A Generalized Gibrat's Law

(Technical Appendix)

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This appendix proves some additional results for the case in which X_t is a diffusion. Let X_t be a random variable and let $P(x,t)$ denote the probability distribution of X at time t, and $p(x,t)$ its corresponding density. The following three assumptions were discussed above.

Assumption 1: X_t is an i.i.d diffusion process with stationary transition defined by a drift, $\mu(x)$, and diffusion, $\sigma^2(x)$.

Assumption 2: Suppose that for $x \geq m_t$, $m_t \equiv me^{\gamma t}$ and $t \geq 0$:

$$p(x,t) = \delta a m_t^{\delta} x^{-(1+\delta)}. \tag{A.1}$$

where $a \equiv \Pr\{X_t \geq m_t\}$.

Assumption 3:

$$E_t[x\,(\mu(x) - \gamma)\,|X_t \geq m_t] = 0 \text{ for all } t \geq 0. \tag{A.2}$$

Let $p(x_0, x; t)$ be the probability density function of x_t, given that at an earlier time, t_0, $x = x_0$. Off boundaries, the motion of the conditional probability distribution $p(x_0, x; t)$ is described by the Forward Kolmogorov Equation - *FKE:*

$$\frac{\partial}{\partial t} p(x_0; x, t) = \frac{1}{2}\frac{\partial}{\partial x^2}\left[x^2\sigma^2(x)p(x_0; x, t)\right] - \frac{\partial}{\partial x}\left[x\mu(x)p(x_0; x, t)\right].$$

Moreover, a limit distribution satisfies:

$$\frac{\partial}{\partial t} p(x,t) = \frac{1}{2}\frac{\partial}{\partial x^2}\left[x^2\sigma^2(x)p(x,t)\right] - \frac{\partial}{\partial x}\left[x\mu(x)p(x,t)\right], \tag{A.3}$$

The following result is easily verified.

Proposition A.I Let Assumptions 1 and 2 hold. Then $\mu(x) = \gamma$ and $\sigma^2(x) = Ax^{\delta-1}$ satisfies A.3.

Definition: A diffusion process, described by $\mu(x)$and $\sigma^2(x)$, supports a Pareto Equilibrium if it satisfies equations A.1, A.2, and A.3.

0.1. Stationary Case: $\gamma = 0$

Substituting A.1 into A.3 and dropping time subscripts produces

$$\frac{1}{2}\frac{\partial}{\partial x^2}\left[x^{1-\delta}\sigma^2(x)\right] - \frac{\partial}{\partial x}\left[\mu(x)x^{-\delta}\right] = 0. \tag{A.4}$$

Integrating and solving for $\mu(x)$, gives

$$\mu(x) = \frac{1}{2}\left[x\frac{\partial}{\partial x}\sigma^2(x) + (1-\delta)\sigma^2(x) + Ax^{\delta}\right], \tag{A.5}$$

where A is a constant of integration. This equation characterizes the drift, $\mu(x)$, as a function of $\sigma^2(x)$. Multiplying A.5 by x, taking expected value with respect to $P(\cdot)$ conditional on $X_t \geq m$ and using condition A.2 one obtains:

$$E\left[x^2\frac{\partial}{\partial x}\sigma^2(x)\right] + (1-\delta)E\left[x\sigma^2(x)\right] + AE\left[x^{1+\delta}\right] = 0, \tag{A.6}$$

where E is the conditional expectation. The first term of the last expression can be re-expressed as:

$$\begin{aligned}
E[x^2\frac{\partial}{\partial x}\sigma^2(x)] &= \delta a m^{\delta}\int_m^{\infty} x^{1-\delta}\frac{\partial}{\partial x}\sigma^2(x)dx \\
&= \delta a m^{\delta}\left[x^{1-\delta}\sigma^2(x)\right]_m^{\infty} - \delta\left(1-\delta\right)am^{\delta}\int_m^{\infty} x^{-\delta}\sigma^2(x)dx \\
&= \delta a m^{\delta}\left[x^{1-\delta}\sigma^2(x)\right]_m^{\infty} - \left(1-\delta\right)E\left[x\sigma^2(x)\right].
\end{aligned}$$

Plugging this result into A.6 produces

$$\delta a m^{\delta}\left[x^{1-\delta}\sigma^2(x)\right]_m^{\infty} + AE\left[x^{1+\delta}\right] = 0.$$

Now, $Ex^{1+\delta} = \int_m^{\infty} x^{1+\delta}\delta a m^{\delta}x^{-\delta-1}dx = \delta a m^{\delta}\int_m^{\infty} dx = \delta a m^{\delta}\left[x\right]_m^{\infty}$. Thus, we can write the previous equation as $\delta m^{\delta}\left[x^{1-\delta}\sigma^2(x) + Ax\right]_m^{\infty} = 0$, or

$$\sigma^2(m) = -Am^{\delta} + m^{\delta-1}\lim_{x\to\infty} x\left[x^{-\delta}\sigma^2(x) + A\right]. \tag{A.7}$$

The following is proposition provide a general characterization of the type of diffusions supporting Pareto Distributions.

Proposition A.II A diffusion process with drift $\mu(x)$ and diffusion $\sigma^2(x)$ supports a Pareto equilibrium if and only if $\sigma^2(x)$ is a positive differentiable function satisfying A.7, and $\mu(x)$ satisfies A.5.

Proof. For sufficiency, notice that equations A.1, A.2, and A.3 are satisfied if A.5 and A.7 are satified. Necessity has already been established since A.5 and A.7 were obtained from A.1, A.2, and A.3. ■

Example. Let $r(x)$ be a positive constinously differentiable function satisfying $r(m) = 0$, and $\lim_{x\to\infty} x^{1-\delta} r(x) = 0$. Then, a diffusion process with drift $\mu(x) = \frac{1}{2}\left[x r'(x) + (1-\delta) r(x)\right]$ and variance $\sigma^2(x) = \beta x^{\delta-1} + r(x)$ supports a Pareto distribution.

0.2. Non-Stationary Case: $\gamma > 0$

Theorem A.III Let X_t be a random variable satisfying assumptions 1, 2, and 3. If $\gamma > 0$ then, $\mu(x) = \gamma$ and $\sigma^2(x) = Ax^{\delta-1} + Bx^{\delta}$ where A and B are non-negative constants.

Proof In our case $p(x,t) = \delta a m_t^{\delta} x^{-\delta-1}$. Then $\frac{\partial}{\partial t} p(x,t) = \gamma\delta p(x,t)$. The KFE reads

$$\gamma\delta^2 a \left(m e^{\gamma t}\right)^{\delta} x^{-\delta-1} = \frac{1}{2}\frac{\partial}{\partial x^2}\left[x^2\sigma^2(x)p(x,t)\right] - \frac{\partial}{\partial x}\left[x\mu(x)p(x,t)\right],$$

and integrating once (with respect to x) produces

$$-\gamma x p(x,t) + \frac{1}{2}A(t) = \frac{1}{2}\frac{\partial}{\partial x}\left[x^2\sigma^2(x)p(x,t)\right] - \mu(x)xp(x,t), \text{ or}$$

$$[\mu(x) - \gamma]xp(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\left[x^2\sigma^2(x)p(x,t)\right] - \frac{1}{2}A(t). \tag{A.8}$$

Moreover, integrating in the interval $[m_t, \infty)$ we have

$$\int_{m_t}^{\infty} [\mu(x) - \gamma]xp(x,t)dx = \frac{1}{2}\left[x^2\sigma^2(x)p(x,t) - A(t)x\right]_{m_t}^{\infty}. \tag{A.9}$$

Now, according to A.2 the left hand side of the previous equation must be zero for all t. Below we show that there are only two possible cases: either $A(t) = 0$ for all t or $A(t) \neq 0$ for all t. Consider first the case $A(t) = 0$ for all t. Then the following equality must hold for all t: $\frac{1}{2}\left[x^2\sigma^2(x)p(x,t)\right]_{m_t}^{\infty} = 0$ or

$$\sigma^2(m_t)m_t = m_t^\delta \lim_{v\to\infty} v^{1-\delta}\sigma^2(v) \text{ for all } t.$$

Define $\beta = \lim_{v\to\infty} v^{1-\delta}\sigma^2(v)$. Then, $\sigma^2(m_t) = \beta m_t^{\delta-1}$ for all $t \geq 0$. We can replace the condition "for all t" by the expression "for all m_t", but then it is the same as "for all x" since m_t grows continuously and unboundedly overtime. Thus, we conclude that

$$\sigma^2(x) = \beta x^{\delta-1} \text{ for all } x. \tag{A.10}$$

Rreplacing this expression into A.8 given that $A(t)$ is zero, one confirms that $\mu(x) = \gamma$ for all x. Now consider the case $A(s) \neq 0$ for some $s \geq 0$ in A.8. In that case, A.9 implies $\left[x^{1-\delta}\sigma^2(x)\delta m_t^\delta - A(t)x\right]_{m_t}^{\infty} = 0$ for all t or

$$m_t\left[\sigma^2(m_t)\delta - A(t)\right] = \lim_{x\to\infty} x\left[x^{-\delta}\sigma^2(x)\delta m_t^\delta - A(t)\right] \text{ for all } t. \tag{A.11}$$

This condition requires that $\lim_{x\to\infty} x^{-\delta}\sigma^2(x)\delta m_t^\delta - A(t) = 0$ for all t, or

$$A(t) = \delta h m_t^\delta \text{ for all } t, \tag{A.12}$$

where $h :=\lim_{x\to\infty} x^{-\delta}\sigma^2(x)$. Substituting A.12 into A.1, we obtain

$$m_t\left[\sigma^2(m_t) - hm_t^\delta\right] = \theta m_t^\delta \text{ for all } t$$

where $\theta := \lim_{x\to\infty} x\left[x^{-\delta}\sigma^2(x) - h\right]$. Finally, solving for $\sigma^2(m_t)$ from the previous equation we obtain

$$\sigma^2(x) = hx^\delta + \theta x^{\delta-1}. \tag{A.13}$$

Finally, substituting A.12 and A.13 into A.8 confirms that that $\mu(x) = \gamma$. Thus, in any solution, the drift must be γ . The diffusion coefficient, in the other hand, can either have the form A.10 or A.13, but A.10 is a particular case of A.13.