# Electromagnetic Waves

Maxwell's Equations in Free Space with no free charges or currents:

Gauss $$\nabla \cdot \vec{\mathbf{E}} = 0$$

No Monopoles $$\nabla \cdot \vec{\mathbf{B}} = 0$$

Faraday $$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Ampere-Maxwell $$\nabla \times \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{\mathbf{E}}$$

Take the time derivative of Faraday:

$$\nabla \times \frac{\partial \vec{\mathbf{E}}}{\partial t} = -\frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

Substitute in Ampere-Maxwell

$$\left(\frac{1}{\mu_0 \varepsilon_0}\right) \nabla \times \nabla \times \vec{\mathbf{B}} = -\frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

| | 1st derivatives | 2nd derivatives | |
|---|---|---|---|
| Function: | $\frac{d}{dx}f(x)$ | $\frac{d^2}{dx^2}f(x)$ | |
| Field: | $\nabla s(x,y,z)$ | $\nabla \cdot \nabla s$ | $\nabla \times \nabla s$ |
| | $\nabla \cdot \vec{\mathbf{V}}(x,y,z)$ | $\nabla(\nabla \cdot \vec{\mathbf{V}})$ | |
| | $\nabla \times \vec{\mathbf{V}}(x,y,z)$ | $\nabla \cdot \nabla \times \vec{\mathbf{V}}$ | $\nabla \times \nabla \times \vec{\mathbf{V}}$ |

$$\nabla \times \nabla \times \vec{\mathbf{V}} = \nabla(\nabla \cdot \vec{\mathbf{V}}) - \nabla^2 \vec{\mathbf{V}}$$

$$\nabla \cdot \nabla s$$

$$\nabla \cdot \left( \frac{\partial s}{\partial x}\hat{i} + \frac{\partial s}{\partial y}\hat{j} + \frac{\partial s}{\partial z}\hat{k} \right)$$

$$\left( \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left( \frac{\partial s}{\partial x}\hat{i} + \frac{\partial s}{\partial y}\hat{j} + \frac{\partial s}{\partial z}\hat{k} \right)$$

$$\frac{\partial^2 s}{\partial x} + \frac{\partial^2 s}{\partial y} + \frac{\partial^2 s}{\partial z}$$

$$\nabla^2 s$$

$$\nabla\left(\nabla \cdot \vec{\mathbf{V}}\right)$$

$$\nabla\left(\nabla \cdot \vec{\mathbf{V}}\right) = \nabla\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right)\hat{i} + \left(\frac{\partial^2 V_x}{\partial y \partial x} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right)\hat{j} + \left(\frac{\partial^2 V_x}{\partial z \partial x} + \frac{\partial^2 V_y}{\partial z \partial y} + \frac{\partial^2 V_z}{\partial z^2}\right)\hat{k}$$

$$\nabla \times \nabla \times \vec{\mathbf{V}}$$

$$\nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\nabla \times \left[ \left( \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{i} - \left( \frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) \hat{j} + \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k} \right]$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} & -\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{vmatrix}$$

$$\left(\frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_z}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z^2}\right)\hat{i}$$

$$\left(-\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial z \partial y} - \frac{\partial^2 V_y}{\partial z^2}\right)\hat{j}$$

$$+ \quad \left(-\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial z} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z}\right)\hat{k}$$

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$$\left(\frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_z}{\partial z \partial x} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial x^2} - \frac{\partial^2 V_x}{\partial x^2}\right)\hat{i}$$

$$\left(-\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial z \partial y} - \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial^2 V_y}{\partial y^2} - \frac{\partial^2 V_y}{\partial y^2}\right)\hat{j}$$

$$+ \quad \left(-\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_x}{\partial x \partial z} - \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2} - \frac{\partial^2 V_z}{\partial z^2}\right)\hat{k}$$

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$$\nabla\left(\nabla \cdot \vec{\mathbf{V}}\right) - \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_x}{\partial x^2}\right)\hat{i} - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2} + \frac{\partial^2 V_y}{\partial y^2}\right)\hat{j} - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right)\hat{k}$$

$$\nabla\left(\nabla \cdot \vec{\mathbf{V}}\right) - \underbrace{\nabla^2 \vec{\mathbf{V}}}$$

"Vector Laplacian"

Substitute in Ampere-Maxwell

$$\left(\frac{1}{\mu_0 \varepsilon_0}\right)\nabla \times \nabla \times \vec{\mathbf{B}} = -\frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

$$\nabla\left(\nabla \cdot \vec{\mathbf{B}}\right) - \nabla^2 \vec{\mathbf{B}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

0

$$\boxed{\nabla^2 \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}}$$

3D Wave equation for **B**

Similar derivation for the **E** field:

$$\nabla^2 \vec{\mathbf{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

3D Wave equation for **E**

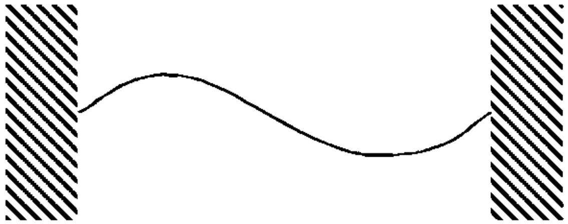
$\varepsilon_0$ = permittivity of free space = $8.85 \times 10^{-12}$ C/Nm$^2$

$\mu_0$ = permeability of free space = $4\pi \times 10^{-7}$ N/A$^2$

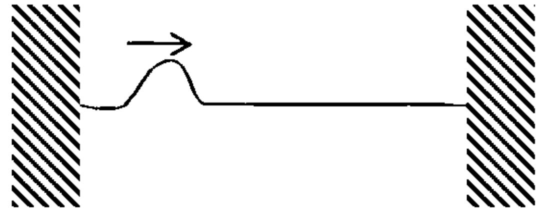
$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3.0 \times 10^8 \text{ m/s}$$

Stretched String:

Standing Waves



Traveling Waves



Simple Guess: traveling wave with E field along $x$, moving along $z$:

$$\vec{\mathbf{E}} = E_{0x}\cos(kz - \omega t)\hat{x} + 0\hat{y} + 0\hat{z}$$

Wave Equation now:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

What about B?

Faraday's Law: $$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\frac{\partial E_x}{\partial z}\hat{y} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$-kE_{0x}\sin(kz - \omega t)\hat{y} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\frac{kE_{0x}}{\omega}\cos(kz - \omega t)\hat{y} = \vec{\mathbf{B}}$$

Note:

$$B_{0y} = \frac{E_{0x}}{c}$$

$E_x, B_y$ are in phase

$$\vec{\mathbf{E}} \perp \vec{\mathbf{B}}$$