$.

In case $[A : M]_{PP} < \infty$, for any $i \in \{1, \dots, k\}$ and $x \in A$, we have the following inequality.

$$E_M(x q_i) \geq \frac{x q_i}{[A : M]_{PP}}$$

Because the left hand side is equal to $\text{tr}(q_i) \sum_{j=1}^k \phi_j(E_{M q_i}(x_i)) q_j$, we obtain

$$\text{tr}(q_i) E_{M q_i}(x_i) \geq \frac{x q_i}{[A : M]_{PP}}.$$

Thus $[A q_i : M q_i]_{PP} < \infty$. Since $[A q_i : M q_i]_{PP} = [A q_i : M q_i]$, we have $[A q_i : M q_i] < \infty$. Let i_0 be an index with

$$\max_{1 \leq i \leq k} \frac{[A q_i : M q_i]}{\text{tr}(q_i)} = \frac{[A q_{i_0} : M q_{i_0}]}{\text{tr}(q_{i_0})}.$$

Let $e_i \in A q_i$ be a Jones projection of $M q_i \subset A q_i$ for the downward basic construction. Then we have

$$E_M(e_{i_0}) = \left(\max_{1 \leq i \leq k} \frac{[A q_i : M q_i]}{\text{tr}(q_i)} \right)^{-1} \geq \left(\max_{1 \leq i \leq k} \frac{[A q_i : M q_i]}{\text{tr}(q_i)} \right)^{-1} e_{i_0}.$$

Thus we obtain

$$[A : M]_{PP} = \max_{1 \leq i \leq k} \frac{[A q_i : M q_i]}{\text{tr}(q_i)}.$$

The identity

$$[A : M] = \sum_{i=1}^k [q_i A q_i : M q_i]$$

implies that $[A : M] = \infty$ is equivalent to $[A : M]_{PP} = \infty$.

(2) Since A is a finite von Neumann algebra and $\dim(M' \cap A) = \infty$, for any $\varepsilon > 0$, there exists a non-zero projection $e \in M' \cap A$ such that $0 \neq \text{tr}(e) \leq \varepsilon$. Since the square

$$\begin{array}{ccc} M & \subset & A \\ \cup & & \cup \\ M' \cap M = \mathbf{C} & \subset & M' \cap A \end{array}$$

is a commuting square, we have $0 \neq E_M(e) = \text{tr}(e) \leq \varepsilon 1_M$. Lemma 5.3 implies the first equality $[A : M]_{PP} = \infty$.

In the rest of the proof (2), the idea is given by [PP, page 71]. They use Jones' identity. In our case A is not a II_1 factor, then we cannot use it in his original form. Thus we replace the identity as follows.

We suppose $\dim(M' \cap A) = \infty$ and $[A : M] < \infty$. For any non-zero projection $p \in M' \cap A$, we have

$$\begin{aligned} [Ap : Mp] &= \dim_{M_p} L^2(pAp) \\ &= \dim_M L^2(A) \operatorname{tr}_{M'}(p) \operatorname{tr}_{M'}(p'). \end{aligned}$$

Here p' means right multiplication of p . We remark that M' is a II_1 factor by the assumption $[A : M] < \infty$. For any m , there exist projections $p_1, \dots, p_m \in M' \cap A$ such that $\sum_{i=1}^m p_i = 1$. Then we have $[A : M] = \sum_{i=1}^m [Ap_i : Mp_i] / \operatorname{tr}(p_i) \geq m$. Therefore $[A : M] = \infty$, which is a contradiction. \square

We need the next two lemmas to show the invariance of the Jones indices under taking ultraproducts.

Lemma 5.5 [PP, Proposition 1.10] *Let $\mathcal{N} \subset \mathcal{M}$ be II_1 factors. Then $[\mathcal{M}^\omega : \mathcal{N}^\omega] = [\mathcal{M} : \mathcal{N}]$*

This identity also holds in the case of the infinite index. Thanks to the above lemma, the next one is quite natural, where we have dropped the factoriality of Q .

Lemma 5.6 *Let Q be a finite von Neumann algebra with a fixed trace tr , and P be a II_1 factor. We have $[Q : P] = [Q^\omega : P^\omega]$.*

Proof When $\dim(P' \cap Q) = \infty$, by the central freedom lemma, we have $P^{\omega'} \cap Q^\omega = (P' \cap Q)^\omega$ then $\dim(P^{\omega'} \cap Q^\omega) = \infty$. Thus $[Q^\omega : P^\omega] = \infty = [Q : P]$ by Proposition 5.4.

When $\dim(P' \cap Q) < \infty$, there exist minimal central projections $q_1, \dots, q_n \in Q$ such that $Q = \bigoplus_{i=1}^n Qq_i$. Then we have

$$\begin{aligned} [Q^\omega : P^\omega] &= [\bigoplus_{i=1}^n Q^\omega q_i : \bigoplus_{i=1}^n P^\omega q_i] = \sum_{i=1}^n [Q^\omega q_i : P^\omega q_i] \\ &= \sum_{i=1}^n [Qq_i : Pq_i] = [Q : P] \end{aligned}$$

The third equality owes to Lemma 5.5. \square

In order to prove the direction (2) \Rightarrow (1) of Theorem 5.1, we shall prove a more general statement as follows.

Theorem 5.7 *Let A be a finite von Neumann algebra with a fixed trace tr , and $M \subset A$ be a hyperfinite II_1 factor. Then we have $[A^\omega : M^\omega] = [M' \cap A^\omega : M' \cap M^\omega]$.*

First we consider the case when $[A : M] < \infty$, and use the idea of the Pimsner–Popa basis. Here we recall their statements as below.

Proposition 5.8 [PP, Proposition 1.3] *Let $\mathcal{N} \subset^F \mathcal{M}$ be II_1 factors with the trace-preserving conditional expectation F from \mathcal{M} onto \mathcal{N} . When $[\mathcal{M} : \mathcal{N}] < \infty$, there exists a family $\{m_j\}_{1 \leq j \leq n+1}$ of elements in \mathcal{M} , where n is the integer part of $[\mathcal{M} : \mathcal{N}]$, such that*

- (a) $F(m_j^* m_k) = 0$, $j \neq k$,
- (b) $F(m_j^* m_j) = 1$, $1 \leq j \leq n$,
- (c) $F(m_{n+1}^* m_{n+1})$ is a projection of trace $[\mathcal{M} : \mathcal{N}] - n$.

Now we fix some notations as follows. Since M is hyperfinite we may represent $M = \otimes_{n=1}^{\infty} M_2(\mathbf{C})$ and set $A_m := \otimes_{n=1}^m M_2(\mathbf{C})$, $A := (M' \cap M_{\infty})' \cap M_{\infty}$ and $B := J_A M' J_A$ on $L^2(A)$. We easily notice that the following squares are commuting squares.

$$\begin{array}{ccc}
 M^{\omega} & \subset & A^{\omega} \\
 \cup & & \cup \\
 \vdots & & \vdots \\
 \cup & & \cup \\
 A'_k \cap M^{\omega} & \subset & A'_k \cap A^{\omega} \\
 \cup & & \cup \\
 A'_{k+1} \cap M^{\omega} & \subset & A'_{k+1} \cap A^{\omega} \\
 \cup & & \cup \\
 \vdots & & \vdots \\
 \cup & & \cup \\
 M' \cap M^{\omega} & \subset & M' \cap A^{\omega}
 \end{array}$$

Our aim is to show the non-degeneracy of the next commuting square.

$$\begin{array}{ccc}
 M^{\omega} & \subset & A^{\omega} \\
 \cup & & \cup \\
 M' \cap M^{\omega} & \subset & M' \cap A^{\omega}.
 \end{array}$$

One can find a similar situation in [P5, Proposition 2.2 and Theorem 2.9], but we do not assume the factoriality of A nor the finiteness of the index. The hyperfiniteness of M is essential in main Theorem 5.1. We use the above presentation of M throughout the proof.

Proof (Theorem 5.7, the finite index case) Since M is a II_1 factor and $A_k \simeq M_{2^k}(\mathbf{C})$, the relative commutant $A'_k \cap M$ is also a II_1 factor. Since $\dim(A' \cap A) \leq \dim(M' \cap A) < \infty$, there exist a finite number of minimal central projections $q_1, \dots, q_a \in A$ such that $A = Aq_1 \oplus \dots \oplus Aq_a$. We set each factor $A^{(i)} := Aq_i$ for $1 \leq i \leq a$ and

n_i to be the integer part of $[A^{(i)} : Mq_i]$. Since

$$\begin{aligned} [(A_k q_i)' \cap A^{(i)} : (A_k q_i)' \cap Mq_i] &\leq [A^{(i)} : (A_k q_i)' \cap Mq_i] \\ &= [A^{(i)} : Mq_i] \times [Mq_i : (A_k q_i)' \cap Mq_i] \\ &\leq [A : M] \times 2^{2k} < \infty, \end{aligned}$$

Proposition 5.8 implies the existence of an orthonormal basis $\{m_{i,j}^{(k)}\}_{0 \leq j \leq n_i}$ of $(A_k q_i)' \cap Mq_i \subset (A_k q_i)' \cap A^{(i)}$ such that

- (1) $E_{Mq_i}(m_{i,j}^{(k)*} m_{i,j'}^{(k)}) = 0, \quad j \neq j',$
- (2) $E_{Mq_i}(m_{i,j}^{(k)*} m_{i,j}^{(k)}) = q_i, \quad 0 \leq j < n_i,$
- (3) $E_{Mq_i}(m_{i,n_i}^{(k)*} m_{i,n_i}^{(k)})$ is a projection of trace $[A^{(i)} : Mq_i] - n_i,$
- (4) $\sum_{j=0}^{n_i} m_{i,j}^{(k)} m_{i,j}^{(k)*} = [A^{(i)} : Mq_i] q_i.$

Because

$$A^{(i)} \simeq A_k q_i \otimes ((A_k q_i)' \cap A^{(i)}) \subset \overline{\text{sp}} Mq_i ((A_k q_i)' \cap A^{(i)}),$$

the square

$$\begin{array}{ccc} Mq_i & \xrightarrow{E|_{A^{(i)}}} & A^{(i)} \\ \cup & & \cup \\ (A_k q_i)' \cap Mq_i & \subset & (A_k q_i)' \cap A^{(i)} \end{array}$$

is a non-degenerate commuting square. Therefore, $\{m_{i,j}^{(k)}\}_j$ is also an orthonormal basis of $Mq_i \subset A^{(i)}$, and $[A^{(i)} : Mq_i] = [(A_k q_i)' \cap A^{(i)} : (A_k q_i)' \cap Mq_i]$.

We set $m_{i,j} := \{m_{i,j}^{(k)}\}_k$, then $\{m_{i,j}\}_j$ is an orthonormal basis of both $(Mq_i)' \cap M^\omega q_i \subset (Mq_i)' \cap A^{(i)\omega}$ and $M^\omega q_i \subset A^{(i)\omega}$.

Therefore, we have

$$\begin{aligned} [M' \cap A^\omega : M' \cap M^\omega] &= \sum_{i=1}^a [(Mq_i)' \cap A^{(i)\omega} : (Mq_i)' \cap M^\omega q_i] \\ &= \sum_{i=1}^a \frac{\text{tr}(\sum_{j=0}^{n_i} m_{i,j} m_{i,j}^*)}{\text{tr}(q_i)} = \sum_{i=0}^a [A^{(i)\omega} : M^\omega q_i] = [A^\omega : M^\omega]. \end{aligned}$$

□

Proof (Theorem 5.7, the infinite index case with $\dim(M' \cap A) = \infty$) Since A is a finite von Neumann algebra and $\dim(M' \cap A) = \infty$, for any $\varepsilon > 0$, there exists a non-zero projection $e \in M' \cap A$ such that $0 \neq \text{tr}(e) \leq \varepsilon$. Since the square

$$\begin{array}{ccc} M & \subset & A \\ \cup & & \cup \\ M' \cap M = \mathbf{C} & \subset & M' \cap A \end{array}$$

is a commuting square, we have $0 \neq E_M(e) = \text{tr}(e) \leq \varepsilon 1_M$. Then if we set $\tilde{e} := \{e\}_k \in M' \cap A^\omega$, we have $\tilde{e} \neq 0$ and $E_{M' \cap M^\omega}(\tilde{e}) = E_{M^\omega}(\tilde{e}) \leq \varepsilon$. By Proposition 5.3, we obtain $[M' \cap A^\omega : M' \cap M^\omega]_{PP} = \infty$. By Proposition 5.4 (2), we have $[M' \cap A^\omega : M' \cap M^\omega] = \infty$.

□

The next proposition will play an important role for the rest of the proof of Theorem 5.7. Pimsner and Popa have mentioned this statement as a Remark to Theorem 2.2 and Lemma 2.3 in [PP] and say that this follows from a maximality argument. We shall include a full proof for the sake of completeness.

Proposition 5.9 [PP, Remark 2.4] *If $[A : M] = \infty$ and $M' \cap A = \mathbf{C}$, in particular, A is a factor, then, for any $\varepsilon \in (0, 1)$, there exists a non-zero projection $e \in A$ such that*

$$\mathrm{tr}(\chi_{\{\varepsilon\}}(E_M(e))) \geq 1 - \varepsilon.$$

We prove that the maximal projection which satisfies $E_M(e) \leq \varepsilon 1_M$ is a one we desire. This is based on the techniques in [PP, Theorem 2.2].

Proof By Zorn's lemma, there exists a maximal family of mutually orthogonal projections $\{e_i\}_{i \in I}$ in A such that $E_M(e) \leq \varepsilon 1_M$ where we have set $0 \neq e := \sum_{i \in I} e_i$.

Suppose that the conclusion of the proposition dose not hold, i.e.,

$$\mathrm{tr}\left(\int_0^\varepsilon \chi_{[0, \varepsilon)}(\lambda) dE(\lambda)\right) > \varepsilon,$$

where $\int_0^\varepsilon \lambda dE(\lambda)$ is the spectral decomposition of $E_M(e)$. Then there exists $\varepsilon_0 \in (0, \varepsilon)$ such that

$$\mathrm{tr}\left(\int_0^\varepsilon \chi_{[0, \varepsilon_0]}(\lambda) dE(\lambda)\right) > \varepsilon.$$

We set $f_0 := \int_0^\varepsilon \chi_{[0, \varepsilon_0]}(\lambda) dE(\lambda) \in M$. We need the following claim.

Claim 5.10 *There exists a non-zero projection $q \in A$ such that $q \leq 1 - e$, $q \leq f_0$ and $E_M(q) \leq (\varepsilon - \varepsilon_0)f_0$.*

If we accept this claim we easily obtain a proof of this proposition as below. We have

$$\begin{aligned} E_M(q \vee e) &= E_M(q) + E_M(e) \leq (\varepsilon - \varepsilon_0)f_0 + E_M(e) \\ &\leq (\varepsilon - \varepsilon_0)f_0 + E_M(e)(1 - f_0) + E_M(e)f_0 \\ &\leq (\varepsilon - \varepsilon_0)f_0 + \varepsilon(1 - f_0) + \varepsilon_0 f_0 = \varepsilon, \end{aligned}$$

which contradicts the maximality of $\{e_i\}_{i \in I}$. □

Proof (Claim 5.10) We set $q_0 := f_0 \wedge (1 - e) \in A$, then we obtain

$$\begin{aligned} \mathrm{tr}(q_0) &= \mathrm{tr}(f_0) + \mathrm{tr}(1 - e) - \mathrm{tr}(f_0 \vee (1 - e)) \\ &\geq \mathrm{tr}(f_0) + \mathrm{tr}(1 - e) - 1 > \varepsilon + 1 - \varepsilon - 1 = 0, \end{aligned}$$

which means $q_0 \neq 0$. We denote $J_A M' J_A$ on $L^2(A)$ by B and the Jones projection in B by e_0 , then B is a type II_∞ factor. Let ϕ be a semifinite trace on B satisfying $\phi(e_0) = 1$. Since $0 < \mathrm{tr}(q_0) \leq 1$ and $(q_0 A q_0)' \cap (q_0 B q_0) = q_0 (A' \cap B) q_0 = \mathbf{C} q_0 \simeq \mathbf{C}$ by [P1,

Lemma 2.1], we know that $q_0 B q_0$ is a type II_∞ factor. Since $\text{tr}(E_M(q_0)) = \text{tr}(q_0) \neq 0$, we have $(e_0 q_0)(e_0 q_0)^* = e_0 q_0 e_0 = E_M(q_0) e_0 \neq 0$. Therefore, $q_0 e_0 q_0 = (e_0 q_0)^*(e_0 q_0) \neq 0$ and $q_0 e_0 q_0 \in (q_0 B q_0)_+$. By the normalization of ϕ , we have

$$\begin{aligned} 0 &< \|q_0 e_0 q_0\|_\phi = \phi(q_0 e_0 q_0 e_0 q_0) \\ &= \phi(e_0 q_0 e_0 q_0 e_0) = \phi(E_M(q_0) e_0 E_M(q_0) e_0) \\ &= \phi(E_M(q_0)^2 e_0) = \text{tr}(E_M(q_0)^2) < \infty. \end{aligned}$$

Next we apply [PP, Lemma 2.3] to the inclusion $q_0 A q_0 \subset q_0 B q_0$ with ε replaced by $\sqrt{\text{tr}(q_0)(\varepsilon - \varepsilon_0)^2 / \|q_0 e_0 q_0\|_\phi}$. (We repeat exactly the same arguments as in [PP, page 72] below.) We obtain projections $f_1, \dots, f_n \in q_0 A q_0$ such that $\sum_{i=1}^n f_i = q_0$ and

$$\begin{aligned} \left\| \sum_{i=1}^n f_i (q_0 e_0 q_0) f_i \right\|_\phi^2 &< \frac{\text{tr}(q_0)(\varepsilon - \varepsilon_0)^4}{\|q_0 e_0 q_0\|_\phi^2} \|q_0 e_0 q_0\|_\phi^2 \\ &= \text{tr}(q_0)(\varepsilon - \varepsilon_0)^4 = \sum_{i=1}^n \text{tr}(f_i)(\varepsilon - \varepsilon_0)^4. \end{aligned}$$

Since the left hand side equals to $\sum_{i=1}^n \|f_i q_0 e_0 q_0 f_i\|_\phi^2 = \sum_{i=1}^n \|f_i e_0 f_i\|_\phi^2$, there exists j such that

$$\|f_j e_0 f_j\|_\phi^2 < \text{tr}(f_j)(\varepsilon - \varepsilon_0)^4.$$

We set $p := \chi_{[0, (\varepsilon - \varepsilon_0)]}(E_M(f_j)) \in M$. By the normalization of ϕ and $0 < \varepsilon - \varepsilon_0 < 1$, we have

$$\begin{aligned} \text{tr}(f_j)(\varepsilon - \varepsilon_0)^4 &> \|f_j e_0 f_j\|_\phi^2 = \phi(f_j e_0 f_j e_0 f_j) \\ &= \phi(e_0 f_j e_0 f_j e_0) = \|e_0 f_j e_0\|_\phi^2 \\ &= \|E_M(f_j) e_0\|_\phi^2 = \|E_M(f_j)\|_2^2 \\ &= \text{tr}(E_M(f_j) E_M(f_j)^*) \geq \text{tr}((1 - p) E_M(f_j) E_M(f_j)^*) \\ &\geq (\varepsilon - \varepsilon_0)^2 \text{tr}(1 - p) \geq (\varepsilon - \varepsilon_0)^4 \text{tr}(1 - p), \end{aligned}$$

which means $\text{tr}(p + f_j) > 1$. If we set $q := p \wedge f_j \in A$, we have

$$\text{tr}(q) = \text{tr}(p) + \text{tr}(f_j) - \text{tr}(p \vee f_j) \leq \text{tr}(p) + \text{tr}(f_j) - 1 > 0.$$

Thus,

$$\begin{aligned} 0 \neq q &= p \wedge f_j \leq f_j \leq q_0 \leq 1 - e, \\ 0 \neq q &\leq q_0 \leq f_0 \end{aligned}$$

and

$$\begin{aligned} E_M(q) &= E_M(p \wedge f_j) \leq E_M(p f_j p) \\ &= p E_M(f_j) p \leq (\varepsilon - \varepsilon_0) p. \end{aligned}$$

Multiplying the both hand sides of the third inequality by f_0 from the left and the right, we obtain

$$E_M(q) = E_M(f_0 q f_0) \leq (\varepsilon - \varepsilon_0) f_0 p f_0 \leq (\varepsilon - \varepsilon_0) f_0.$$

□

We need the next lemma as a preparation for Lemma 5.12.

Lemma 5.11 *Let $P \subset Q$ be type II_1 factors with $\dim(P' \cap Q) < \infty$. Let $p_1, \dots, p_n \in P' \cap Q$ be projections such that $\sum_{i=1}^n p_i = 1$. Then we obtain the following identity.*

$$[Q : P] = \sum_{i=1}^n [p_i Q p_i : P p_i] / \text{tr}_Q(p_i)$$

Proof When $[Q : P] < \infty$, Jones has already proved in [J2, Lemma 2.2.2].

When $[Q : P] = \infty$, we include a proof here for the sake of completeness, though it has been noted in [PP, page 61].

For any $x \in Q$, we have

$$\begin{aligned} E_P(p_i x p_i) &= \sum_{j=1}^n p_j E_P(p_i x p_i) p_j \geq p_i E_P(p_i x p_i) p_i \\ &= E_{P p_i}(p_i x p_i) \text{tr}(p_i) \geq \frac{\text{tr}(p_i) p_i x p_i}{[p_i Q p_i : P p_i]_{PP}}. \end{aligned}$$

For any $x \in Q$ and $y \in P$, we have

$$\text{tr}_P(E_P(p_i x p_j) y) = \text{tr}_Q(p_i x p_j y) = \text{tr}_Q(x y p_j p_i).$$

Therefore, for any $x \in Q$, we obtain

$$E_P(x) = \sum_{i=1}^n E_P(p_i x p_i).$$

By the above arguments, if $[p_i Q p_i : P p_i]_{PP} = [p_i Q p_i : P p_i] < \infty$ for all i , we have

$$\begin{aligned} E_P(x) &= \sum_{i=1}^n E_P(p_i x p_i) \geq \sum_{i=1}^n \frac{\text{tr}(p_i) p_i x p_i}{[p_i Q p_i : P p_i]} \\ &\geq \min_{1 \leq i \leq n} \left(\frac{\text{tr}(p_i)}{[p_i Q p_i : P p_i]} \right) \sum_{i=1}^n p_i x p_i \geq \frac{1}{n} \min_{1 \leq i \leq n} \left(\frac{\text{tr}(p_i)}{[p_i Q p_i : P p_i]} \right) x, \end{aligned}$$

which contradicts $[Q : P] = \infty$. The last inequality holds by [GHJ, Definition 3.7.5] and [Jol, Corollaire 2.3]. Thus, there exists an index i_0 such that $[p_{i_0} Q p_{i_0} : P p_{i_0}] = \infty$. \square

Thanks to the following lemma, we can reduce the case of $\dim(M' \cap A) < \infty$ to the case of $\dim(M' \cap A) = 1$, which is the assumption of Proposition 5.9.

Lemma 5.12 *Let A be a finite von Neumann algebra with a fixed trace tr , and $M \subset A$ be a type II_1 factor with $[A : M] = \infty$ and $\dim(M' \cap A) < \infty$. Then there exist a minimal central projection $q \in A' \cap A$ and a minimal projection $p \in (Mq)' \cap (qAq)$ such that qAq is a type II_1 factor, $[pqAqp : pqMqp] = \infty$ and $(pqMqp)' \cap (pqAqp) = \mathbf{C}$.*

Proof Since A is a finite von Neumann algebra and $\dim(A' \cap A) \leq \dim(M' \cap A) < \infty$, there exist minimal central projections $q_1, \dots, q_n \in A' \cap A$ such that $A = \bigoplus_{i=1}^n q_i A q_i$. We have ${}_M L^2(A) \simeq \bigoplus_{i=1}^n {}_M L^2(q_i A q_i)$ as a left M -module, that is,

$$\begin{aligned} \infty &= [A : M] = \sum_{i=1}^n \dim_M L^2(q_i A q_i) \\ &= \sum_{i=1}^n \dim_{M q_i} L^2(q_i A q_i) = \sum_{i=1}^n [q_i A q_i : M q_i]. \end{aligned}$$

Then there exists an index j such that $[q_j A q_j : M q_j] = \infty$. Since we have

$$\begin{aligned} \dim((M q_j)' \cap (q_j A q_j)) &= \dim(M' \cap (q_j A q_j)) \leq \dim(\bigoplus_{i=1}^n (M' \cap q_i A q_i)) \\ &= \dim(M' \cap A) < \infty, \end{aligned}$$

there exists a finite number of minimal projections $p_1, \dots, p_m \in (M q_j)' \cap (q_j A q_j)$ such that $\sum_{i=1}^m p_i = 1$. By Lemma 5.11, we obtain

$$\infty = [q_j A q_j : M q_j] = \sum_{i=1}^m [p_i q_j A q_j p_i : M q_j p_i] / \text{tr}(p_i).$$

Then there exists an index i such that $[p_i q_j A q_j p_i : M q_j p_i] = \infty$ and $(p_i q_j M q_j p_i)' \cap (p_i q_j A q_j p_i) = \mathbf{C}$ by [P1, Lemma 2.1] and the minimality of p_j . \square

Lemma 5.13 *Let \mathcal{N} be a type II_1 factor and $\mathcal{M} \supset \mathcal{N}$ be a finite von Neumann algebra. If $f_0 \in \mathcal{N}' \cap \mathcal{M}$ is a projection and $f \leq f_0$ is a projection, then we obtain the following.*

$$E_{\mathcal{N}}(f) f_0 = \text{tr}(f_0) E_{\mathcal{N} f_0}(f)$$

This is also noted in [PP, page 71], but we include a proof.

Proof For any $x \in \mathcal{N}$ we have

$$\begin{aligned} \text{tr}_{\mathcal{N} f_0}(E_{\mathcal{N} f_0}(f) x) \text{tr}_{\mathcal{M}}(f_0) &= \text{tr}_{\mathcal{N} f_0}(f f_0 x) \text{tr}_{\mathcal{M}}(f_0) \\ &= \text{tr}_{\mathcal{M}}(f f_0 x) = \text{tr}_{\mathcal{M}}(f x) \\ &= \text{tr}_{\mathcal{N}}(E_{\mathcal{N}}(f) x) = \text{tr}_{\mathcal{N} f_0}(E_{\mathcal{N}}(f) x f_0). \end{aligned}$$

Since $f_0 \in \mathcal{N}' \cap \mathcal{M}$, we obtain the identity $E_{\mathcal{N}}(f) f_0 = \text{tr}(f_0) E_{\mathcal{N} f_0}(f)$. \square

Proof (Theorem 5.7, the infinite index case with $\dim(M' \cap A) < \infty$) By Proposition 5.12, there exist projections $p \in A' \cap A$ and $q \in (M q)' \cap (q A q)$ such that $[p q A q p : p q M q p] = \infty$ and $(p q M q p)' \cap (p q A q p) = \mathbf{C}$. And by Lemma 5.9, for any $\varepsilon > 0$, there exists a non-zero projection $e \in p q A q p$ such that $E_{p q M q p}(e) \leq \varepsilon q p$ and $\text{tr}_{p q M q p}(\chi_{\{\varepsilon\}}(E_{p q M q p}(e))) \geq 1 - \varepsilon$. Then by Lemma 5.13 (here we use the idea of [PP]), we have $E_M(e) q p = \text{tr}_A(q p) E_{p q M q p}(e) \leq \varepsilon \text{tr}_A(q p) q p$.

Since $qp \in M' \cap A$, M is isomorphic to Mqp , thus $E_M(e) \leq \varepsilon \text{tr}_A(qp)1_M$. We also have,

$$\begin{aligned} \text{tr}_M(E_M(e)) &= \text{tr}_A(e) = \text{tr}_A(eqp) = \text{tr}_{pqAqp}(eqp)\text{tr}_A(pq) \\ &= \text{tr}_{pqMqp}(E_{pqMqp}(e))\text{tr}_A(pq) \geq \varepsilon(1 - \varepsilon)\text{tr}_A(pq) \neq 0. \end{aligned}$$

Since $pq \in M' \cap A = (A'_k \cap M)' \cap (A'_k \cap A)$, we have

$$(pq(A'_k \cap M)qp)' \cap (pq(A'_k \cap A)qp) = \mathbf{C}$$

and

$$[pq(A'_k \cap A)qp : pq(A'_k \cap M)qp] = \infty$$

in the same way. Then for any ε , there exists a non-zero projection $e_k \in pq(A'_k \cap A)qp \subset A'_k \cap A$ such that $E_{A'_k \cap M}(e_k) \leq \varepsilon \text{tr}_A(qp)1$ and $\text{tr}(e_k) \geq \varepsilon(1 - \varepsilon)\text{tr}(pq) \neq 0$.

By setting $e := \{e_k\}_k \in M' \cap A^\omega$, we have $e \neq 0$ and $E_{M' \cap M^\omega}(e) \leq \varepsilon 1_{M' \cap M^\omega}$. Thanks to Propositions 5.3 and 5.4, we obtain $[M' \cap A^\omega : M' \cap M^\omega] = \infty$. \square

Proof (Theorem 5.1 (2) \Rightarrow (1)) Thanks to Theorem 5.7 and Lemma 5.6, we obtain

$$(M' \cap M_\infty)' \cap M_\infty = A = M.$$

\square

Remark So far we have considered only the larger algebra of $M' \cap N^\omega \subset M' \cap M^\omega$ and proved the equivalence between the double commutant property of $M' \cap M^\omega$ and Popa's strong amenability. It is known that when $N \subset M$ has an infinite depth, we have $[M' \cap M^\omega : M' \cap N^\omega] = \infty$. Thus if we define P_2 by the Jones basic construction of $M' \cap N^\omega \subset M' \cap M^\omega$, it does not make sense to consider the double commutant properties of P_k ($k \geq 2$), because $P_2 \not\subset M_\infty^\omega$.

As for the smaller algebra, we have not considered yet. In general, the condition $((M' \cap N^\omega)' \cap M_\infty^\omega)' \cap M_\infty^\omega = M' \cap N^\omega$ does not imply the strong amenability. (We recall that the converse direction always holds, see [EK, Section 15.5].) For example, let R_0 be a hyperfinite II_1 factor. We set $G := \text{PSL}(2, \mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, $R := \otimes_{g \in G} R_0 (\simeq R_0)$, and α to be an outer action of G on R defined by the Bernoulli shift. We restrict the action α to $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$ regarded as subgroups of G , and set $N := R^{\mathbf{Z}/2\mathbf{Z}}$ and $M := R \rtimes (\mathbf{Z}/3\mathbf{Z})$. Since G is non-amenable, by [J1, Proposition 2], we have $N^\omega \cap M' = (R_\omega)^G = \mathbf{C}$. This example in [B, page 211] is due to Jones and based on [J1]. Then $N^\omega \cap M' = \mathbf{C} = (N^\omega \cap M')^{cc}$. If the subfactor were strong amenable, the generating property (see [P4, Theorem 4.2.1]) would imply

$$\begin{aligned} (N \subset M) &\simeq \left(\overline{\bigcup_{k=1}^{\infty} N'_k \cap N} \subset \overline{\bigcup_{k=1}^{\infty} N'_k \cap M} \right) \\ &\simeq \left(\overline{\bigcup_{k=1}^{\infty} (R \otimes N_k)' \cap (R \otimes N)} \subset \overline{\bigcup_{k=1}^{\infty} (R \otimes N_k)' \cap (R \otimes M)} \right) \\ &\simeq (R \otimes N \subset R \otimes M), \end{aligned}$$

where $\cdots N_2 \subset N_1 \subset N \subset M$ is a generating tunnel. Then we have

$$\begin{aligned} \mathbf{C} &\simeq N^\omega \cap M' \simeq (R \otimes N)^\omega \cap (R \otimes M)' \\ &\supset R_\omega \otimes \mathbf{C} \simeq R_\omega, \end{aligned}$$

which is a contradiction.

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