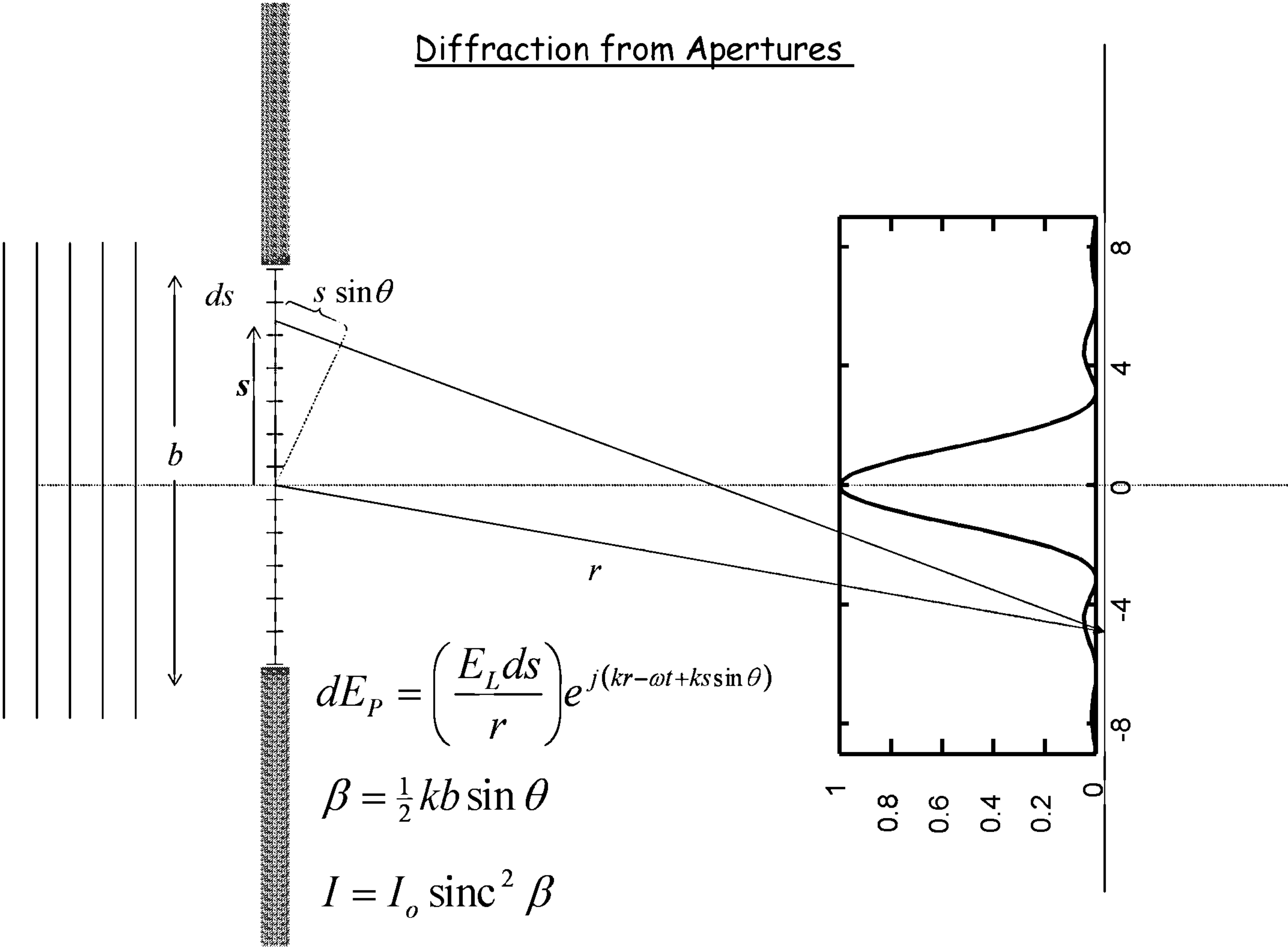
Diffraction from Apertures

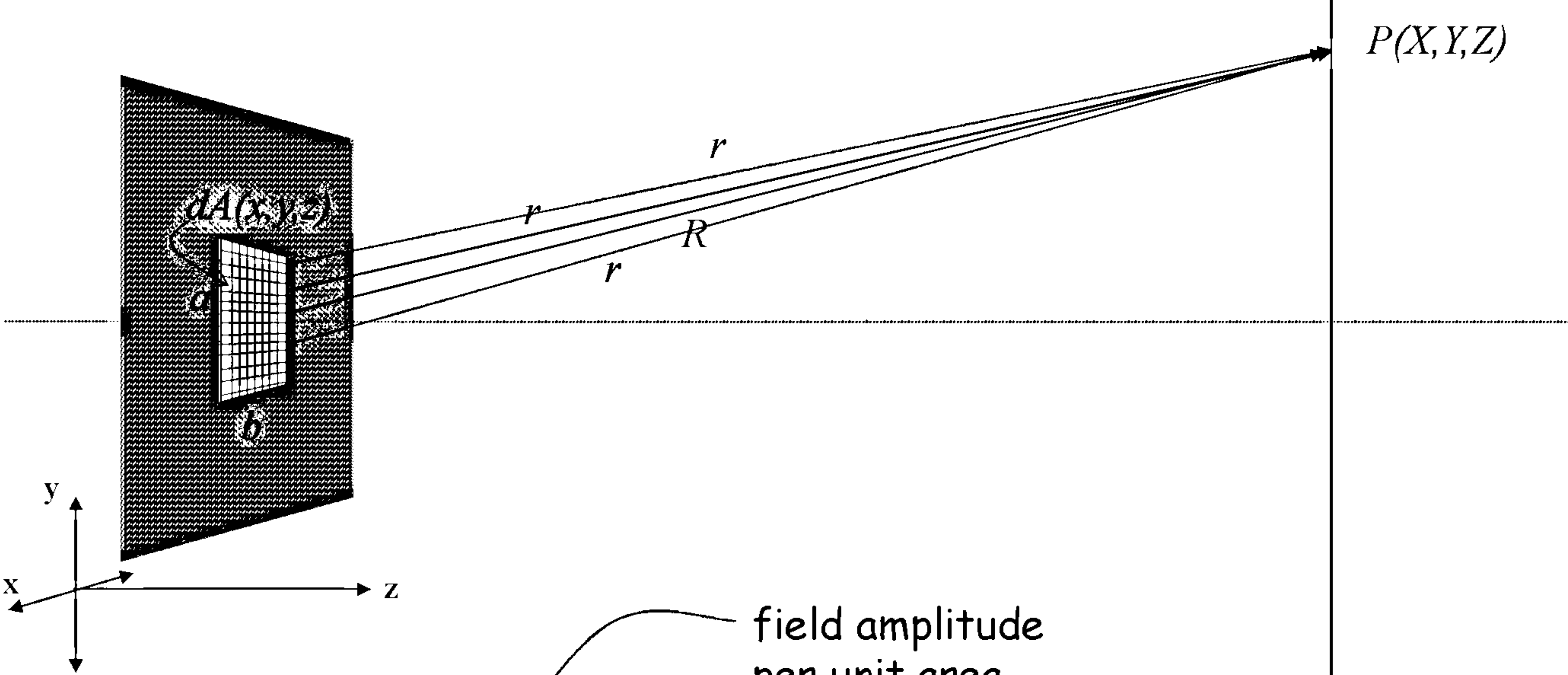


$$dE_P = \left(\frac{E_L ds}{r}\right) e^{j(kr - \omega t + ks\sin\theta)}$$

$$\beta = \tfrac{1}{2} kb \sin\theta$$

$$I = I_o \,\mathrm{sinc}^2\, \beta$$

Any Aperture



field amplitude per unit area

$$dE_P = \left(\frac{E_A dA}{r} \right) e^{j(\omega t - kr)}$$

Phase differences in *r*'s

$$R = \sqrt{X^2 + Y^2 + Z^2}$$

$$r = \sqrt{(X - x)^2 + (Y - y)^2 + Z^2}$$

$$r \approx R$$

They are close enough for the amplitude term
(the difference is small compared to the screen distance)

$$dE_P = \left(\frac{E_A dA}{R} \right) e^{j(\omega t - kr)}$$

They are *not* close enough for the phase term
(the difference is *not* small compared to the wavelength)

$$r = \sqrt{X^2 + Y^2 + Z^2 - 2Xx + x^2 - 2Yy + y^2}$$

$$r = R\sqrt{1 - 2(Xx + Yy)/R^2 + (x^2 + y^2)/R^2}$$

$$r \approx R\sqrt{1 - 2(Xx + Yy)/R^2}$$

binomial expansion:

$$r \approx R\left(1 - (Xx + Yy)/R^2\right)$$

$$dE_P = \left(\frac{E_A dA}{R}\right) e^{j\left(\omega t - kR\left[1 - (Xx + Yy)/R^2\right]\right)}$$

$$dE_P = \left(\frac{E_A dA}{R}\right) e^{j\left(\omega t - kR\left[1-(Xx+Yy)/R^2\right]\right)}$$

$$dE_P = \left(\frac{E_A dA}{R}\right) e^{j(\omega t - kR)} e^{jk(Xx+Yy)/R}$$

$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \iint_{aperture} e^{jk(Xx+Yy)/R} dA$$

For any shape!

Rectangular:

$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{jk(Xx+Yy)/R} dx dy$$

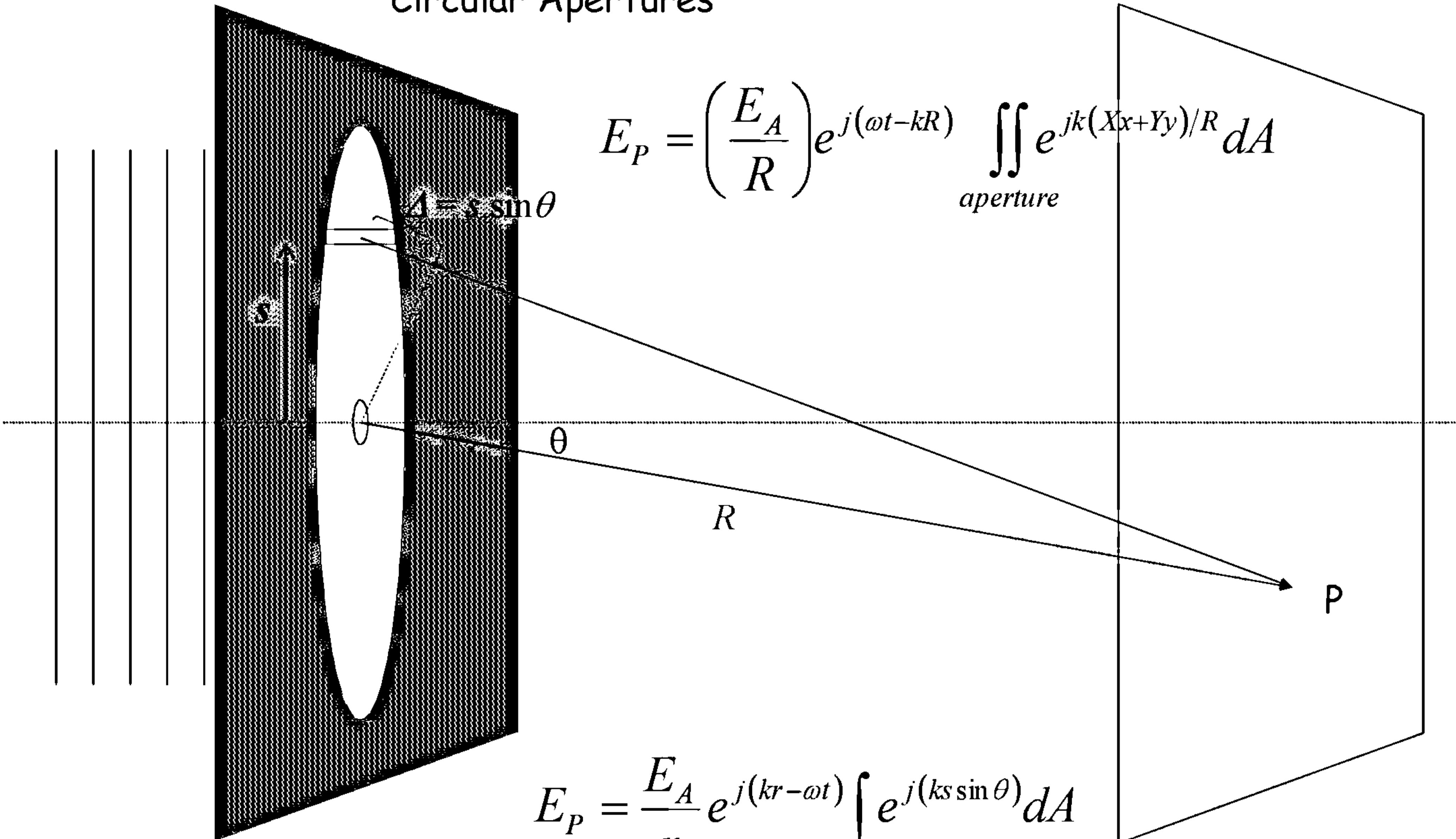
$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{\frac{jkYy}{R}} dy \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{\frac{jkXx}{R}} dx$$

$$\beta = \tfrac{1}{2} kb \sin\theta = \tfrac{1}{2} kb \frac{Y}{R} \qquad \alpha = \tfrac{1}{2} ka \sin\theta = \tfrac{1}{2} ka \frac{X}{R}$$

From here its just like the single slit:

$$I(X,Y) = I_o \operatorname{sinc}^2 \beta \operatorname{sinc}^2 \alpha$$

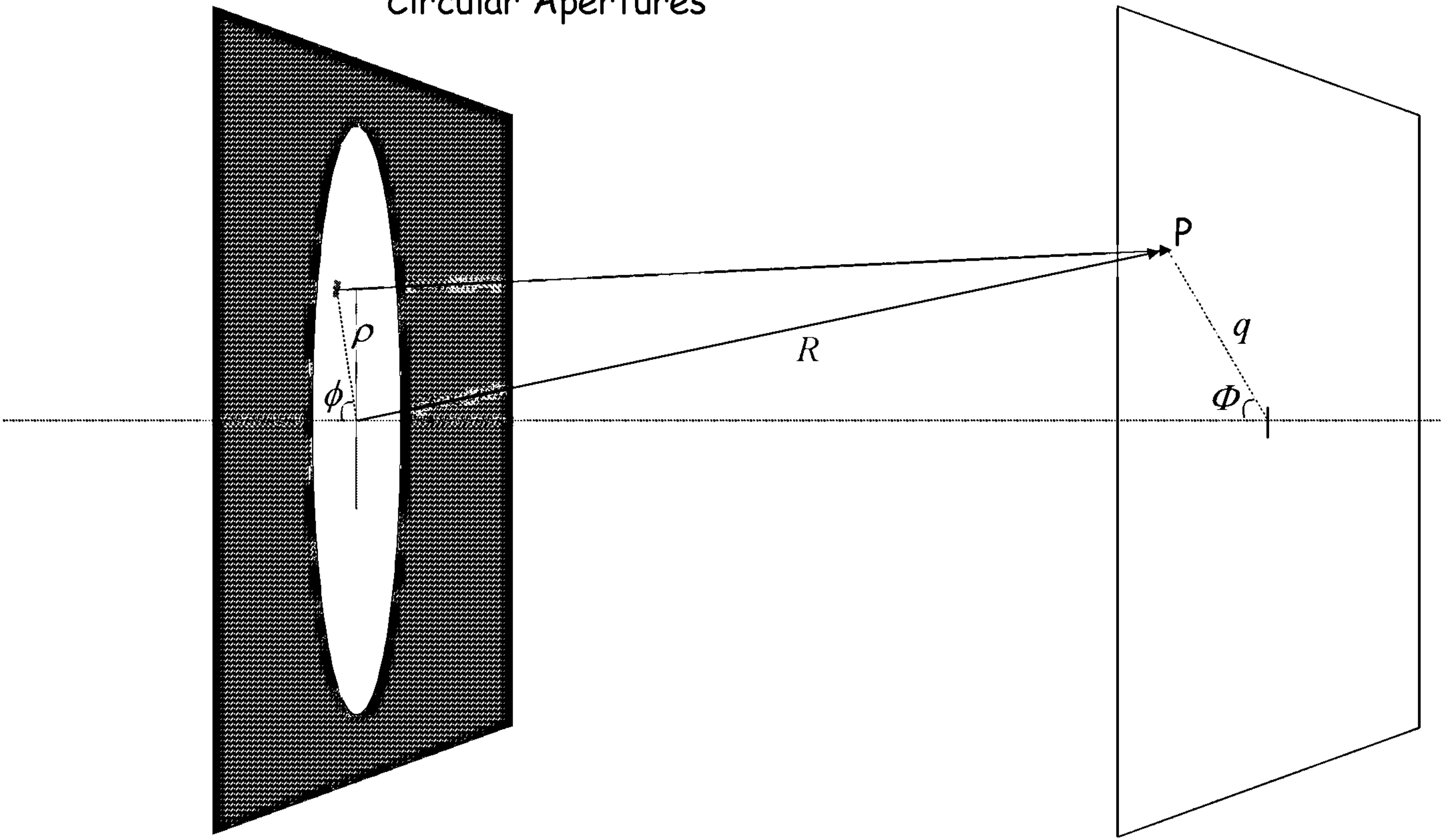
Circular Apertures



$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \iint_{aperture} e^{jk(Xx+Yy)/R} dA$$

$$E_P = \frac{E_A}{r} e^{j(kr-\omega t)} \int_A e^{j(ks\sin\theta)} dA$$

Circular Apertures



$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} e^{jk(\rho q/R)\cos(\phi-\Phi)} \rho \, d\phi \, d\rho$$

Φ is arbitrary, might as well make it zero:

$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} e^{jk(\rho q/R)\cos(\phi)} d\phi \, \rho \, d\rho$$

$$x = k\rho q / R$$

$$E_P = \left(\frac{E_A}{R}\right) e^{j(\omega t - kR)} \int_{\rho=0}^{a} \boxed{\int_{\phi=0}^{2\pi} e^{jx\cos(\phi)} d\phi} \rho \, d\rho$$

Bessel Function!

Bessel function of the first kind (J) of order zero (0)

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\cos\phi}\, d\phi$$

Bessel functions $J_m(x)$ solve this differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - m^2\right)y = 0$$

Recurrence relations let you solve integrals and derivatives:

$$\frac{d}{dx}\left[x\, J_1(x)\right] = x\, J_0(x) \qquad x\, J_1(x) = \int_0^x x'\, J_0(x')dx'$$

They can be expressed as series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{-1^n}{n!\,\Gamma(n+1)}\left(\tfrac{1}{2}\, x\right)^{2n}$$

$$E_P = 2\pi\left(\frac{E_A}{R}\right)e^{j(\omega t - kR)}\int_{\rho=0}^{a} J_0(x)\,\rho\,d\rho$$

$$E_P = 2\pi\left(\frac{E_A}{R}\right)e^{j(\omega t - kR)}\int_{x=0}^{x=kaq/R} J_0(x)\;\frac{xR}{kq}\frac{R}{kq}\,dx$$

$$E_P = 2\pi\left(\frac{E_A}{R}\right)e^{j(\omega t - kR)}\left(\frac{R}{kq}\right)^2 (kaq/R)\,J_1(kaq/R)$$

$$E_P = 2\left(\frac{E_A A}{R}\right)e^{j(\omega t - kR)}\left(\frac{R}{kaq}\right)J_1(kaq/R)$$

$$I_P = \tfrac{1}{2}\varepsilon_o c 4\left(\frac{E_A A}{R}\right)^2 \left[\frac{J_1(kaq/R)}{kaq/R}\right]^2$$

$$I_P = I_o\left[\frac{2J_1(kaq/R)}{kaq/R}\right]^2$$