PHYS 201 Problem Set

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1. In class we showed that you can solve the differential equation $-kx = m\ddot{x}$ by "guessing" the solution $x(t) = A\sin(Bt + C)$. According to uniqueness and completeness, if this guess solves the equation of motion and has the correct number of adjustable constants (two, in this case), then it is the one and only solution. To be more precise, it *describes* the one and only solution. There are, of course, other mathematical expressions that could describe the same solution. For *each* of the expressions below (**i**) demonstrate that it solves the differential equation, (**ii**) find which constants (A, B, or C) are defined by parameters in the equation of motion and which are free to be defined by initial conditions, and (**iii**) find the free constants for the initial conditions $x(0) = x_o$ and $\dot{x}(0) = v_o$, and (**iv**) write the complete solution. Finally, plot the two solutions assuming $\omega_o = x_o = v_o = 1$ to confirm that they are identical.

a. $x(t) = A\cos(Bt + C)$ (20 pts.)
b. $x(t) = A\sin(Bt) + C\cos(Bt)$ (20 pts.)
c. plot (10 pts.)

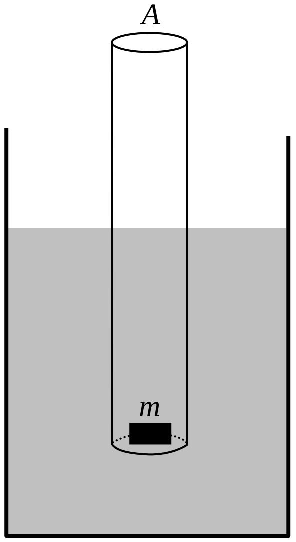
2. There is a very large pendulum at the Houston Museum of Natural Science. Either visit the museum to inspect the pendulum, or watch the video on the class web page. What is the length of the pendulum? (15 pts)

3. Consider the cylindrical "floater" of circular cross section A shown below which sits in a body of water and is free to oscillate up and down.

a. At its point of stable equilibrium, how high is the water level above the bottom of the floater? (10 pts)

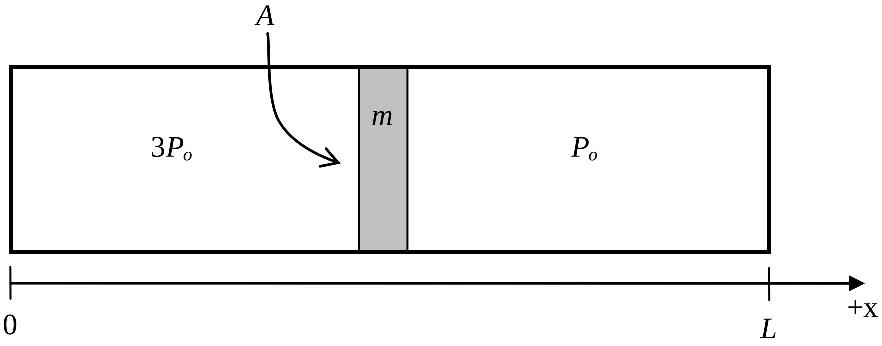
b. Derive the equation of motion for this object as it bobs up and down in the water. Is it linear? (10 pts)

Hint: You can assume that all of the weight of the floater is due to the mass at the bottom, and that this weight keeps it from tipping sideways.



4. A frictionless piston of mass *m* and cross-sectional area *A* is surrounded by two sealed chambers of total length *L*. When the piston is centered, the chambers are at pressures $3P_o$ and P_o. You may ignore the small finite width of the piston.

- **a.** Find the point of stable equilibrium along the x-axis. (10 pts)
- **b.** Derive the equation of motion for the piston. (10 pts)
- **c.** Linearize the equation of motion if necessary (Hint: *It is necessary!*). (10 pts)
- **d.** Give the expression for the natural frequency of SHM for the piston. (10 pts)



5. In the "Feel the Beat!" demo, we listened simultaneously to pitches at 200 Hz and 202 Hz to hear beats. We found that the beats occurred at the difference between the two frequencies, but according to our formula:

$$x_{1+2}=2A\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

the beat frequency is *half* the difference. Explain this apparent inconsistency in a few sentences or a paragraph. A figure might be wise… (15 pts.)

6. In class I said that an underdamped free oscillator's amplitude decays by 1/e in "roughly" Q/π cycles. Show that the exact answer is that it will decay from an initial amplitude of A_0 to an amplitude of A_0/e in:

$$\frac{Q}{\pi}\sqrt{1-\frac{1}{4Q^2}}\quad \text{cycles.}$$ (20 pts.)

7. Consider a simple harmonic oscillator with mass *m*, spring constant *k*, and damping parameter *γ*, initial conditions $x(0)=0$ and $\dot{x}(0)=0$, and a driving force $F(t)=F_o\cos(\omega t)$ beginning at $t=0$. To preserve the grader's sanity, set $F_o=m=k=1$. Set $\gamma=0.1$. Assume that the drive frequency is exactly equal to the natural frequency ω_o. Using the full solution (transient plus steady state):

$$z(t)=Ae^{-\frac{\gamma}{2}t}e^{j\left(\sqrt{\omega_0^2-\frac{\gamma^2}{4}}t+\phi\right)}+\frac{F_o/m}{\sqrt{\left(\omega_o^2-\omega^2\right)^2+(\gamma\omega)^2}}e^{j\left(\omega t-\tan^{-1}\left(\frac{\gamma\omega}{\omega_o^2-\omega^2}\right)\right)}$$

- **a.** Solve for the free parameters *A* and ϕ from the initial conditions, and write the complete expression for the motion. (10 pts.)
- **b.** Sketch or plot the motion from the point $t=0$ to steady state. (10 pts.)

c. How do the transient and steady state solutions, which have finite sinusoidal amplitudes, achieve a condition of zero position and velocity at $t = 0$?. (10 pts.)

8. In class I said that resonance for a damped, driven oscillator actually occurs below the natural frequency. From the expression for the steady state amplitude of a driven simple harmonic oscillator (see French, equation 4-11), show that the drive frequency that will give maximum amplitude response is given by:

$$\omega_m = \omega_0\sqrt{1-\frac{1}{2Q^2}} \quad \text{(20 pts.)}$$

9. According to classical electromagnetic theory an accelerated electron radiates energy at the rate Ke^2a^2/c^3, where $K = 6 \times 10^9$ N-m²/C², e = electronic charge (C), a = instantaneous acceleration (m/s²), and c = speed of light (m/s).

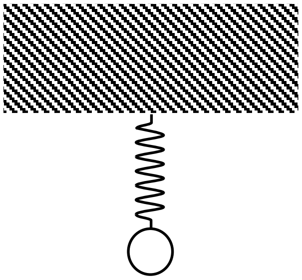
a. If an electron were oscillating along a straight line with angular frequency ω and amplitude A, how much energy would it radiate away during 1 cycle? (Assume the motion is described adequately by $x = A\sin(\omega t)$ during one cycle). (10 pts.)

b. Derive an expression for the Q of this oscillator? (10 pts.)

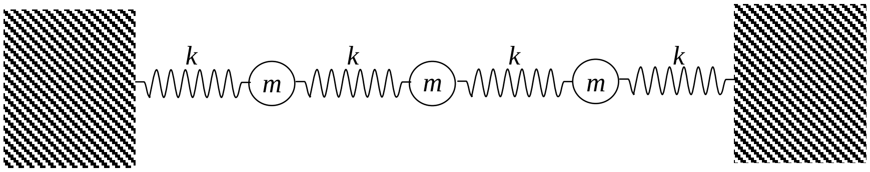
c. How many periods of oscillation would elapse before the energy of the motion was down to half the initial value? (10 pts.)

d. Putting for ω a typical optical frequency (i.e., for visible light) estimate numerically the approximate Q and "half-life" of the radiating system. (5 pts.)

10. You will get 1 pt for each independent degree of freedom you can describe for the spherical mass shown below, which is hanging from a spring attached to a fixed ceiling:



11. Determine the normal frequencies of this one dimensional system, ignoring friction and gravity. (30 pts.)



12. For each of the possible wave forms below, indicate which satisfy the wave equation, and which represent *reasonable* waveforms for actual waves on a string. For those which do represent waves, find the speed and direction of propagation, and sketch the waveform at $t=0$. The quantities a, b, c, A, and v are all positive constants.

a. $y(x,t)=(ax+bt+c)^2$ (4 pts.)

b. $y(x,t)=\frac{1}{(ax^2+b)}$ (4 pts.)

c. $y(x,t)=A\sin\left(\frac{x}{a}+\frac{t}{b}\right)$ (4 pts.)

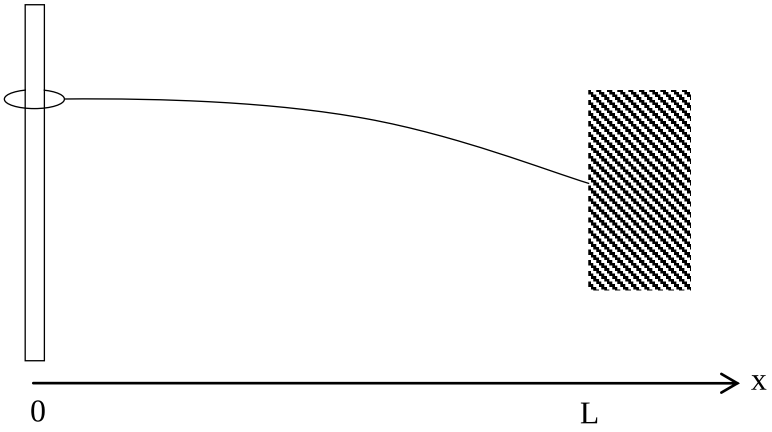
d. $y(x,t)=A\sin(ax^2+bt^2)$ (4 pts.)

13. It is observed that a pulse requires 0.1 second to travel from one end to the other of a long string. The tension in the string is provided by passing the string over a pulley to a weight which has 100 times the mass of the spring.

(a) What is the length of the string? (10 pts.)

(b) What is the equation of the third normal mode? (10 pts.)

14. Consider a string extending form x = 0 to x = L as shown below. At x = L the string is fixed to a rigid wall, and at x = 0 the string is attached to a ring of negligible mass which is free to slide on a friction-free rod. (friction free, so no transverse force).



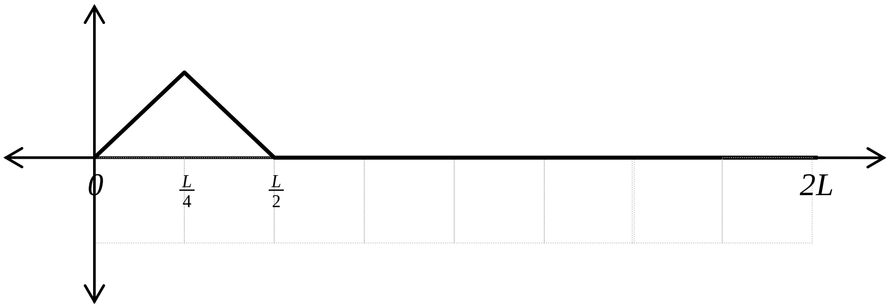
a. Using the appropriate boundary conditions, give the solution $y_n(x,t)$ for the n^{th} normal mode, and give explicitly the frequency ωn. (20 pts.)

b. Sketch the patterns for the three lowest allowed modes. (10 pts.)

Hints: Think carefully about the x=0 boundary condition. The size of the rod and massless ring are negligible. Perhaps a free body diagram would be in order!

15. (20 pts) A stretched string of mass m, length L, and tension T is driven by two sources, one at each end. The sources both have the same frequency ν and amplitude A, but are exactly 180 degrees out of phase with respect to one another. (Each end is an antinode). What is the smallest normal mode frequency of the string? This is French problem 6-5. (20 pts.)

16. Find the amplitudes A_o, A_n, and B_n needed to describe the triangular pulse we discussed in class:



with the following Fourier series:

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Assume the slopes of the sides of the pulse are +c and –c. (25 pts.)

17. Based on what you know about vibrations of stretched strings, explain why the sound a guitar string makes depends on where you pluck it. (*Hint: assume the string is clamped at each end and has the same length no matter where it is plucked. You do not need to _solve_ any equations to answer this question, but your detailed answer may include one or two*). (15 pts.)

18. The motion of ripples of short wavelength (< 1 cm) on water is controlled by surface tension. The phase velocity of such ripples is given by:

$$v_{ph} = \sqrt{\frac{2\pi S}{\rho\lambda}}$$

Where S is the surface tension and ρ the density of water.

- **a.** Show that the group velocity for a disturbance made up of wavelengths close to a given λ is equal to $3v_{ph}/2$. (10 pts.)
- **b.** What does this imply about the observed motion of a group of ripples traveling over a water surface? (5 pts.)
- **c.** If the group consists of just two waves, of wavelength 0.99 cm and 1.01 cm, what is the distance between crests of the group? (10 pts.)

19. When we derived the wave equation for a stretched string, we only considered the tension force. A curved segment of the string will also want to straighten due to the stiffness of the material. This "bending force" depends on the 4th spatial derivative:

$$F_{bending} = -T\Delta x\alpha \frac{\partial^4 y}{\partial x^4}$$

where T and Δx are the same as in our prior derivation of the wave equation. Derive a new wave equation using this force (in addition to the tension force from before) and show that the solutions have the following dispersion:

$$\omega = v\sqrt{k^2 + \alpha k^4}$$

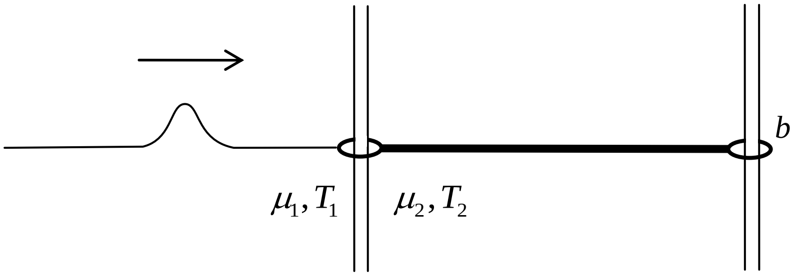
20. A harmonic wave traveling along a string (1) is incident at an interface with a lighter string (2), with the density of the second string, $\mu_2 = \mu_1/4$. For the part of the wave that is transmitted into the second string, indicate if the following increase, decrease, or remain the same. If the quantities change, indicate by what factor they change.

a. The velocity of propagation (5 pts.)
b. The wavenumber *k* (5 pts.)
c. The wavelength λ (5 pts.)
d. The frequency ω (5 pts.)

For the part of the wave that is reflected at the interface back to the first string, indicate if the following increase, decrease, or remain the same. If the quantities change, indicate by what factor they change.

e. The velocity of propagation (5 pts.)
f. The wavenumber *k* (5 pts.)
g. The wavelength λ (5 pts.)
h. The frequency ω (5 pts.)

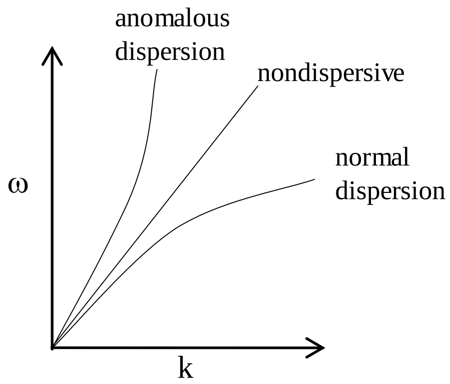
21. Two stretched strings are connected by a frictionless loop so that they may have independent mass densities *and* tensions. The right string terminates at a frictional loop with damping constant *b*. Given defined values for μ_1, T_1, find *two* distinct conditions that would return a single inverted pulse with ½ the incident pulse amplitude. Give the value of *b* for each case in terms of μ_1 and T_1. (20 pts.)



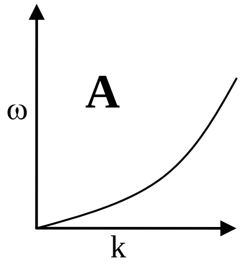
22. You are standing 100 yards from someone who is hitting two objects together to create a very sharp impulse waveform in the air pressure that moves to your ear.

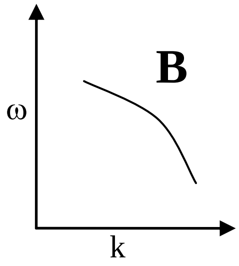
Describe the sound you hear if the dispersion of sound waves in air

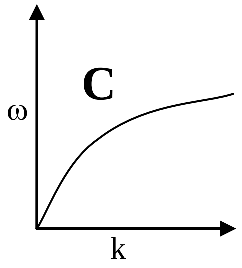
a. is non-dispersive (5 pts)
b. exhibits normal dispersion (5 pts)
c. exhibits anomalous dispersion (5 pts)

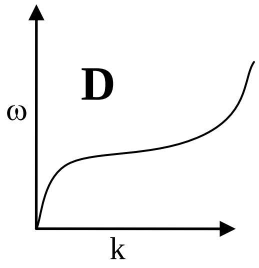


23. Consider the following dispersion relations:









a) Identify which will support a pulse moving in a direction opposite to its Fourier components?

b) On curve D, identify a point in ω-k space where the group and phase velocities are equal.

c) Identify which curve best corresponds to the following dispersive wave equation (*k* is the wave number). Assume the constant *A* provides the proper units.

$$\frac{\partial^2 f(x,t)}{\partial x^2} = Ak\frac{\partial^2 f(x,t)}{\partial t^2}$$

24. In class we derived the wave equation for sound in air and found that the velocity is:

$$v = \left\{ \frac{d\rho}{dP}(P_o) \right\}^{-1/2}$$

and using the ideal gas law $PV=NRT$ we found a simple expression for the velocity (assuming constant temperature).

a) Show that the expression yields v = 290 m/s (approximately) when you plug in typical ambient values for gas density and pressure. (10 pts)

b) Actually, that number is not correct! The speed of sound in air is about 340 m/s. If you allow for the fact that the temperature changes when the pressure changes, but the effect is too fast for heat to flow, the flow is "adiabatic". For adiabatic flow the ideal gas law leads to:

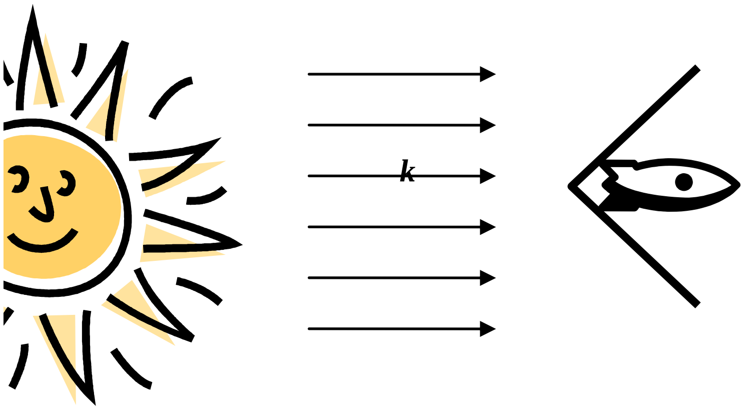
$$\frac{P}{\rho^{\gamma}} = \text{constant}.$$

In air $\gamma = 1.4$ (approximately). Show that this leads to correct velocity of sound in air. (10 pts)

25. Show that the second field derivatives $\nabla \times \nabla s$ and $\nabla \cdot \nabla \times \vec{\mathbf{V}}$ are equal to zero. Use a general scalar field s and a general vector field $\vec{\mathbf{V}} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$. (5 pts each)

26. What is the *minimum* number of numerical parameters needed to *completely* characterize an electromagnetic plane wave in all space and at all times in a vacuum with no free charges or currents? List your parameters. (10 pts.)

27. A spacecraft wishes to accelerate using radiation pressure from the sun. The ship is approximately at the same distance as Earth orbit, where the incident irradiance is 1.5 kW/m². The ship deploys a large solar sail with a 90 degree fold as shown in the figure below. The sail is made of perfectly reflecting material.



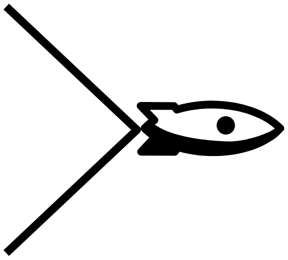
a. If the cross-sectional area of the sail (seen by the sun) is 10^4 m^2, what is the force exerted on the sail? (6 pts.)

b. If the sail is deployed for an hour, how much momentum has the spacecraft gained? (6 pts.)

c. If the mass of the spacecraft is 600 kg, what is its velocity after an hour? (6 pts.)

d. If the sail described above were totally absorbing, how would the force change? (6 pts.)

e. If the orientation of the sail were reversed, as shown below, how does the net force acting on the sail change (assume it is still totally reflecting)? (6 pts.)



28. A 60 kg astronaut is stranded in deep space, 20 m from her spacecraft and at rest with respect to the spacecraft. The only means of propulsion she has is a 1.5 kW laser which weighs 10 kg and is self-powered by a battery. What is the shortest time in which she can reach her spacecraft using the laser for propulsion? If she has 5 hours of air with her, does she reach her ship before she runs out of air? (20 pts.)

29. Specify the polarization state of each of the following Jones vectors, the angle relative to the horizontal axis (linear and elliptical only), and the direction (left or right) for circular or elliptical polarization. (35 pts.)

$$\text{(a)}\begin{bmatrix}3j\\ j\end{bmatrix}\qquad \text{(b)}\begin{bmatrix}j\\ 1\end{bmatrix}\qquad \text{(c)}\begin{bmatrix}4j\\ 5\end{bmatrix}\qquad \text{(d)}\begin{bmatrix}3\\ 0\end{bmatrix}$$

$$\text{(e)}\begin{bmatrix}2\\ 2j\end{bmatrix}\qquad \text{(f)}\begin{bmatrix}2\\ 3\end{bmatrix}\qquad \text{(g)}\begin{bmatrix}2\\ 6+8j\end{bmatrix}$$

30. Write the normalized Jones vectors for each of the following waves, and the angle from the x-axis in the case of linear or elliptical polarization. (24 pts.)

a. $\vec{\mathbf{E}} = E_o\cos(kz-\omega t)\hat{i} - E_o\cos(kz-\omega t)\hat{j}$

b. $\vec{\mathbf{E}} = E_o\sin 2\pi\left(\dfrac{z}{\lambda}-vt\right)\hat{i} - E_o\sin 2\pi\left(\dfrac{z}{\lambda}-vt\right)\hat{j}$

c. $\vec{\mathbf{E}} = E_o\sin(kz-\omega t)\hat{i} + E_o\sin(kz-\omega t-\tfrac{\pi}{4})\hat{j}$

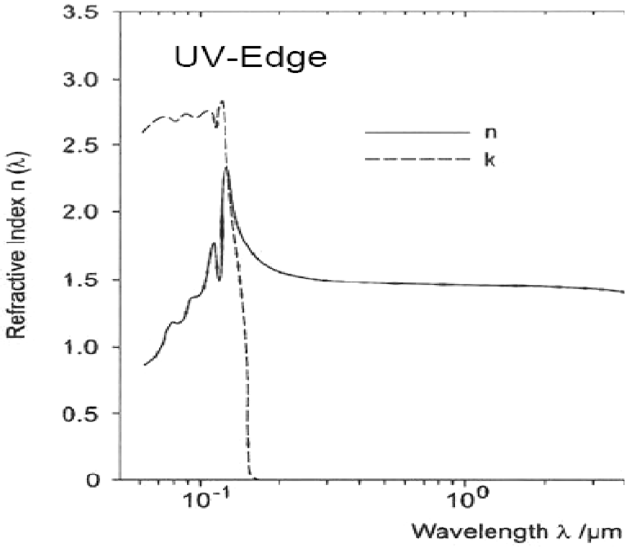
d. $\vec{\mathbf{E}} = E_o\cos(kz-\omega t)\hat{i} + E_o\cos(kz-\omega t+\tfrac{\pi}{2})\hat{j}$

31. An EM plane wave enters a dielectric medium. Given the properties of EM plane waves and the relationship between the applied EM field and the polarization of a material that we discussed in class, show that $\nabla \cdot \vec{\mathbf{P}} = 0$. (20 pts.)

32. In class we derived an expression for the effect of a dielectric on the dispersion curve for light:

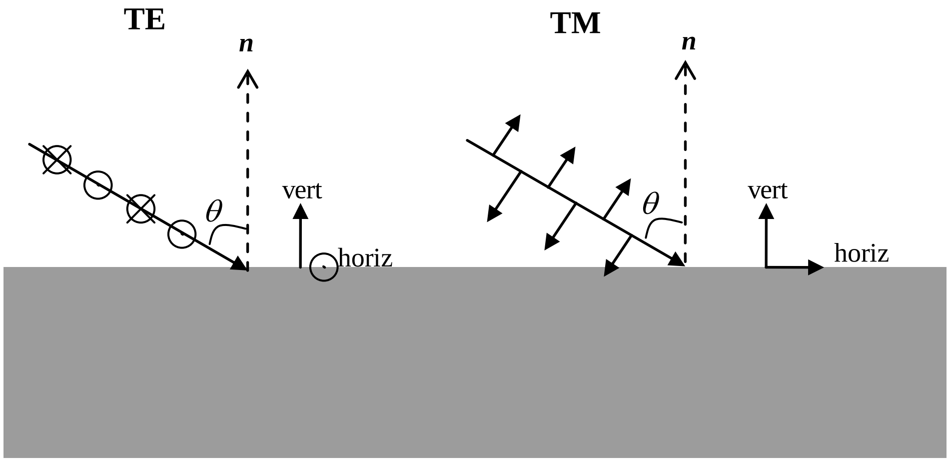
$$\omega = \frac{c}{\sqrt{1 + \dfrac{Ne^2}{m\varepsilon_o\left(\omega_o{}^2 - \omega^2 - j\omega\gamma\right)}}}k$$

where the part in the denominator is the complex refractive index. A plot of the spectral dependence of the real and imaginary part of the refractive index shows that this resonance occurs in the UV (*the x-axis is the vacuum wavelength in micrometers*) (20 pts.)



Based on (i) the wavelength of the peak, (ii) the mass of the electron, and (iii) a rough estimate for the density of electrons in glass, estimate the stiffness of the "electromagnetic spring" which is pulling the electrons back towards the nucleus.

33. Describe TE and TM polarized light of amplitude E_o incident on a plane with Jones vector notation using the coordinate systems given. (10 pts.)



34. *r* and *t* are the reflection coefficients for the electric field of an EM plane wave at a dielectric interface. *R* and *T* are the reflectance and transmissivity for the irradiance of the EM plane wave. To conserve energy, $R + T = 1$, and we have said that $R = r^2$. In a few sentences and equations, explain why it is not simply true that $T = t^2$. (10 pts.)

35. For sufficiently large angles of incidence, we get "total internal reflection", which must mean that $R = 1$. Starting with the Fresnel equation for TM mode and writing it this way:

$$r = \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} = \frac{-a + \mathrm{j}b}{a + \mathrm{j}b}$$

show that total internal reflection occurs. (25 pts.)
Hint: don't forget complex conjugates..

36. (20 pts) A laser beam from a 1 mW He-Ne laser (632.8 nm) is directed onto a parallel film with an incident angle of 45 degrees. Assume a beam diameter of 1 mm and a film index of 1.414. Determine:
a. the amplitude of the E-vector of the incident beam. (10 pts.)
b. the angle of refraction of the laser beam into the film. (10 pts.)

37. (15 pts) A beam of light (treat it as a plane wave) enters a dielectric slab of thickness *t* at an angle of incidence θ from the normal, At what angle does it emerge from the slab on the other side?

38. (40 pts) Review the section on phase changes on reflection in Pedrotti section 23-3

(a) For the Fresnel rhomb shown, use the equations given to confirm that θ = 53degrees will produce circular light if illuminated with light polarized 45 degrees to the plane of incidence.

(b) Why is this device useful over a wider wavelength range than a wave plate that relies on birefringence?

(c) How could you produce elliptically polarized light with this Fresnel rhomb?

39. (25 pts) Your research advisor has an excess of funding so you are asked to make a polarizer out of a parallel slab of diamond (n = 2.44, assume no dichroism or birefringence).

a) If your incident beam consists of unpolarized light of irradiance I_0, what irradiance of linearly polarized light can you make considering only a *single* external reflection (ignore the reflection off of the back, which you will consider next)? Draw a schematic of your polarizer and indicate the relevant angles.

b) If you now consider the first internal reflection from the back of the diamond slab, what will be the total reflected irradiance (do not consider interference)?

40. (20 pts) Using Jones calculus, show that the effect of a Half Wave Plate on light linearly polarized a inclination angle α is to rotate the plane of polarization through an angle of 2α. (The HWP may be used in this way as a "laser line rotator", allowing the plane of polarization of a laser beam to be rotated without having to rotate the laser.

41. (10 pts) How thick should a half-wave plate of mica be in an application where laser light of 633 nm is used? Appropriate refractive indices for mica are 1.599 and 1.594.

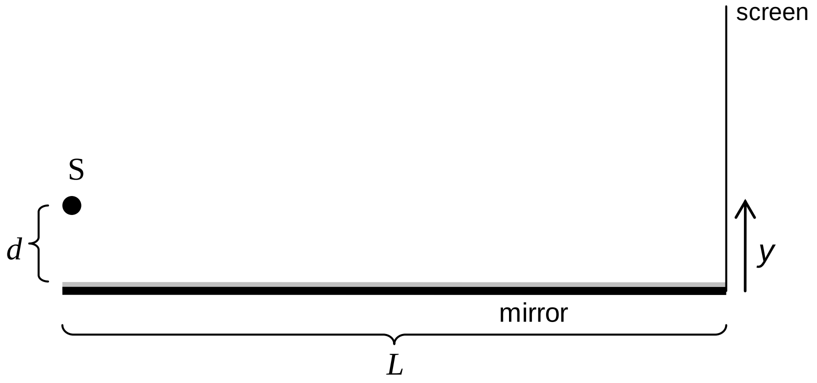
42. Use Fermat's Principle to prove Snell's law. (20 pts.)

43. Show that in Young's double slit experiment, if the top slit is covered with a dielectric film of thickness *t* and index of refraction *n*, the entire interference pattern is shifted upward by:

$$\theta_{shift} \approx \frac{(n-1)t}{a}$$

where *a* is the separation between the slits. Assume small angles, assume the film is thin compared to the slit to screen distance, and don't consider reflections in the dielectric film. (20 pts.)

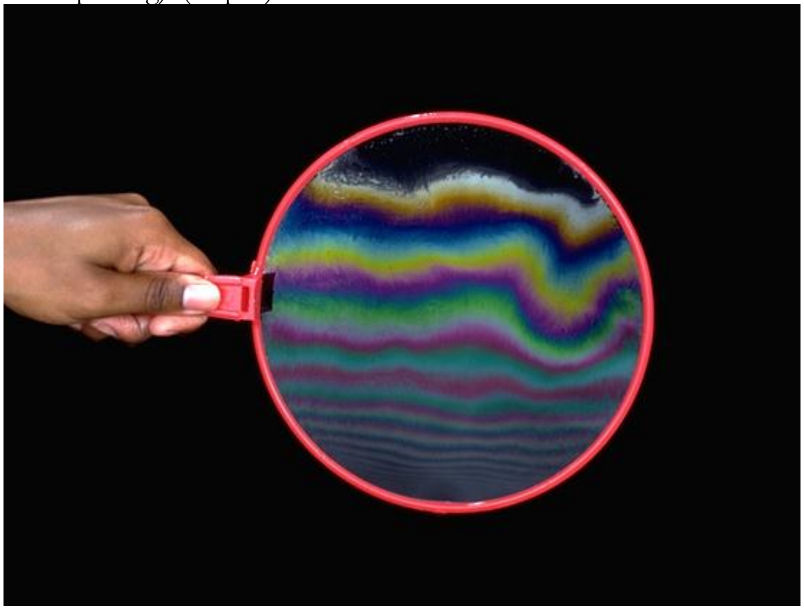
44. A point source of light S is emitting a single wavelength □$_o$ and is situated a small distance *d* above a plane mirror. A screen stands normal to the mirror at a distance L from S with $L >> d$. Find the intensity of light on the screen as a function of height y above the mirror. (25 pts.)



45. Design a thin dielectric film to minimize reflection for white light at normal incidence on glass (n = 1.515) in water (n = 1.33). Determine values for the optimum film thickness and index of refraction. Bonus +3:recommend a dielectric for this purpose. (15 pts.)

46. This is an image of a soap film taken by reflected white light. Explain:

- **a.** why it is colorful. (10 pts.)
- **b.** why some regions are black (there *is* a stable soap film there, it is *not* rupturing). (10 pts.)



47. A thin film of MgF_2 (n = 1.38) is deposited onto glass so that its antireflection wavelength is 580 nm under normal incidence. What wavelength is minimally reflected when the light is incident instead at 45 degrees? (15 pts.)

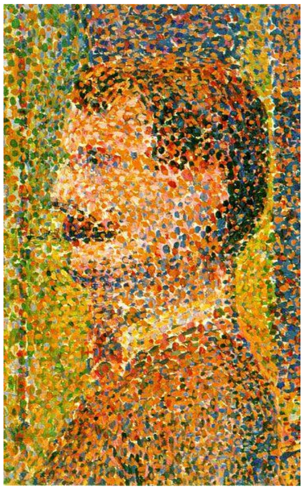
48. In viewing the far-field diffraction pattern of a single slit illuminated by a two-color source, one finds that the fifth minimum of one wavelength component coincides exactly with the fourth minimum of the pattern due to a wavelength of 620 nm. What is the other wavelength? (20 pts.)

49. Calculate the rectangular slit widths that will produce central maxima in their far field diffraction patterns having an angular widths of 30 degrees, 45 degrees, 90 degrees, and 180 degrees. (20 pts.)

50. Compare the peak irradiance of the first two secondary maxima of a circular diffraction pattern to those of a single slit diffraction pattern. (10 pts.)

51. Make a rough sketch for the irradiance pattern from seven equally spaced slits having a separation-to-width ratio of 4. Label points on the x-axis with corresponding values of α and β. (30 pts.)

52. The impressionist painter Georges Seurat used a technique call "pointillism" in which his paintings are composed of small red, green, and blue dots about 2 mm apart so that they are blended by the eye to make other colors. For this effect to occur due to diffraction, calculate the minimum viewing distance. Use the wavelength of visible light that requires the greatest distance so that your answer will work for all visible wavelengths. Assume the pupil of the eye has a diameter of 5 mm. Do you think the effect is due to diffraction? (30 pts.)



53. (30 pts) You are looking at the oranges and blues of a sunset through a polarizer. By rotating the polarizer you can see if the light that reaches your eye is polarized. Assume that there is no volcanic ash, dust, smog, or clouds in the air. Will the orange light be polarized? Why/Why not? Will the blue light be polarized? Why/Why not?

54. (10 pts) The ozone layer high in the earth's atmosphere protects us from UV light from the sun. Based on what you have learned in PHYS 201, is this be due to absorption or scattering? Explain.