The Algebraic Crossing Number of 3-braids

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Abstract

By proving sharpness of the Morton-Franks-Williams inequality, we show that the maximal Bennequin number is realized at their minimal braid representatives for most of the knots and links of braid index 3.

1 Introduction

There is a conjecture on range of the braid indices and the algebraic crossing numbers of closed braid representatives of a given knot as we state in Conjecture 1.1.

For an oriented braid diagram D let b_D denote the number of the braid strands and c_D the *algebraic crossing number* namely the number of positive crossings minus the number of negative crossings.

Let $\mathcal{K}$ be an oriented knot or a link. In this paper, for simplicity, a link means an $n \geq 1$ component knot or link. Let $\mathcal{B}_\mathcal{K}$ be the set of closed braid diagram of $\mathcal{K}$. Then the *braid index* $b_\mathcal{K}$ of $\mathcal{K}$ is $\min\{b_D | D \in \mathcal{B}_\mathcal{K}\}$. Let $\Phi : \mathcal{B}_\mathcal{K} \to \mathbb{N} \times \mathbb{Z}$ be the map defined by $\Phi(D) := (b_D, c_D)$ for $D \in \mathcal{B}_\mathcal{K}$.

Conjecture 1.1 [5] *There exists a unique $c_\mathcal{K} \in \mathbb{Z}$ such that*

$$\Phi(\mathcal{B}_\mathcal{K}) = \{(b_\mathcal{K} + x + y, c_\mathcal{K} + x - y) \mid x, y \in \mathbb{N}\}, \tag{1.1}$$

the subset of the infinite quadrant region shaded in Figure 1.

Remark 1.2 *If this conjecture is true then the maximal* Bennequin *number $\beta_\mathcal{K}$ of $\mathcal{K}$ is realized at the minimal braid index and $\beta_\mathcal{K} = c_\mathcal{K} - b_\mathcal{K}$.*

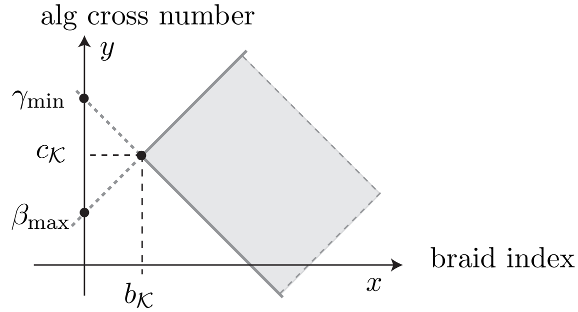

Figure 1: The region of braid representatives of $\mathcal{K}$

Conjecture 1.1 is inspired by a conjecture by V. F. R. Jones and the Morton-Franks-Williams (MFW) inequality.

Conjecture 1.3 [4] **Jones' conjecture:** *For any* $\mathcal{K}$ *there exists a unique*$c_{\mathcal{K}}$ *such that if* $D_1, D_2 \in \mathcal{B}_{\mathcal{K}}$ *have* $b_{D_1} = b_{D_2} = b_{\mathcal{K}}$ *then* $c_{D_1} = c_{D_2} = c_{\mathcal{K}}$*.*

In otherwords, $\{(b_{\mathcal{K}}, c) | c \in \mathbb{Z}\} \cap \Phi_{\mathcal{K}}(\mathcal{B}_{\mathcal{K}}) = \{(b_{\mathcal{K}}, c_{\mathcal{K}})\}$. Conjecture 1.1 implies Conjecture 1.3.

To state the MFW inequality we need the HOMFLY polynomial $P_{\mathcal{K}}(v, z)$ of $\mathcal{K}$. It is defined by the skein relation; $\frac{1}{v} P_{\mathcal{K}_+} - v P_{\mathcal{K}_-} = z P_{\mathcal{K}_0}$ and $P_{\text{unknot}} = 1$. Let $d_{\pm}(\mathcal{K})$ be the maximal (resp. minimal) v-degree of $P_{\mathcal{K}}(v, z)$.

Theorem 1.4 [3] [7] **The Morton-Franks-Williams inequality:** *For any closed braid diagram* $D \in \mathcal{B}_{\mathcal{K}}$ *of knot or link* $\mathcal{K}$ *we have* $c_D - b_D + 1 \leq d_-$ *and* $d_+ \leq c_D + b_D - 1$*. Moreover* $\frac{d_+ - d_-}{2} + 1 \leq b_{\mathcal{K}}$*.*

The inequalities $c_D - b_D + 1 \leq d_-$ and $d_+ \leq c_D + b_D - 1$ correspond to the two boundary lines of the infinite region in Figure 1.

We say the MFW inequality is *sharp* on $\mathcal{K}$ if there exists $D \in \mathcal{B}_{\mathcal{K}}$ such that $c_D - b_D + 1 = d_-$ and $d_+ = c_D + b_D - 1$ hold. If this is the case, $b_D = b_{\mathcal{K}} = \frac{1}{2}(d_+ - d_-) + 1$.

Note that sharpness of the MFW inequality on $\mathcal{K}$ implies the truth of Conjecture 1.1 for $\mathcal{K}$.

It has been known that the MFW inequality is sharp on unlinks, torus links, links with a braid representative of full positive twists with a positive word ($\Delta^{2n} P$) [3], alternating fibred links, 2-bridge links [9] and knots of ≤ 10 crossing in the standard knot table except $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$ [4]. Therefore for these knots and links Conjecture 1.1 is true.

By different approach, we know that Conjecture 1.1 is ture for $9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$, [6]. Also we know that truth of Conjecture 1.1 for $\mathcal{K}, \mathcal{L}$ implies truth of the conjecture for the connect sum $\mathcal{K} \sharp \mathcal{L}$ and the (p, q)-cable of $\mathcal{K}$, see [5].

Comparing to braids of braid index ≥ 4, the 3-braids are well classified and studied. Then we ask:

- Whether Conjecture 1.1 is true for the 3-braids?
- Is the MFW inequality sharp on the 3-braids?

In the next section, we use Murasugi's classification of the 3-braids [8] and answer these questions for most of the classes of the 3-braids (Theorem 2.1 and 2.4). The truth of Conjecture 1.3 for the 3-braids has been known by Birman and Menasco [1].

2 The Morton-Franks-Willams inequality and the 3-braids

Murasugi classified the 3-braids into seven classes $\Omega_0, \cdots, \Omega_6$: [8]. Let $\Delta = \sigma_1\sigma_2\sigma_1 \in B_3$ denote the positive half twist braid word. Then Δ^2 is the positive full twist.

$$\begin{aligned}
\Omega_0 &= \{\Delta^{2n} \mid n \in \mathbb{Z}\} \\
\Omega_1 &= \{\Delta^{2n}\sigma_1\sigma_2 \mid n \in \mathbb{Z}\} \\
\Omega_2 &= \{\Delta^{2n}(\sigma_1\sigma_2)^2 \mid n \in \mathbb{Z}\} \\
\Omega_3 &= \{\Delta^{2n+1} \mid n \in \mathbb{Z}\} \\
\Omega_4 &= \{\Delta^{2n}{\sigma_1}^{-p} \mid n \in \mathbb{Z},\ p \in \mathbb{N}\} \\
\Omega_5 &= \{\Delta^{2n}{\sigma_2}^{q} \mid n \in \mathbb{Z},\ q \in \mathbb{N}\} \\
\Omega_6 &= \{\Delta^{2n}{\sigma_1}^{-p_1}{\sigma_2}^{q_1}\cdots{\sigma_1}^{-p_r}{\sigma_2}^{q_r} \mid n \in \mathbb{Z},\ p_i, q_j \in \mathbb{N}\}
\end{aligned}$$

Our first goal is to prove:

Theorem 2.1 *Conjecture 1.1 is true for classes* Ω_i, $i = 0, 1, 2, 3, 4, 5$ *and*

$$\Omega'_6 := \{{\sigma_1}^{-p_1}{\sigma_2}^{q_1}\cdots{\sigma_1}^{-p_r}{\sigma_2}^{q_r} \mid \ p_i, q_j \in \mathbb{N}\}$$

a special class of Ω_6.

The following is a lemma for Theorem 2.1.

Lemma 2.2 *Conjecture 1.1 is ture for any 3-braids of more than one component.*

Proof of Lemma 2.2 We have already known that the conjecture is true for the unknot and torus links. Suppose that $\mathcal{K}$ is an n-component link. Then $\mathcal{K}$ consists of the unknot U and torus link $T_{2,p}$ where p is an odd number when $n = 2$ and an even number when $n = 3$. Let a be the linking number of U and $T_{2,p}$. Then $\Phi_{\mathcal{K}}(\mathcal{B}_{\mathcal{K}}) = \{(3 + x + y, p + 2n + x - y) \mid x, y \in \mathbb{N}\}$ and the conjecture is true. □

Proof of Theorem 2.1 Recall that Franks and Williams proved that the MFW inequality is sharp on a knot having a braid representative of full positive twists with a positive word ($\Delta^{2n}P$) [3]. Since sharpness of the inequality implies the truth of Conjecture 1.1, the conjecture is true for classes Ω_0, Ω_3. Also, it is true for $\Omega_1, \Omega_2, \Omega_5$ with $n > 0$ and for Ω_4 with $n < 0$.

Suppose $n > 0$. Then $\Delta^{-2n}\sigma_1\sigma_2 = \Delta^{-2(n-1)}(\sigma_2^{-1}\sigma_1^{-2})^2$ and $\Delta^{-2n}(\sigma_1\sigma_2)^2 = \Delta^{-2(n-1)}\sigma_2^{-1}\sigma_1^{-2}$. By the same result in [3], the conjecture is true for Ω_1, Ω_2 with $n < 0$.

Any closed braid of Ω_4 or Ω_5 has more than 1 link component. By Lemma 2.2, the conjecture holds for these classes.

Murasugi proved that the MFW inequality is sharp on any alternating fibred knots and links [9]. Notice that the braid closure of ${\sigma_1}^{-p_1}{\sigma_2}^{q_1}\cdots{\sigma_1}^{-p_r}{\sigma_2}^{q_r} \in \Omega'_6$ ($p_i, q_j \in \mathbb{N}$) is alternating fibred. The fibre surface is the Bennequin surface of the closed braid. Thus the conjecture is true for Ω'_6. □

Birman and Menasco studied the braid foliation of the 3-braids and proved that any knot and link $\mathcal{K}$ of braid index 3 has a unique conjugacy class at braid index 3 unless $\mathcal{K}$ admits the $\pm$-*flype* moves as in Figure 2. See [1] for the precise statement. The block, say $p \in \mathbb{Z}$, is the braiding assignment of

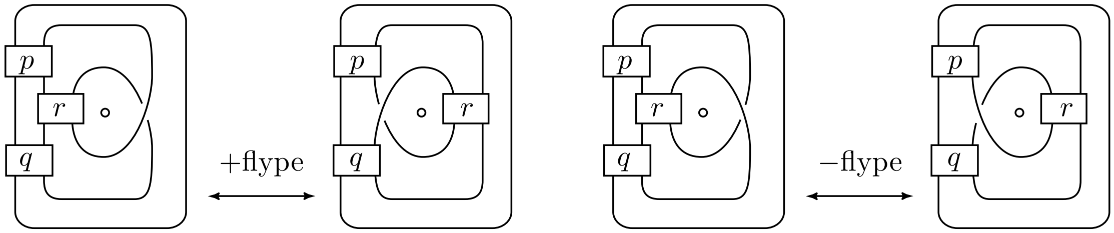

Figure 2: Positive and negative flype moves of $\mathcal{K}(p,q,r,\pm 1)$

p-positive half twists σ_1^p. Let $\mathcal{K}(p,q,r,\pm 1)$ denote the link type of the braid closure of $\sigma_1^p\sigma_2^r\sigma_1^q\sigma_2^{\pm 1}$. Later, Birman and Menasco discovered another interesting property of $\mathcal{K}(p,q,r,-1)$ [2]: The transversal closed braids $\sigma_1^{2p+1}\sigma_2^{2r}\sigma_1^{2q}\sigma_2^{-1}$ and $\sigma_1^{2p+1}\sigma_2^{-1}\sigma_1^{2q}\sigma_2^{2r}$ $(p+1 \neq q \neq r,\ p,q,r > 1)$ in the symmetric contact structure $\xi_{\mathrm{sym}} = \ker(dz + r^2 d\theta)$ of $\mathbb{R}^3$ have the same topological knot type and the same Bennequin number but they are not transversal isotopic.

Proposition 2.3 *The intersection $\{\mathcal{K}(p,q,r,\pm) \mid p,q,r \in \mathbb{Z}\} \cap (\Omega_6 \setminus \Omega_6')$ has infinitely many elements.*

Proof of Proposition 2.3 Let p,q be positive integers. We have $(\Omega_6 \setminus \Omega_6') \ni \Delta^2\sigma_1^{-p}\sigma_2^q = \sigma_2\sigma_1^2\sigma_2\sigma_1^{-p+2}\sigma_2^q = \sigma_1^{-p+2}\sigma_2^{q+1}\sigma_1^2\sigma_2 = \mathcal{K}(-p+2,2,q+1,+1)$. Also $\Delta^{-2}\sigma_1^{-p}\sigma_2^q = \mathcal{K}(-p-2,-2,q-1,-1)$. □

Here is our main result of this paper.

Theorem 2.4 *The MFW inequality is sharp on $\mathcal{K}(p,q,r,\pm)$ for any $p,q,r \in \mathbb{Z}$, i.e. the Conjecture 1.1 is true for $\mathcal{K}(p,q,r,\pm)$.*

Before proving Theorem 2.4, we need study the 2-braid $T_{2,n} = \sigma_1^n$. Let $P_n := P_{T_{2,n}}(v,1)$ the HOMFLY polynomial of $T_{2,n}$ evaluated at $z=1$. Let $\{a_n\}_{n\in\mathbb{Z}}$ be the *Fibonacci sequence* defined by the recursive relation: $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$.

Lemma 2.5 *For all $n \in \mathbb{Z}$ we have $P_n = a_{n+1}v^{n-1} - a_{n-1}v^{n+1}$.*

Proof of Lemma 2.5 By the skein relation of the HOMFLY polynomial, we have $P_{T_{2,n+1}} = vzP_{T_{2,n}} + v^2 P_{T_{2,n-1}}$. Putting $z=1$ we get $P_{n+1} = vP_n + v^2P_{n-1}$. We know that $P_1 = 1$ and $P_0 = \frac{1}{v} - v$. Then for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} &= \begin{pmatrix} v & v^2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} \\ &= \begin{pmatrix} a_{n+1}v^n & a_n v^{n+1} \\ a_n v^{n-1} & a_{n-1}v^n \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{v} - v \end{pmatrix}.\end{aligned}$$

Thus $P_n = a_n v^{n-1} + a_{n-1}v^{n-1} - a_{n-1}v^{n+1} = a_{n+1}v^{n-1} - a_{n-1}v^{n+1}$. □

Proof of Theorem 2.4 Since HOMFLY polynomials satisfy $P_{\mathcal{K}(p,q,r,-1)}(v,z) = P_{\mathcal{K}(-p,-q,-r,1)}(-\frac{1}{v},z)$, if the MFW inequality is sharp on $P_{\mathcal{K}(p,q,r,-1)}$ then so is on $P_{\mathcal{K}(-p,-q,-r,1)}$. Thus we only prove the theorem for $\mathcal{K}_{p,q,r} := \mathcal{K}(p,q,r,-1)$.

Note that:

Claim 2.6 *If $p=-1, q=-1$ or $r=0$ then $\mathcal{K}_{p,q,r}$ can be reduced to a 2-braid. Also if $p=0, q=0$ or $r=1$ then $\mathcal{K}_{p,q,r}$ is the connect sum of two 2-braids. In either case, we know that the MFW inequality is sharp.*

Therefore, we may assume that $p \neq -1, 0$, $q \neq -1, 0$ and $r \neq 0, 1$.

Since $\sigma_1^p\sigma_2^r\sigma_1^q\sigma_2^{-1} = \sigma_1^p\sigma_2^{-1}\sigma_1^q\sigma_2^r = \sigma_1^q\sigma_2^r\sigma_1^p\sigma_2^{-1}$ we have $\mathcal{K}_{p,q,r} = \mathcal{K}_{q,p,r}$. Thus it is enough to study the following six cases:

case	p	q	r
(1)	≥ 1	≥ 1	≥ 2
(2)	≥ 1	≥ 1	≤ -1
(3)	≤ -2	≥ 1	≥ 2
(4)	≤ -2	≥ 1	≤ -1
(5)	≤ -2	≤ -2	≥ 2
(6)	≤ -2	≤ -2	≤ -1

In the following, instead of computing the HOMFLY polynomial $P_{\mathcal{K}_{p,q,r}}(v,z)$, we compute $P_{\mathcal{K}_{p,q,r}}(v,1)$ for each of the six cases and obtain that the v-span of $P_{\mathcal{K}_{p,q,r}}(v,1)$ is 4. This implies that the v-span $(= d_+ - d_-)$ of $P_{\mathcal{K}_{p,q,r}}(v,z)$ is ≥ 4. By the MFW inequality, we have $b_{\mathcal{K}_{p,q,r}} \geq (d_+ - d_-)/2 + 1 \geq 3$. Since it is obvious that $3 \geq b_{\mathcal{K}_{p,q,r}}$, the inequality is sharp on $\mathcal{K}_{p,q,r}$.

Let $X_{p,q,r} := P_{\mathcal{K}_{p,q,r}}(v,1)$. Note that $\mathcal{K}_{p,q,1} = T_{2,p}\sharp T_{2,q}$ and $\mathcal{K}_{p,q,0} = T_{2,p+q}$. By the same argument as in the proof of Lemma 2.5 we have

$$\begin{aligned}
\begin{pmatrix} X_{p,q,r+1} \\ X_{p,q,r} \end{pmatrix} &= \begin{pmatrix} v & v^2 \\ 1 & 0 \end{pmatrix}^r \begin{pmatrix} X_{p,q,1} \\ X_{p,q,0} \end{pmatrix} \\
&= \begin{pmatrix} a_{r+1}v^r & a_r v^{r+1} \\ a_r v^{r-1} & a_{r-1}v^r \end{pmatrix} \begin{pmatrix} P_p \cdot P_q \\ P_{p+q} \end{pmatrix}.
\end{aligned}$$

Thanks to Lemma 2.5 we have

$$\begin{aligned}
X_{p,q,r} = {} & a_r a_{p+1} a_{q+1} v^{p+q+r-3} \\
& + (-a_r a_{p+1} a_{q-1} - a_r a_{p-1} a_{q+1} + a_{r-1} a_{p+q+1}) v^{p+q+r-1} \\
& + (a_r a_{p-1} a_{q-1} - a_{r-1} a_{p+q-1}) v^{p+q+r+1}
\end{aligned}$$

Since we have assumed $p \neq -1$, $q \neq -1$ and $r \neq 0$, the coefficient of the lowest term has $a_r a_{p+1} a_{q+1} \neq 0$. We will show that the coefficient $a_r a_{p-1} a_{q-1} - a_{r-1} a_{p+q-1}$ of the highest term is non-zero.

Let $f(p,q,r) := a_r a_{p-1} a_{q-1} - a_{r-1} a_{p+q-1}$. For $s,t,u > 0$ let $f_i(s,t,u)$ $(i = 1, \cdots 6)$ be as in the following:

$$\begin{array}{lcll}
f_1(s,t,u) & := & f(s,t,u) & \text{if } (s,t,u) \text{ is in Case (1),} \\
f_2(s,t,u) & := & (-1)^u f(s,t,-u) & \text{if } (s,t,-u) \text{ is in Case (2),} \\
f_3(s,t,u) & := & (-1)^s f(-s,t,u) & \text{if } (-s,t,u) \text{ is in Case (3),} \\
f_4(s,t,u) & := & (-1)^{s+u} f(-s,t,-u) & \text{if } (-s,t,-u) \text{ is in Case (4),} \\
f_5(s,t,u) & := & (-1)^{s+t} f(-s,-t,u) & \text{if } (-s,-t,u) \text{ is in Case (5),} \\
f_6(s,t,u) & := & (-1)^{s+t+u} f(-s,-t,-u) & \text{if } (-s,-t,-u) \text{ is in Case (6).}
\end{array}$$

Then we have:

Claim 2.7 *For each* $i = 1, \cdots 6$ *we have*

$$\begin{aligned} f_i(s+2,t,u) &= f_i(s+1,t,u) + f_i(s,t,u) \\ f_i(s,t+2,u) &= f_i(s,t+1,u) + f_i(s,t,u) \\ f_i(s,t,u+2) &= f_i(s,t,u+1) + f_i(s,t,u). \end{aligned} \tag{2.1}$$

Proof of Claim 2.7 We show that equations (2.1) hold for the function f_2.

By the recursive relation $a_{n+2} = a_{n+1} + a_n$ of the sequence, we have the first two equations; $f_2(s+2,t,u) = f_2(s+1,t,u) + f_2(s,t,u)$ and $f_2(s,t+2,u) = f_2(s,t+1,u) + f_2(s,t,u)$.

We see the last equation by

$$\begin{aligned} & f_2(s,t,u+1) + f_2(s,t,u) \\ &\quad = (-1)^{u+1}(a_{-u-1}a_{s-1}a_{t-1} - a_{-u-2}a_{s+t-1}) + (-1)^u(a_{-u}a_{s-1}a_{t-1} - a_{-u-1}a_{s+t-1}) \\ &\quad = (-1)^u((-a_{-u-1} + a_{-u})a_{s-1}a_{t-1} + (a_{-u-2} - a_{-u-1})a_{s+t-1}) \\ &\quad = (-1)^{u+2}(a_{-u-2}a_{s-1}a_{t-1} - a_{-u-3}a_{s+t-1}) \\ &\quad = f_2(s,t,u+2). \end{aligned}$$

Other equations follow by similar computations. □

Claim 2.8 *For each* $i = 1, \cdots, 6$, *we have* $f_i(s,t,u) \neq 0$ *except* $f_1(2,2,3) = 0$. *Therefore, for all* p, q, r ($p \neq -1, 0$, $q \neq -1, 0$, $r \neq 0, 1$ *and* $(p,q,r) \neq (2,2,3)$), *we have* $f(p,q,r) \neq 0$.

Proof of Claim 2.8

Case (1). $[s, t \geq 1,\ u \geq 2.]$ We have

$$f_1(1,1,2) = f_1(2,1,2) = f_1(1,2,2) = f_1(2,2,2) = f_1(1,1,3) = f_1(2,1,3) = f_1(1,2,3) = -1$$

and $f_1(2,2,3) = 0$. Thus by equations (2.1), we have $f_1(s,t,u) < 0$ for all s, t, u, unless $(p,q,r) \neq (2,2,3)$.

Case (2). $[s, t \geq 1,\ u \geq 1.]$ We have $f_2(1,1,1) = f_2(1,2,1) = f_2(2,1,1) = -1$, $f_2(2,2,1) = -3$, $f_2(1,1,2) = f_2(1,2,2) = f_2(2,1,2) = -2$ and $f_2(2,2,2) = -5$. Then by (2.1) we have $f_2(s,t,u) < 0$ for all s, t, u.

Case (3). $[s, u \geq 2,\ t \geq 1.]$ If $s, u \in \{2,3\}$ and $t \in \{1,2\}$ then $f_3(s,t,u) > 0$. Thus by (2.1) we have $f_3(s,t,u) > 0$ for all s, t, u.

Case (4). $[s \geq 2,\ t, u \geq 1.]$ If $s \in \{2,3\}$, $t = 1$ and $u \in \{1,2\}$ then $f_4(s,1,u) > 0$. Thus $f_4(s,1,u) > 0$ for all s and u.

If $s, t \in \{2,3\}$ and $u \in \{1,2\}$ then $f_4(s,t,u) < 0$. Thus $f_4(s,t,u) < 0$ for all s, u and $t \neq 1$.

Case (5). $[s, t, u \geq 2.]$ If $s, t \in \{2,3\}$ and $u = 2$, then $f_5(s,t,2) < 0$. Thus $f_5(s,t,2) < 0$ for all s and t.

If $s, t \in \{2,3\}$ and $u \in \{3,4\}$ then $f_5(s,t,u) > 0$. Thus $f_5(s,t,u) > 0$ for all s, t and $u \neq 2$.

Case (6). $[s, t \geq 2,\ u \geq 1.]$ If $s, t \in \{2,3\}$ and $u \in \{1,2\}$ then $f_6(s,t,u) < 0$. Thus $f_6(s,t,u) < 0$ for all s, t and u. □

The link $\mathcal{K}_{2,2,3}$ is a 3 component link on which the MFW inequality is sharp. Therefore by Claims 2.6 and 2.8, the MFW inequality is sharp on $\mathcal{K}(p,q,r,\pm)$ for all p,q,r. This completes the proof of Theorem 2.4. $\square$

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