The Fundamental Theorem of Calculus

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

(Integral of a derivative over a region is related to values at the boundary)

Dot Product: multiply components and add

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

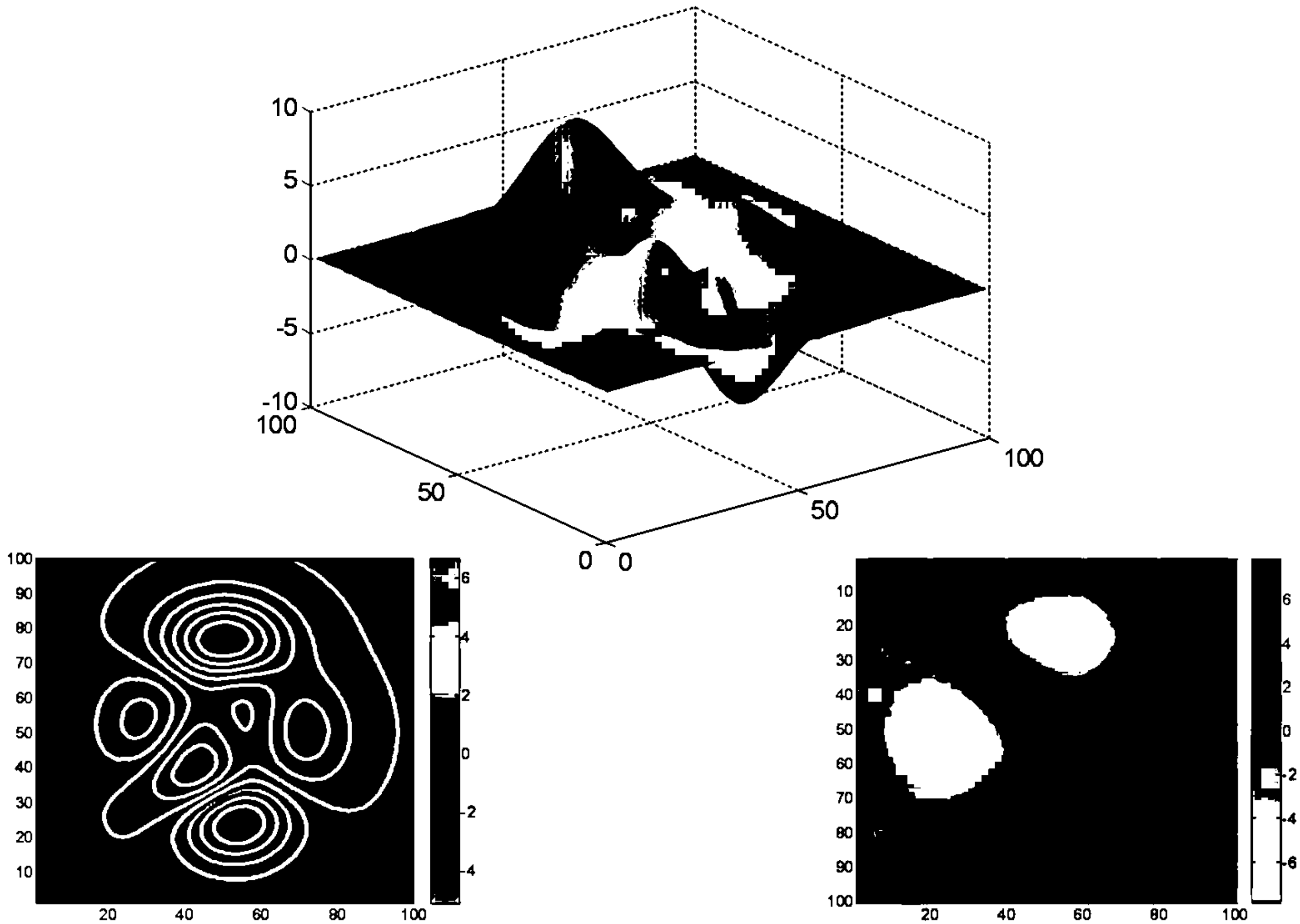
Cross Product: determinant of matrix with unit vector

$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

EM Fields

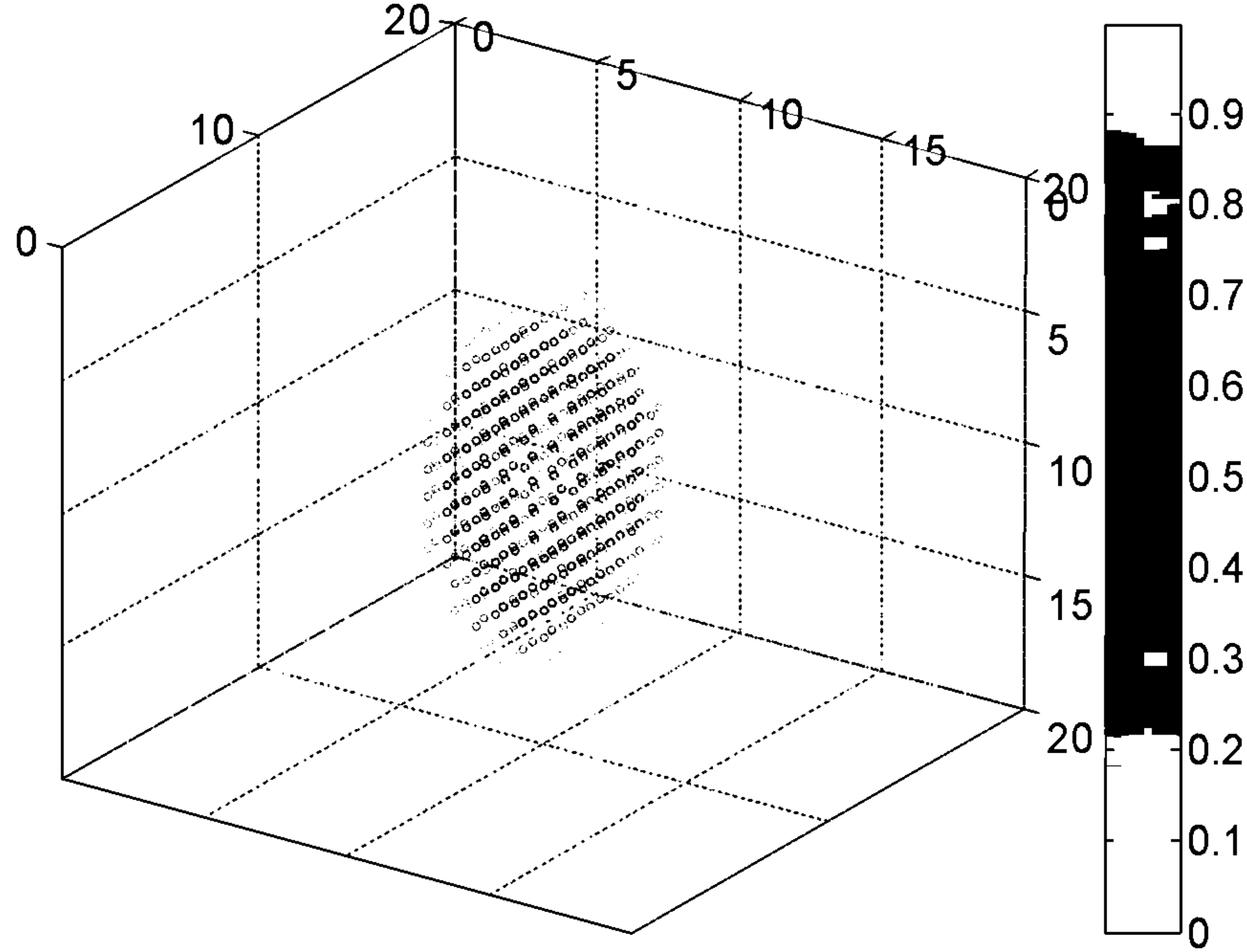
Scalar Field : a scalar quantity defined at every point of a 2D or 3D space.

$S(x,y) = f(x,y)$ Ex: $S(x,y) = x\sin(xy)$



3D scalar field $S(x,y,z) = 1 - e^{-(x-10)^2/3} e^{-(y-10)^2/2} e^{-(z-10)^2/5}$

3D scatter plot with color giving the field value:



Vector Field: a vector quantity defined at every point of a 2D or 3D space.

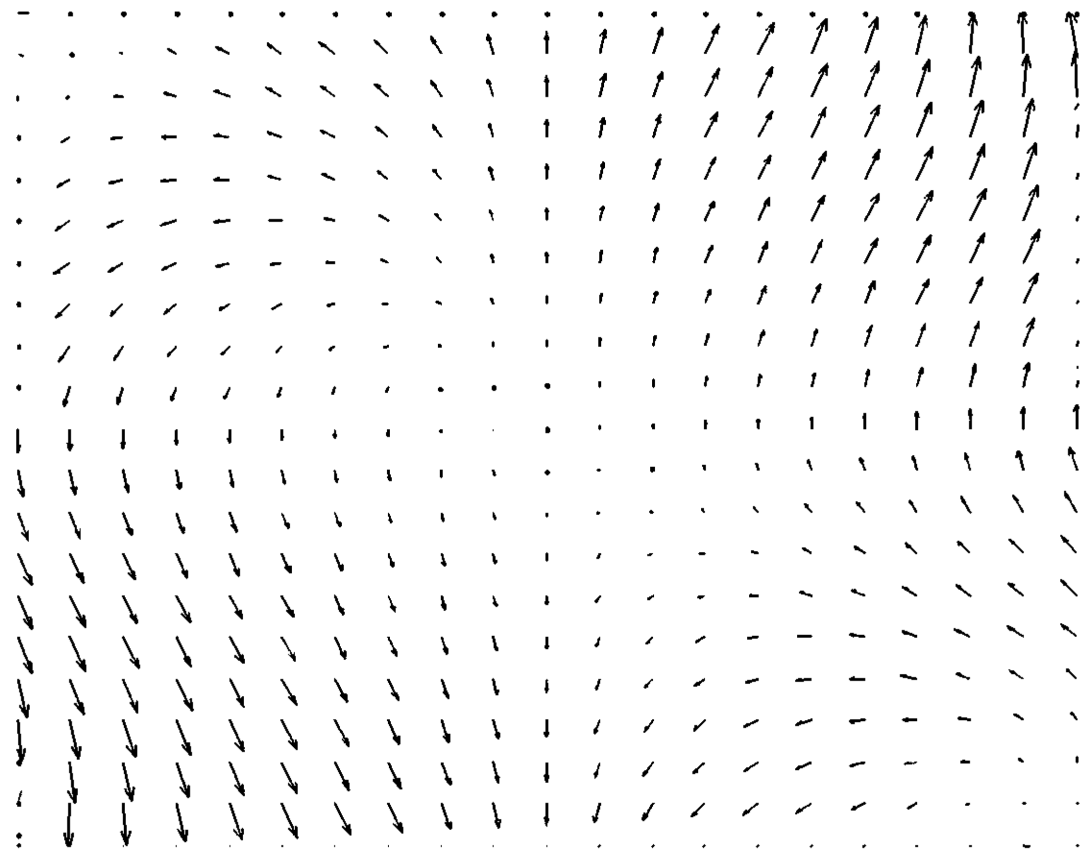
$$\vec{\mathbf{V}}(x, y, z) = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

Functions of (x,y,z)

NOT constants
NOT partial derivatives

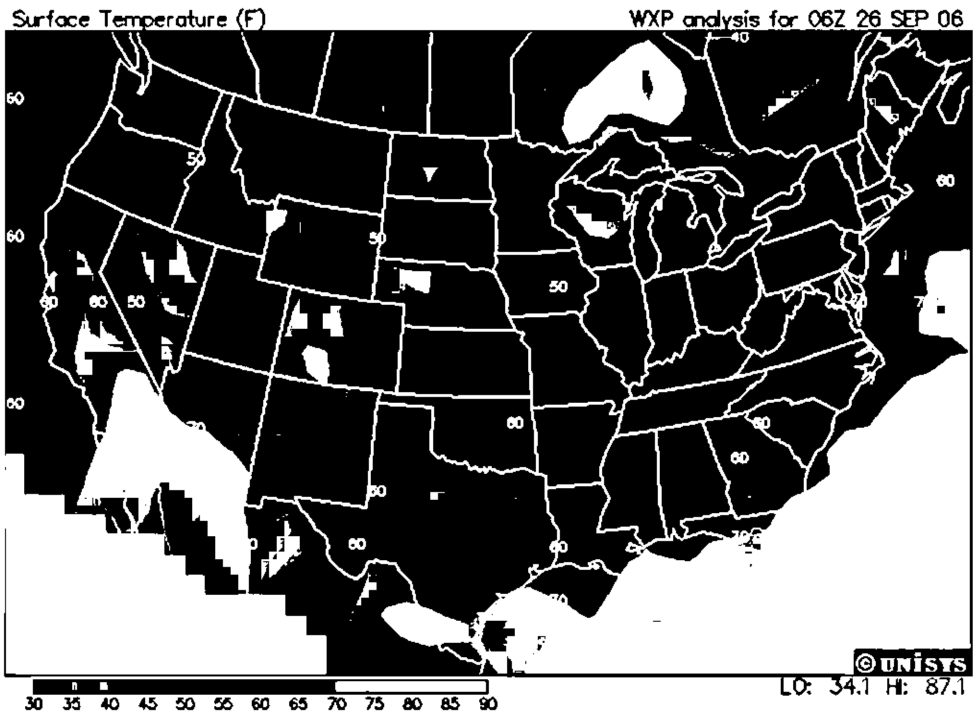
2D Ex:

$$\vec{\mathbf{S}}(x, y) = \sin(xy)\hat{i} + (x + y)\hat{j}$$

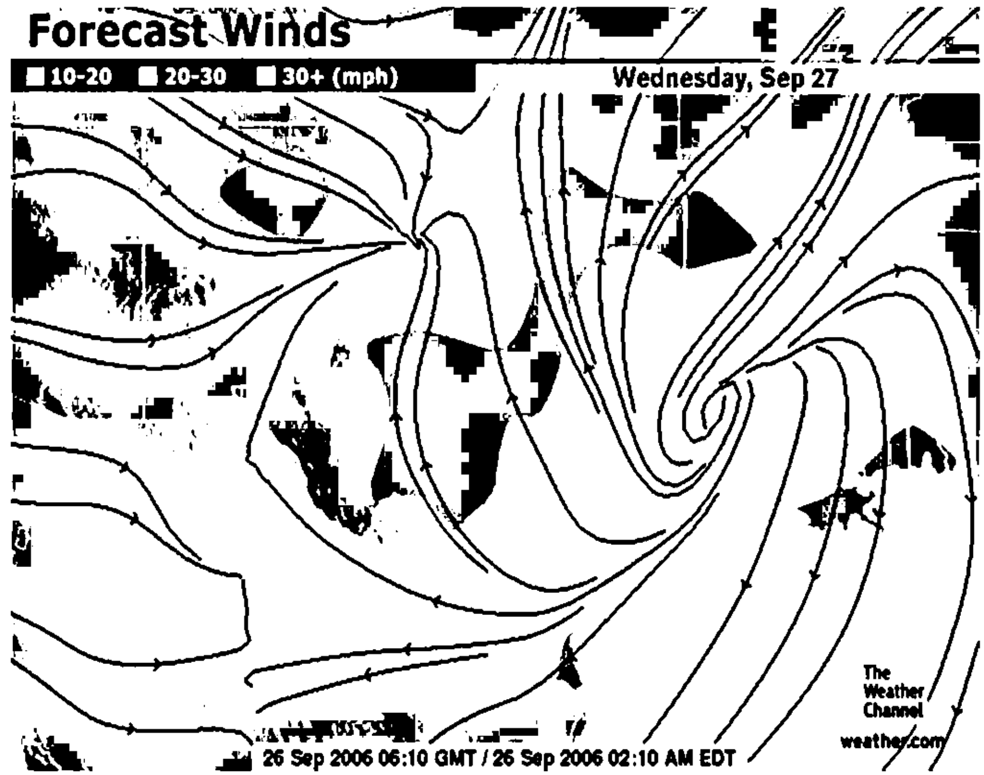


Two Fields

Temperature Map:
a scalar field

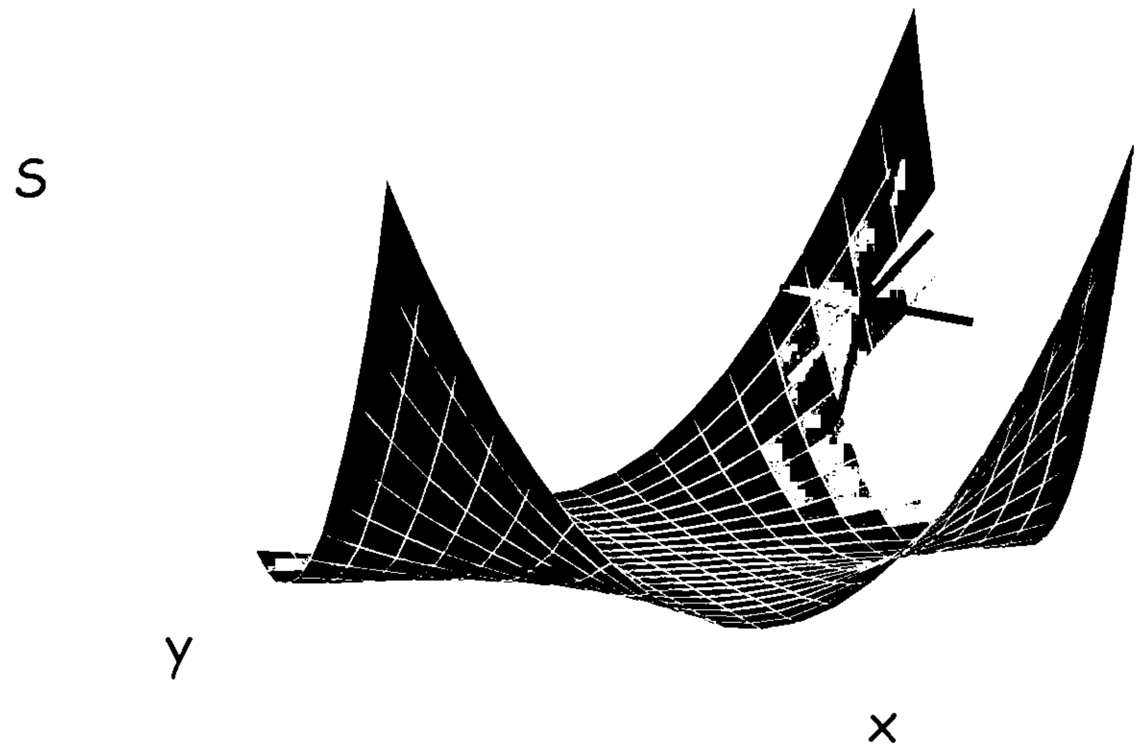


Wind Map:
a vector field



1. Gradient

$$S = x^2 y^2$$



"the derivative of a scalar field"

Derivative (slope) depends on direction!

Total Differential: $$dS = \left(\frac{\partial S}{\partial x}\right)dx + \left(\frac{\partial S}{\partial y}\right)dy$$

Looks like a dot product: $$dS = \left(\frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right)$$

$$dS = (\nabla S)\cdot\left(d\vec{l}\right)$$

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

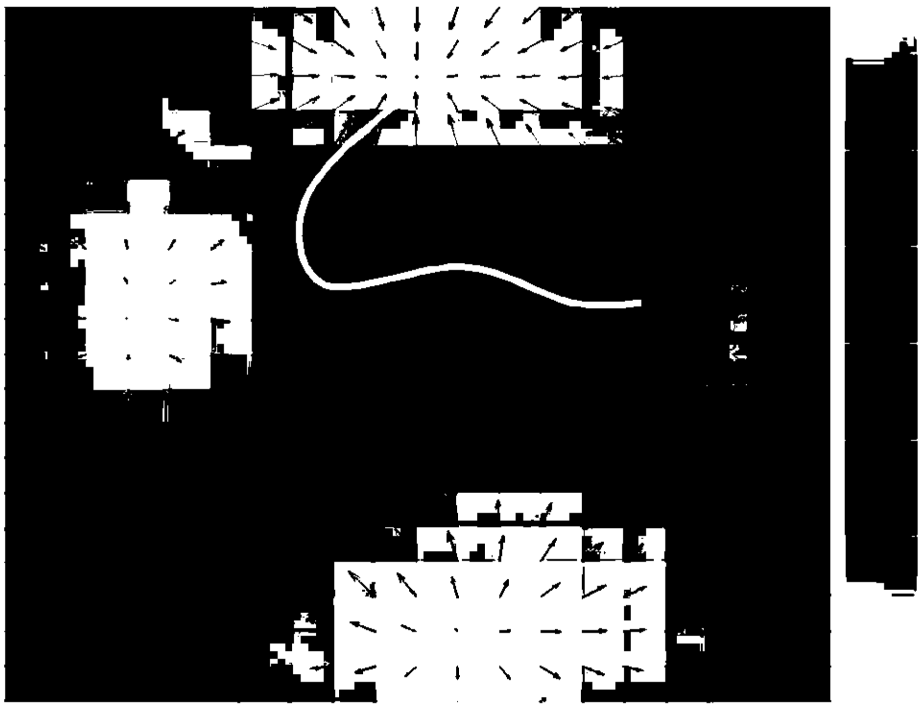
"del"

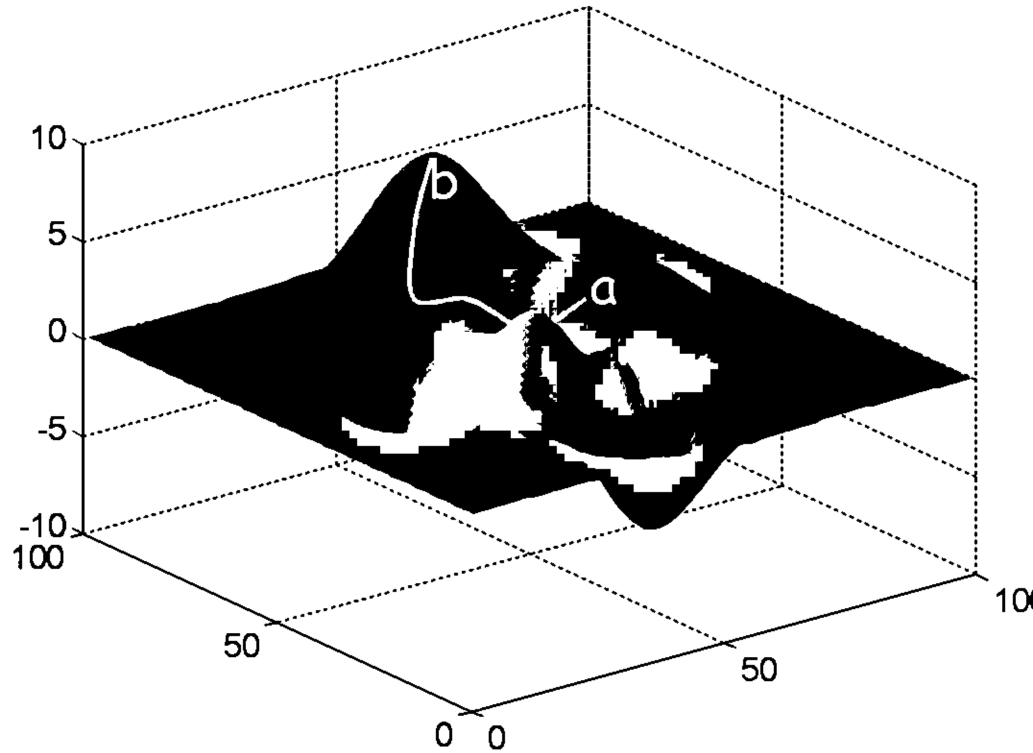
"nabla"

Del is not a vector and it does not multiply a field – it is an operator!

1. The Fundamental Theorem of Gradients

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla S)\cdot d\vec{l} = S(b) - S(a)$$





(Integral of a derivative over a region is related to values at the boundary)

2. Divergence

$$\nabla \cdot \vec{\mathbf{V}}$$

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i}+V_y\hat{j}+V_z\hat{k}\right)$$

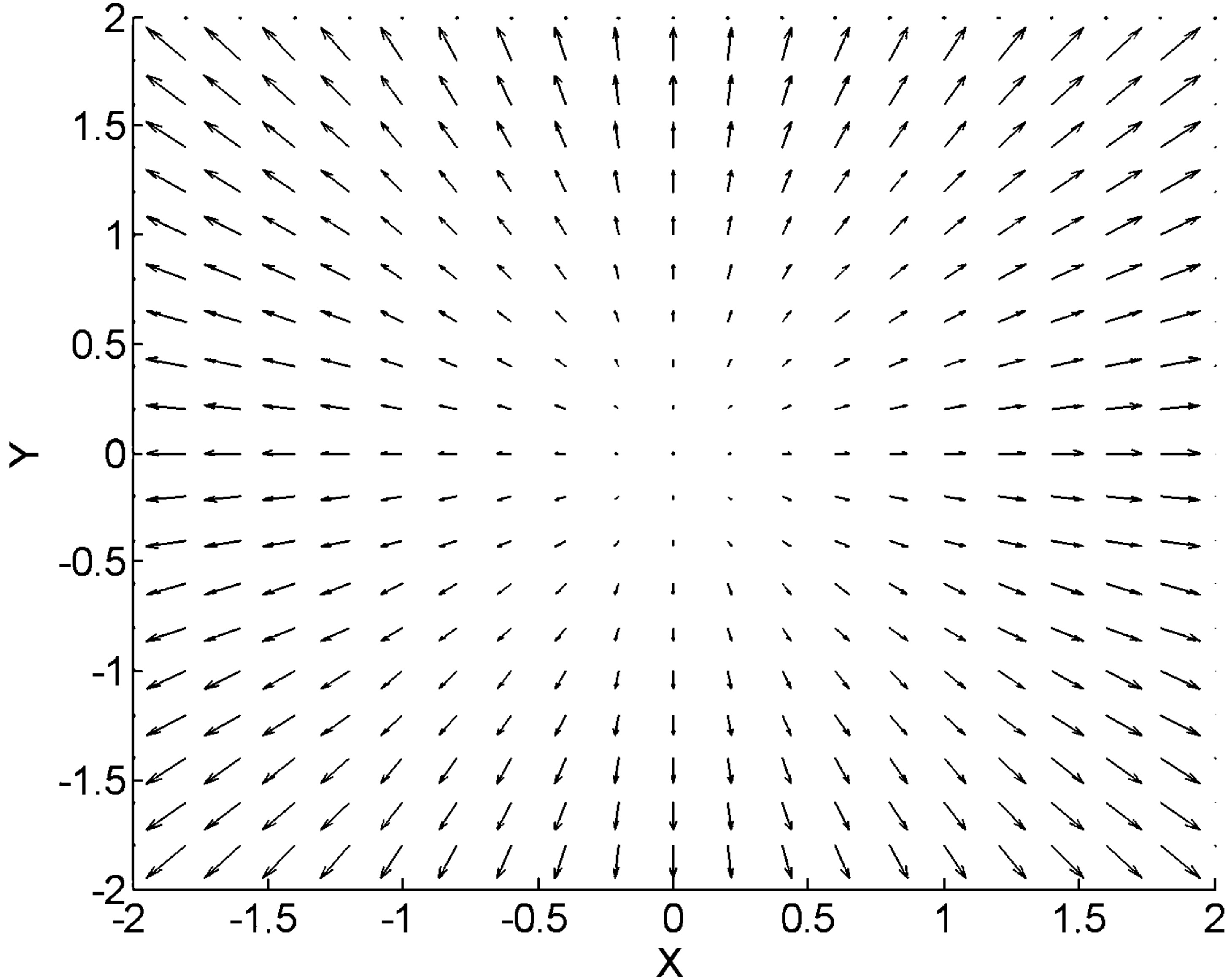
$$\frac{\partial}{\partial x}V_x+\frac{\partial}{\partial y}V_y+\frac{\partial}{\partial z}V_z \quad \text{(a scalar field!)}$$

$$\vec{\mathbf{V}}(x,y)=\cos(xy)\hat{i}+\left(x^2+y^2\right)\hat{j}$$

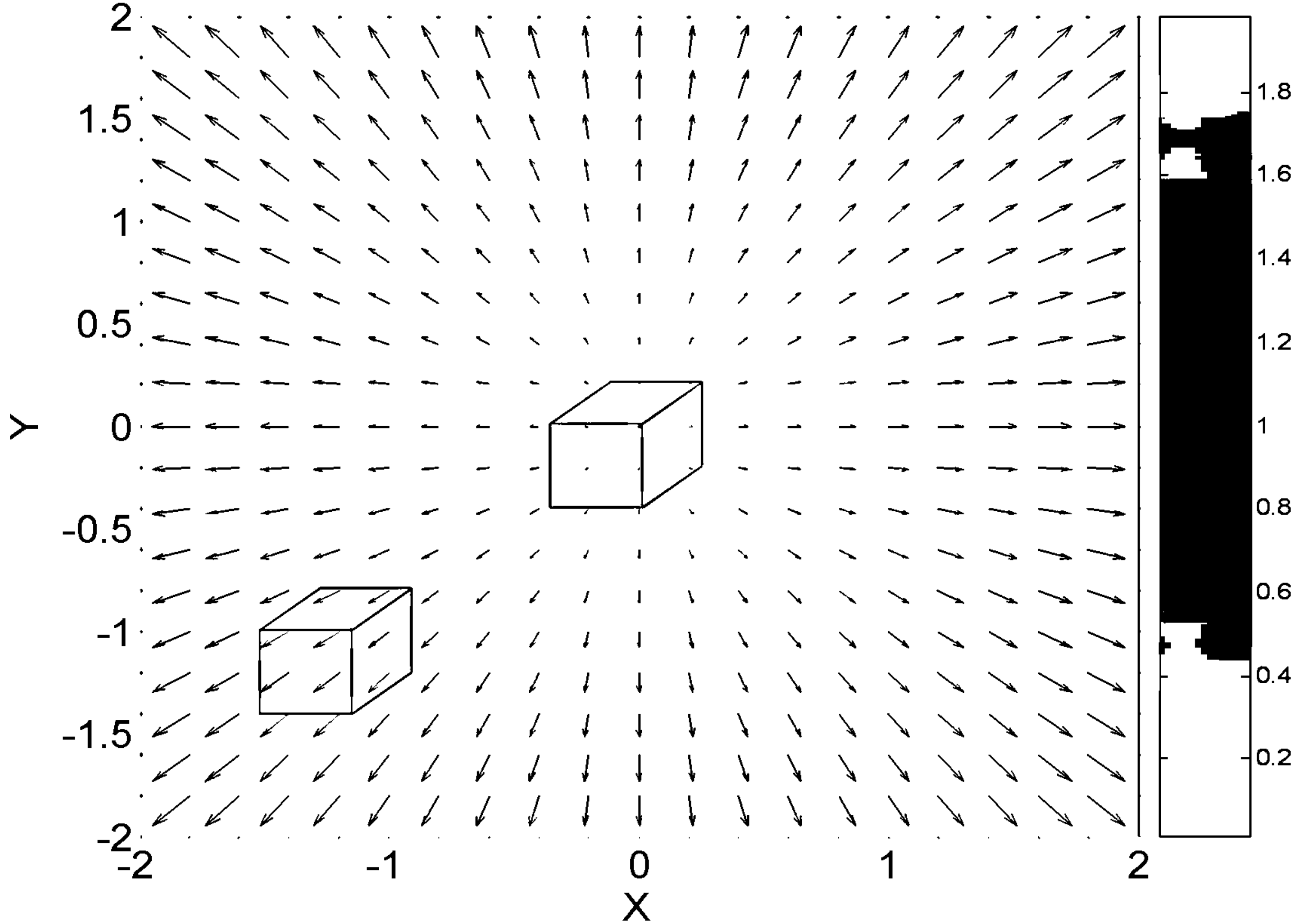
$$\nabla \cdot \vec{\mathbf{V}}=-y\sin(xy)+2y$$

"the *creation or destruction* of a vector field"

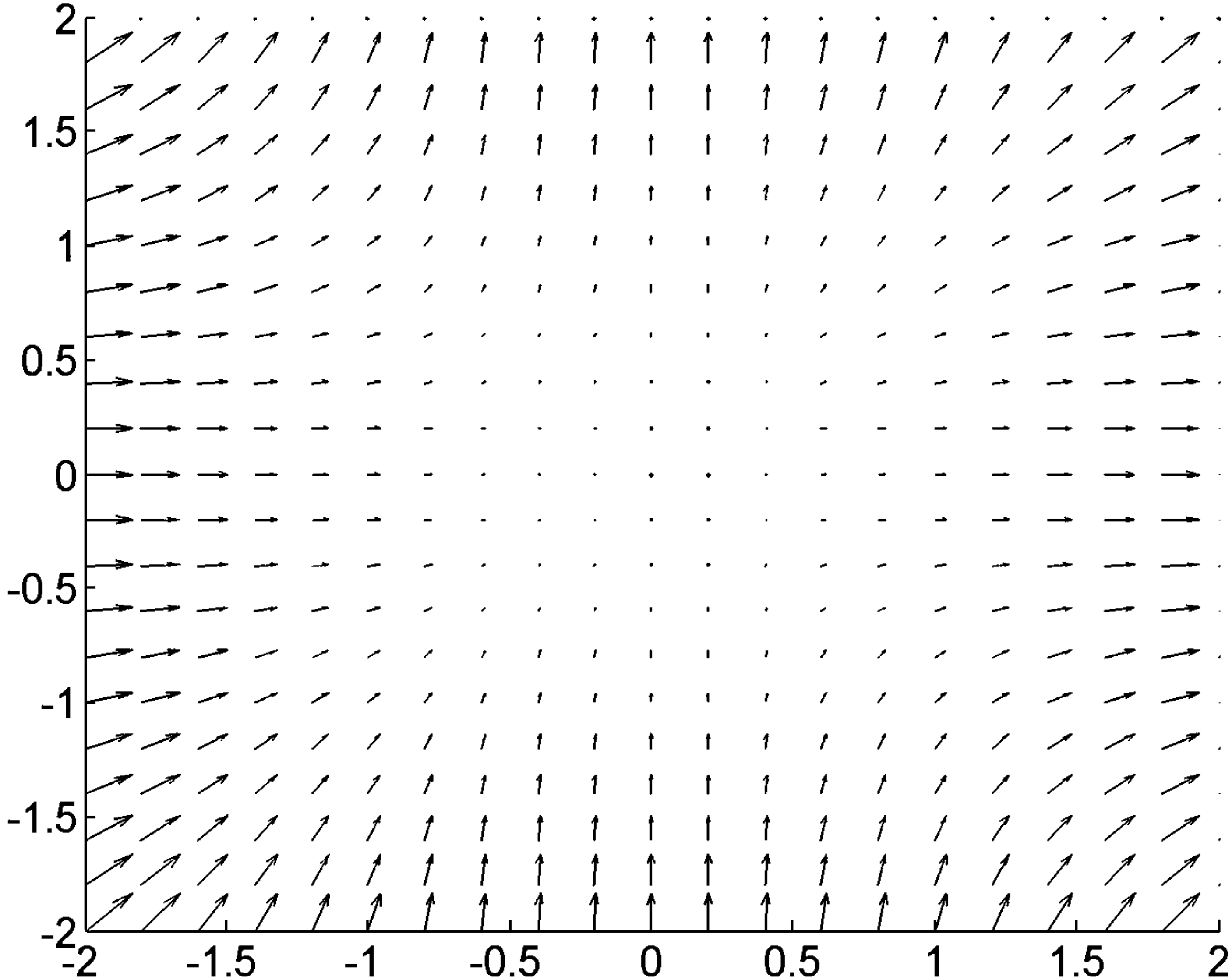
$$\vec{\mathbf{V}} = x\hat{i} + y\hat{j} + 0\hat{k}$$



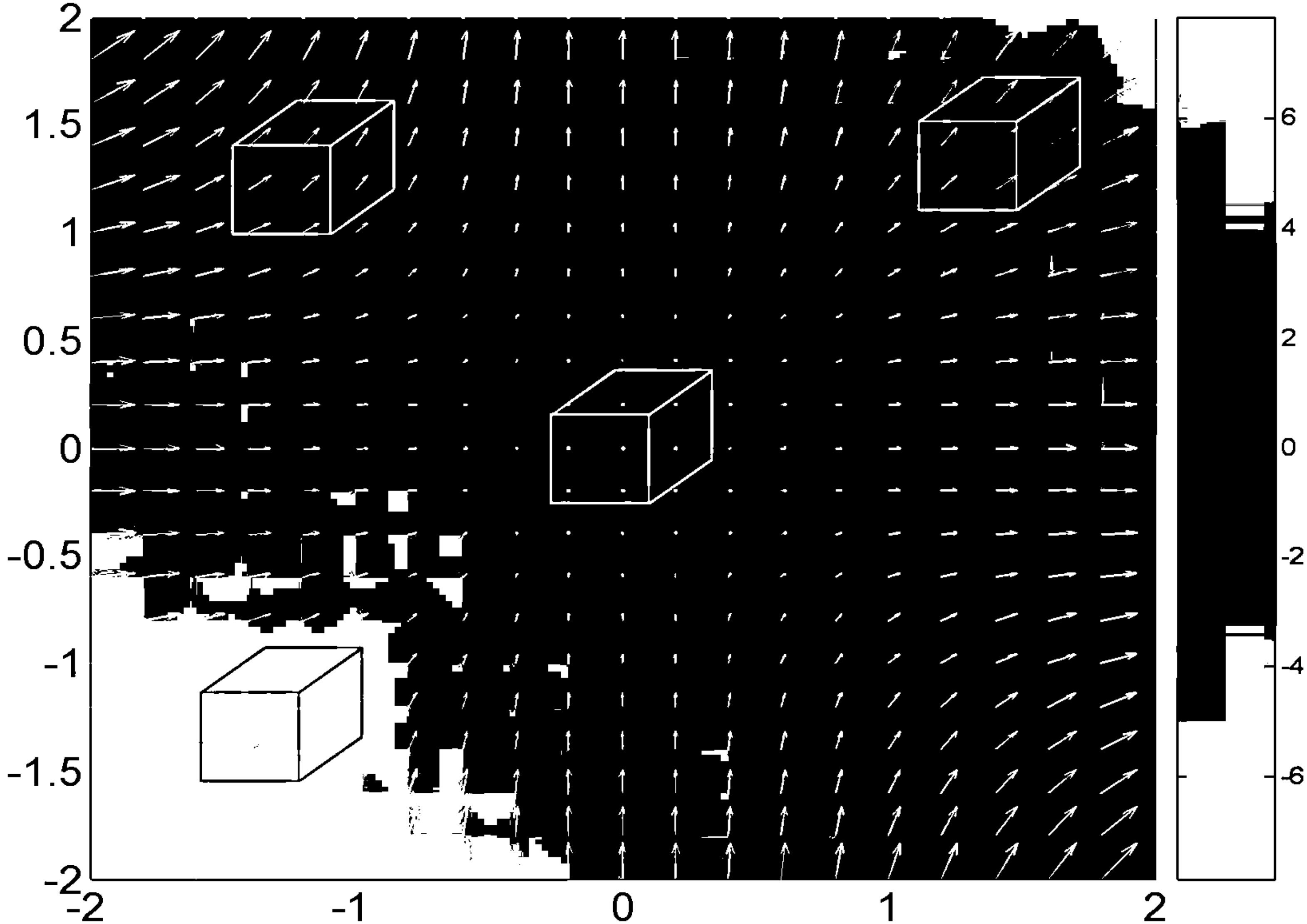
$$\vec{\mathbf{V}} = x\hat{i} + y\hat{j} + 0\hat{k} \qquad \nabla \cdot \vec{\mathbf{V}} = 2$$



$$\vec{\mathbf{V}} = x^2\hat{i} + y^2\hat{j}$$



$$\vec{\mathbf{V}} = x^2\hat{i} + y^2\hat{j} \qquad \nabla \cdot \vec{\mathbf{V}} = 2x + 2y$$

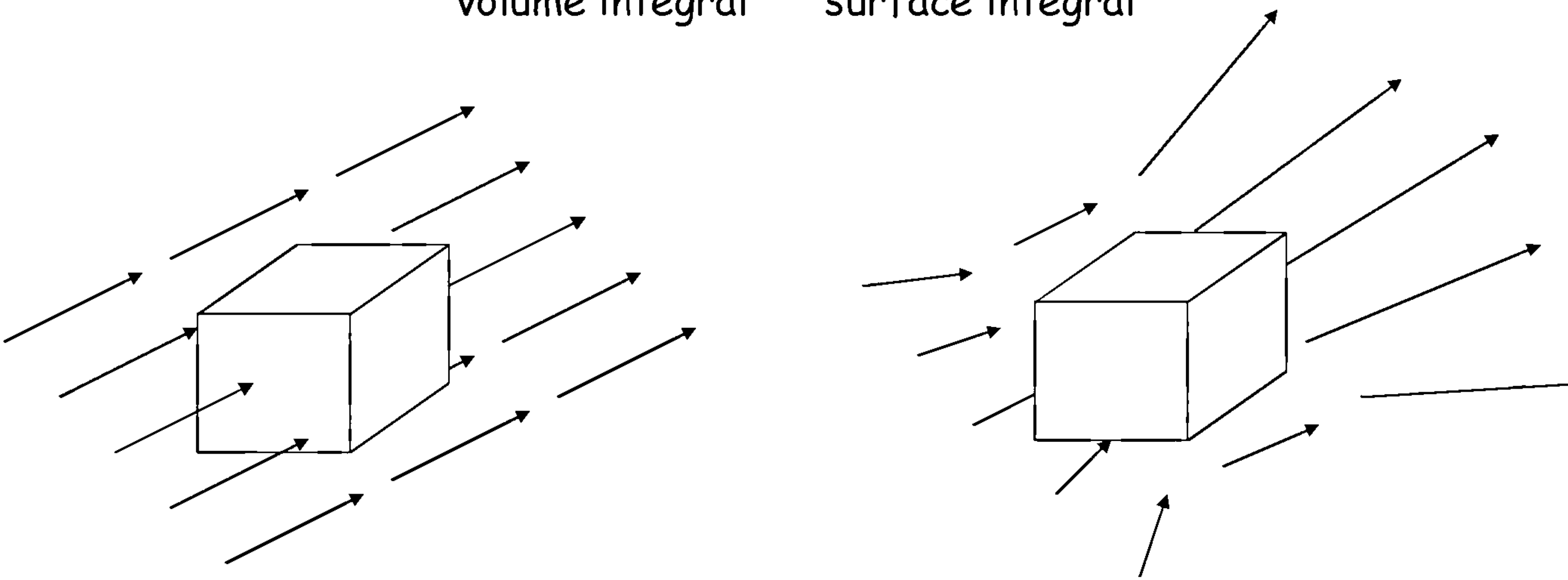


2. The Fundamental Theorem of Divergence

(The Divergence Theorem)

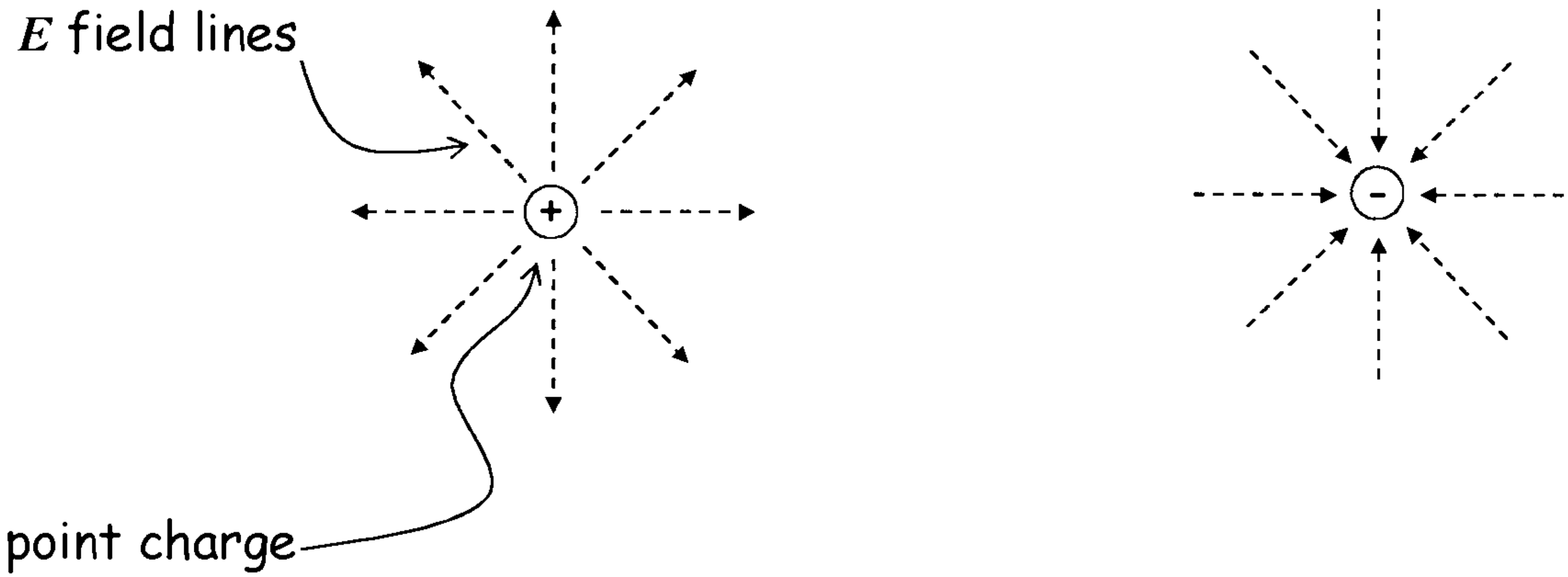
$$\int_V \left(\nabla \cdot \vec{\mathbf{V}}\right) d\tau = \oint_S \vec{\mathbf{V}} \cdot d\vec{a}$$

volume integral surface integral



(Integral of a derivative over a region is related to values at the boundary)

I. Gauss' Law: relation between a charge distribution and the electric field



$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{q}{\varepsilon_0}$$

$$\nabla \cdot \vec{\mathbf{E}} = \frac{1}{\varepsilon_0} \rho$$

Gauss' Law
(differential form)

II. Gauss' Law for Magnetism: relation between magnetic monopole distribution and the magnetic field

$$\oint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = 0 \qquad \nabla \cdot \vec{\mathbf{B}} = 0$$

Cabrera

The Valentine's Day Monopole

First Results from a Superconductive Detector for Moving Magnetic Monopoles

Blas Cabrera

Physics Department, Stanford University, Stanford, California 94305

Received 5 April 1982

A velocity- and mass-independent search for moving magnetic monopoles is being performed by continuously monitoring the current in a 20-cm^2-area superconducting loop. A single candidate event, consistent with one Dirac unit of magnetic charge, has been detected during five runs totaling 151 days. These data set an upper limit of 6.1×10^{-10} cm^{-2} sec^{-1} sr^{-1} for magnetically charged particles moving through the earth's surface.

PRL 48, p1378 (1982)

3. Curl

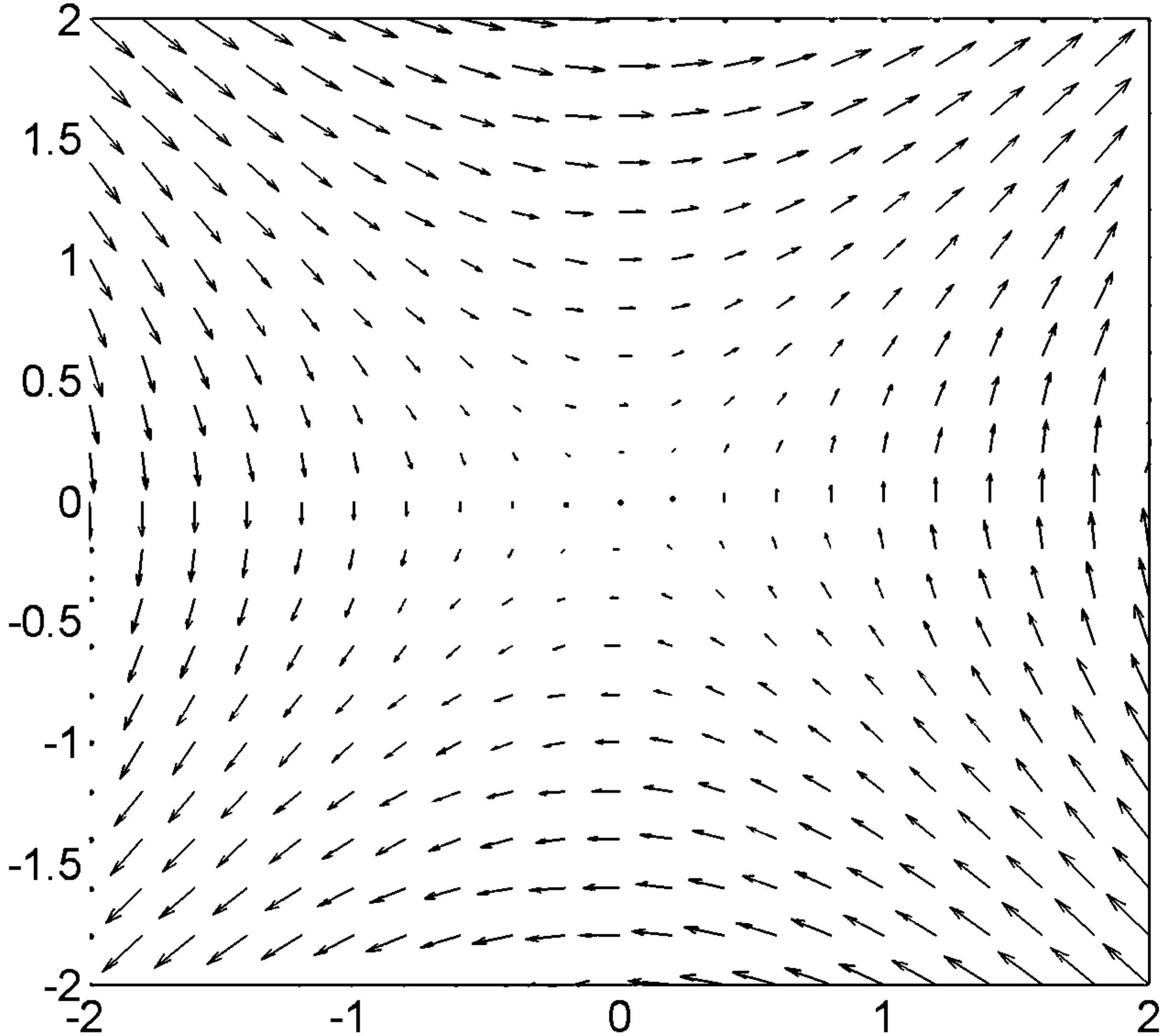
$$\nabla \times \vec{\mathbf{V}}$$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

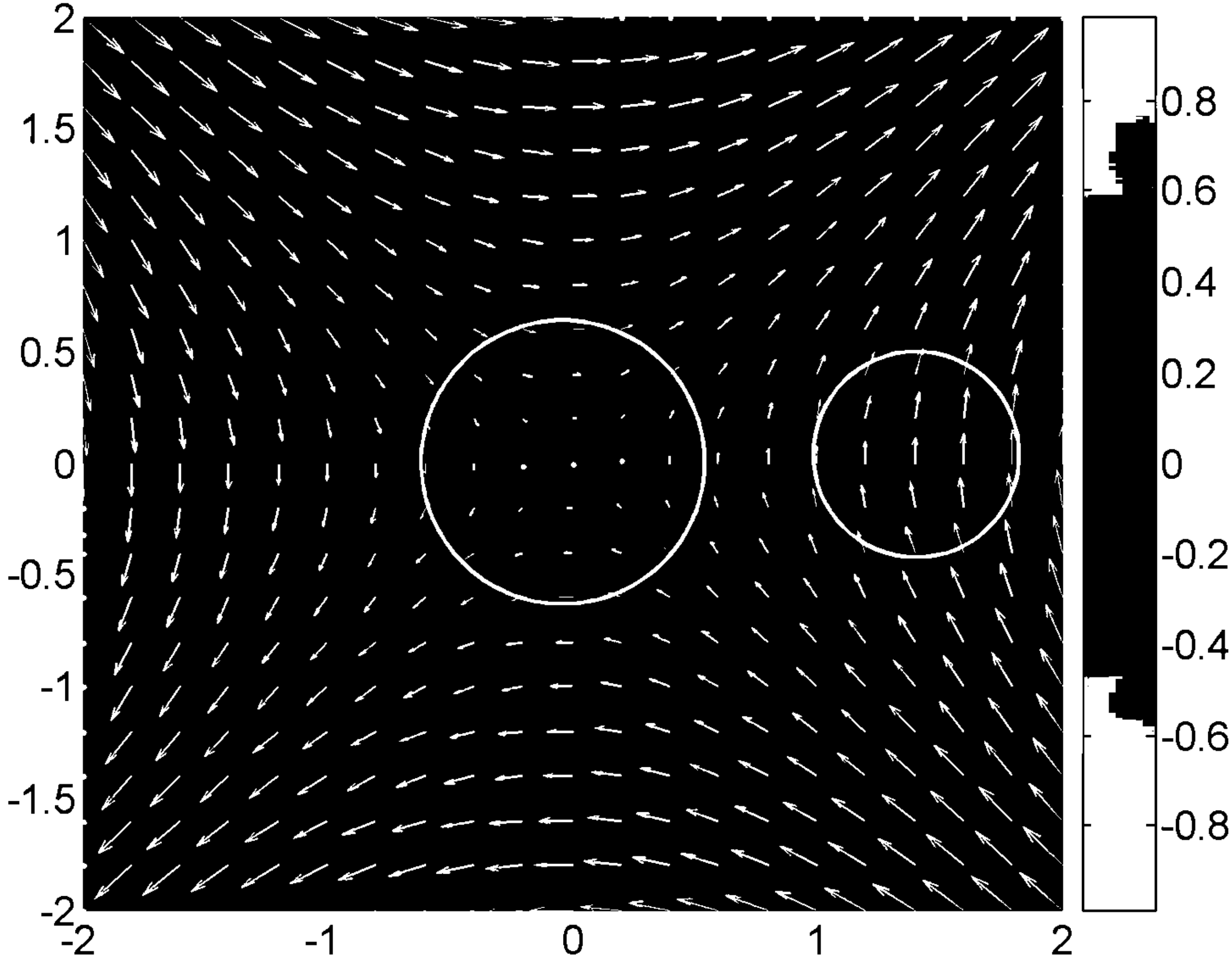
$$\det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{bmatrix}$$

"How much a vector field causes something to twist"

$$\vec{\mathbf{V}} = x\hat{i} + y\hat{j} + 0\hat{k}$$

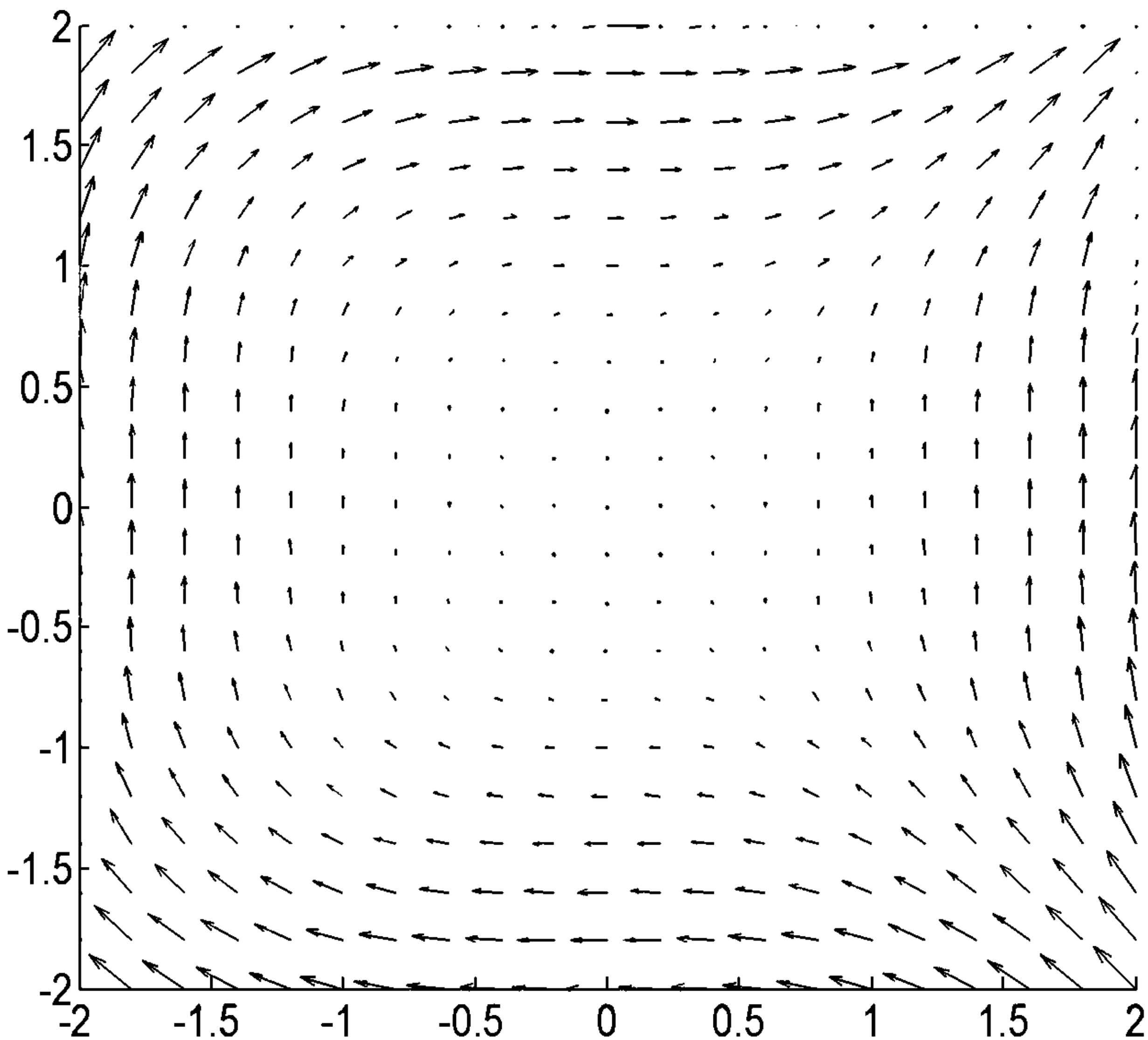


$$\vec{\mathbf{V}} = x\hat{i} + y\hat{j} + 0\hat{k} \qquad \nabla \times \vec{\mathbf{V}} = 1 - 1 = 0$$



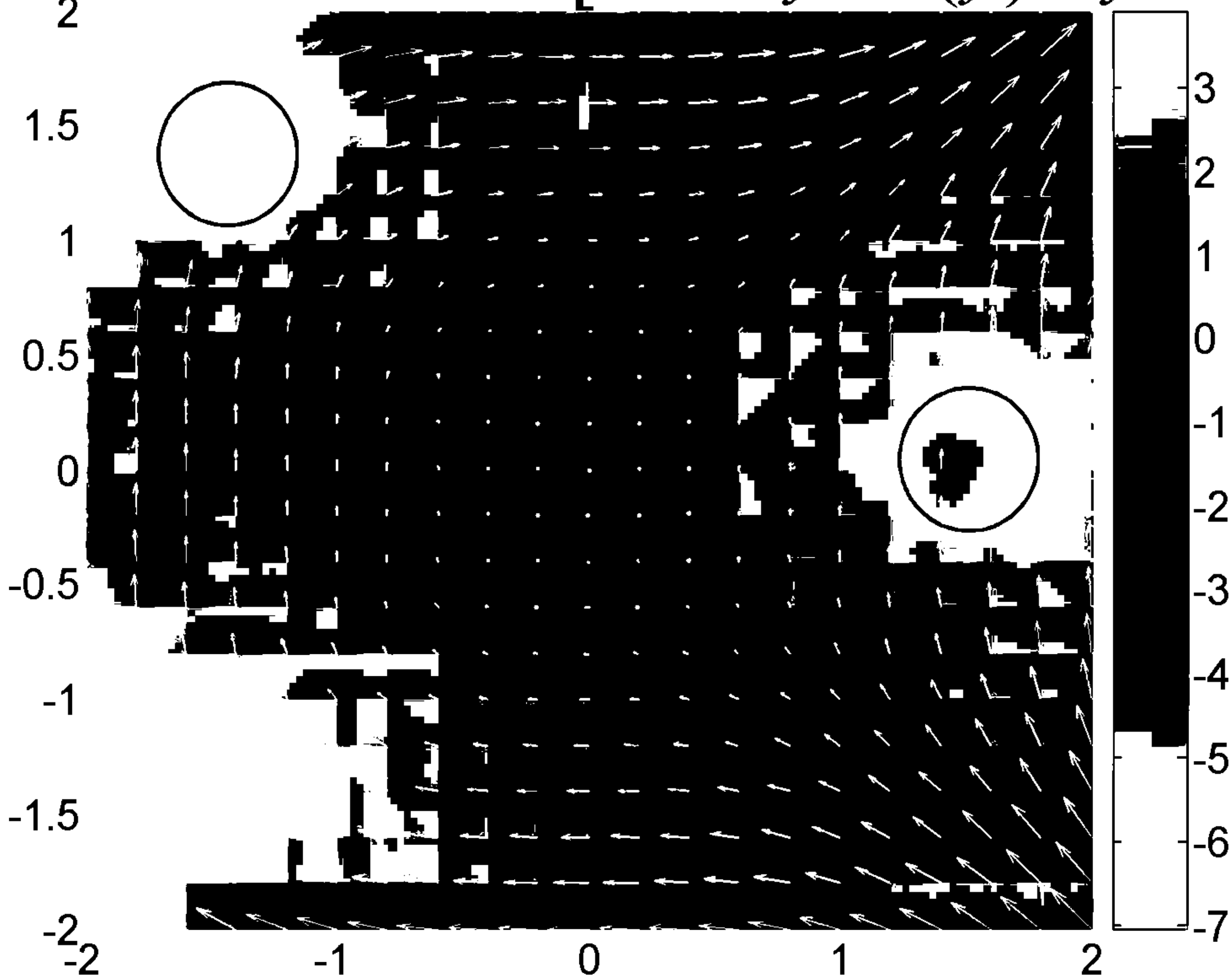
colorplot = z component of curl(V)

$$\vec{\mathbf{V}} = y^2 \sin(y)\hat{i} + x^2\hat{j} + 0\hat{k}$$



$$\vec{\mathbf{V}} = y^2 \sin(y)\hat{i} + x^2 \hat{j} + 0\hat{k}$$

$$\nabla \times \vec{\mathbf{V}} = \left[2x - 2y\sin(y) + y^2 \cos(y)\right]\hat{k}$$



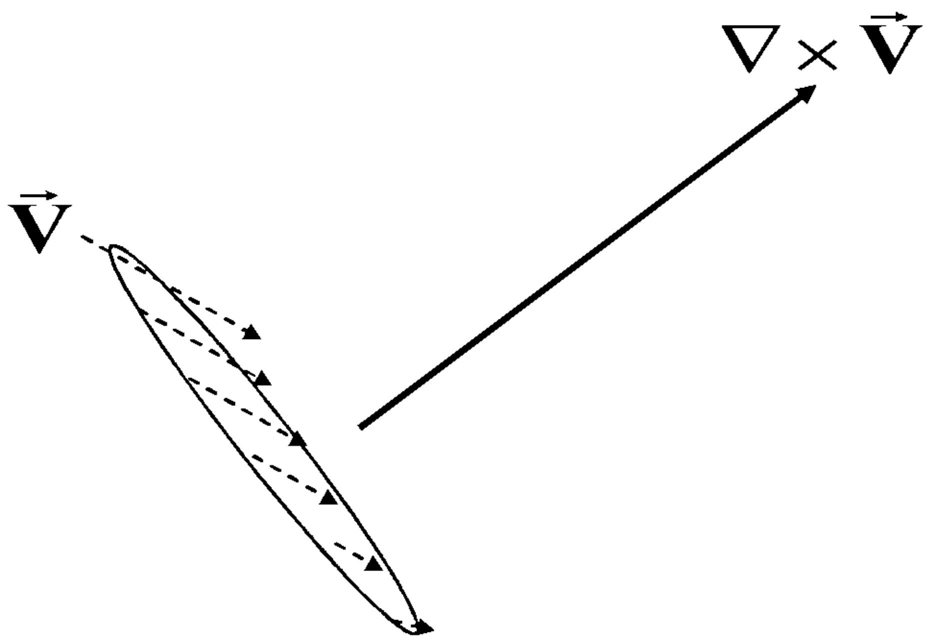
colorplot = z component of curl(V)

3. The Fundamental Theorem of Curl

(Really called Stokes' Theorem)

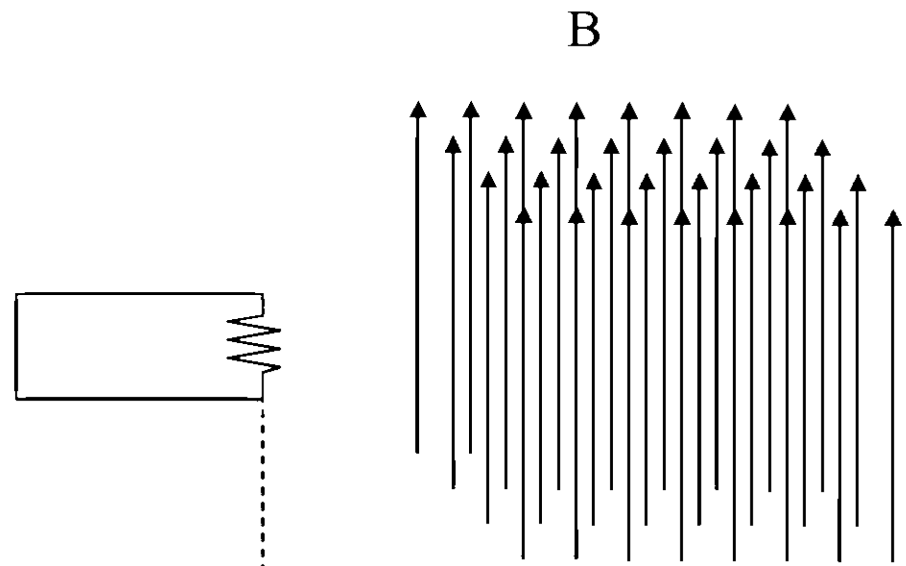
$$\int_S \left(\nabla \times \vec{\mathbf{V}}\right) \cdot d\vec{\mathbf{a}} = \oint_P \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}}$$

open surface integral closed perimeter line integral

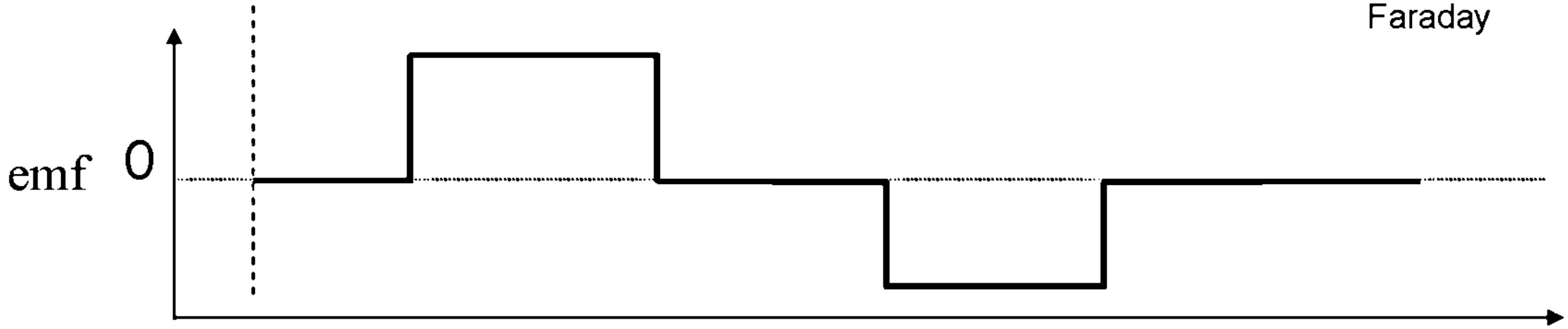


(Integral of a derivative over a region is related to values at the boundary)

III. Faraday's Law: A changing magnetic field induces an electric field.



Faraday

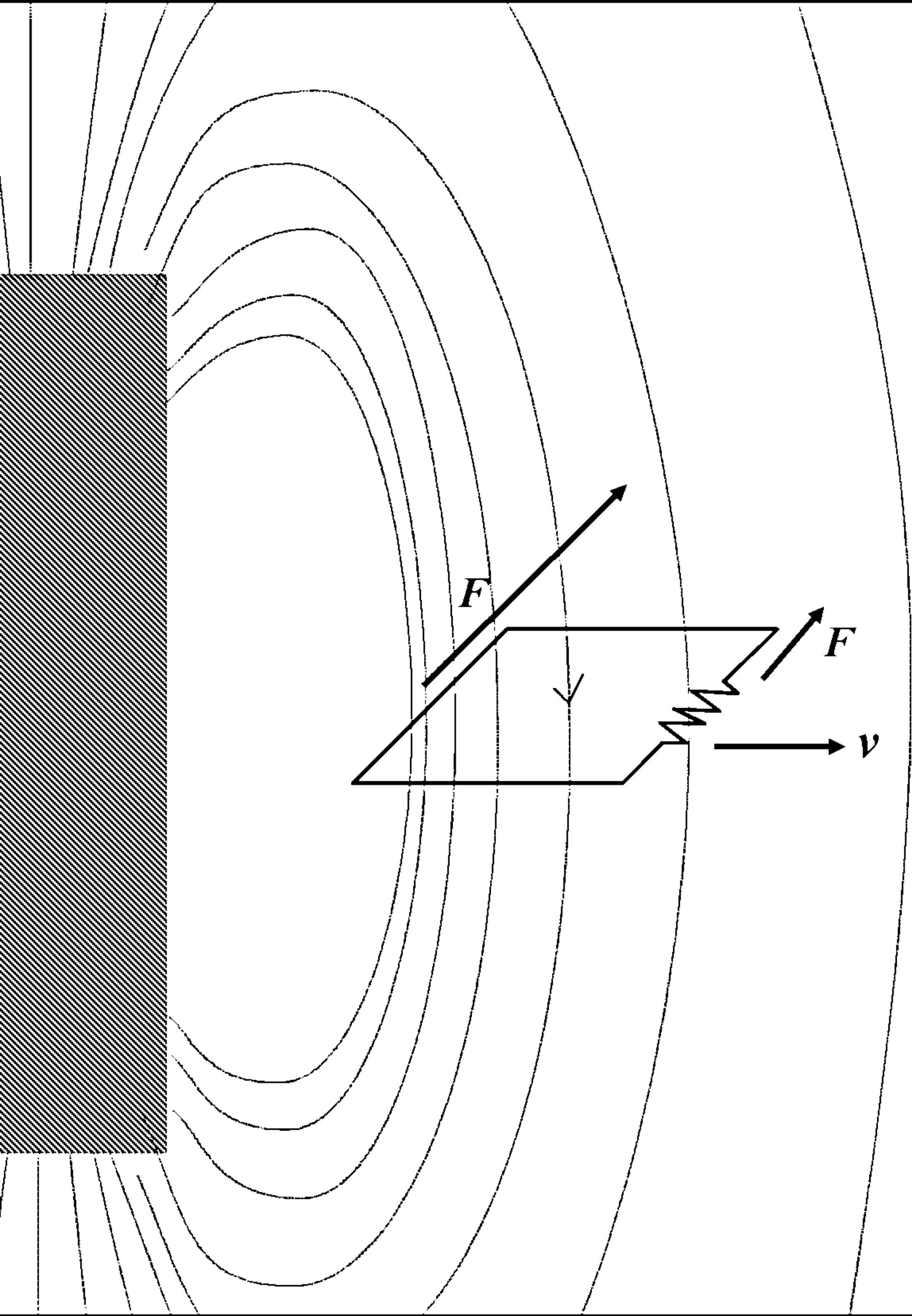


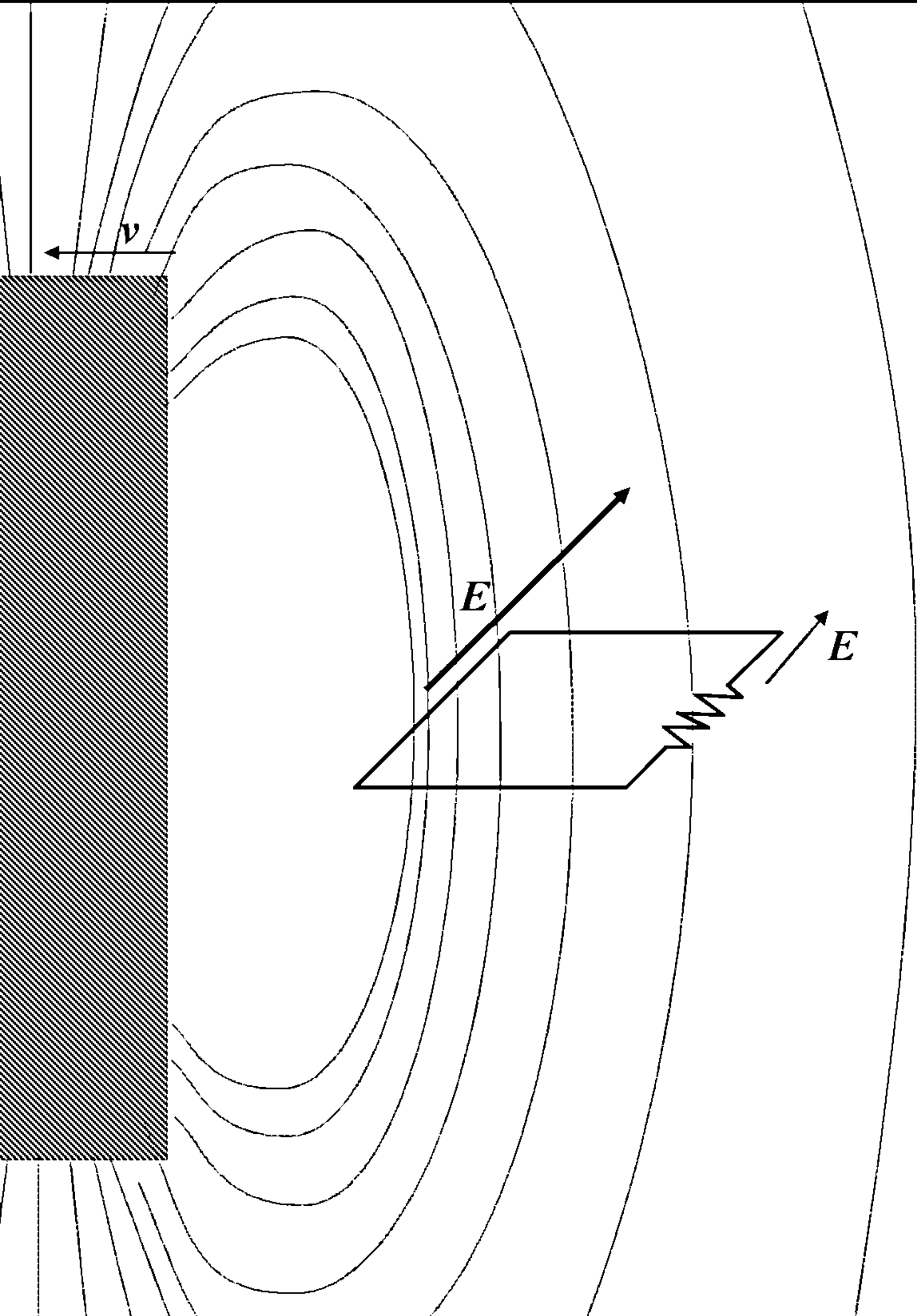
$$emf = -\frac{\partial \Phi_B}{\partial t}$$

Moving coil in a varying B field.
Force on electrons:

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

Forces don't cancel: $emf \neq 0$





Stationary coil with moving *B* source:

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

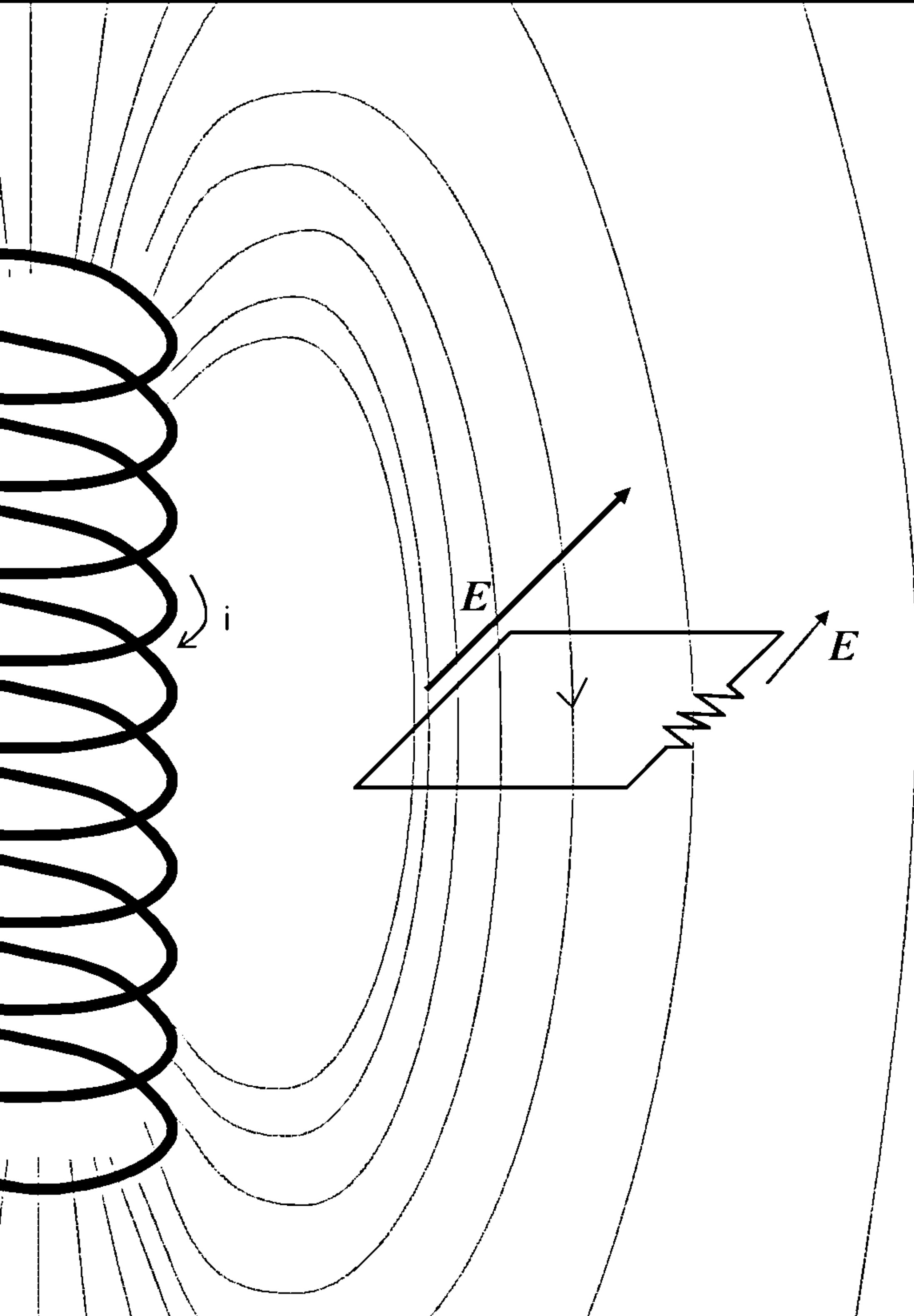
$$\vec{\mathbf{v}} = 0$$

But we still get an *emf* …

Only left with:

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}}$$

Electric field must be created!



Stationary coil and B source, but increasing B strength:

$$emf \neq 0$$

In general:

$$emf = -\frac{\partial \Phi_B}{\partial t}$$

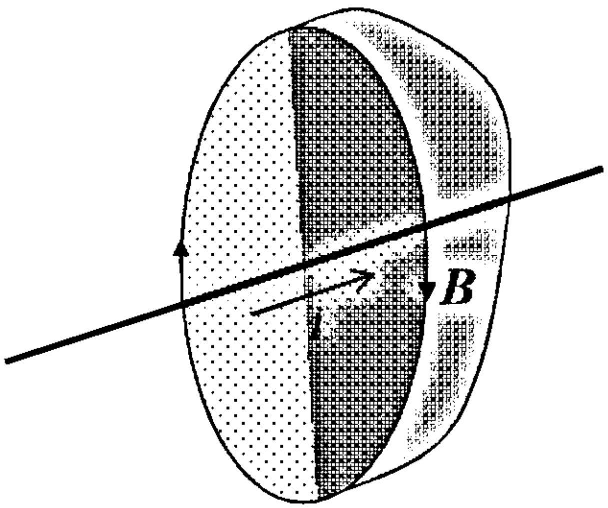
$$\oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

Faraday's Law
(integral form)

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Faraday's Law
(differential form)

IV. Ampere's Law



$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 i_{enclosed}$$

More general:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 \int_S \vec{\mathbf{J}} \cdot d\vec{\mathbf{a}}$$

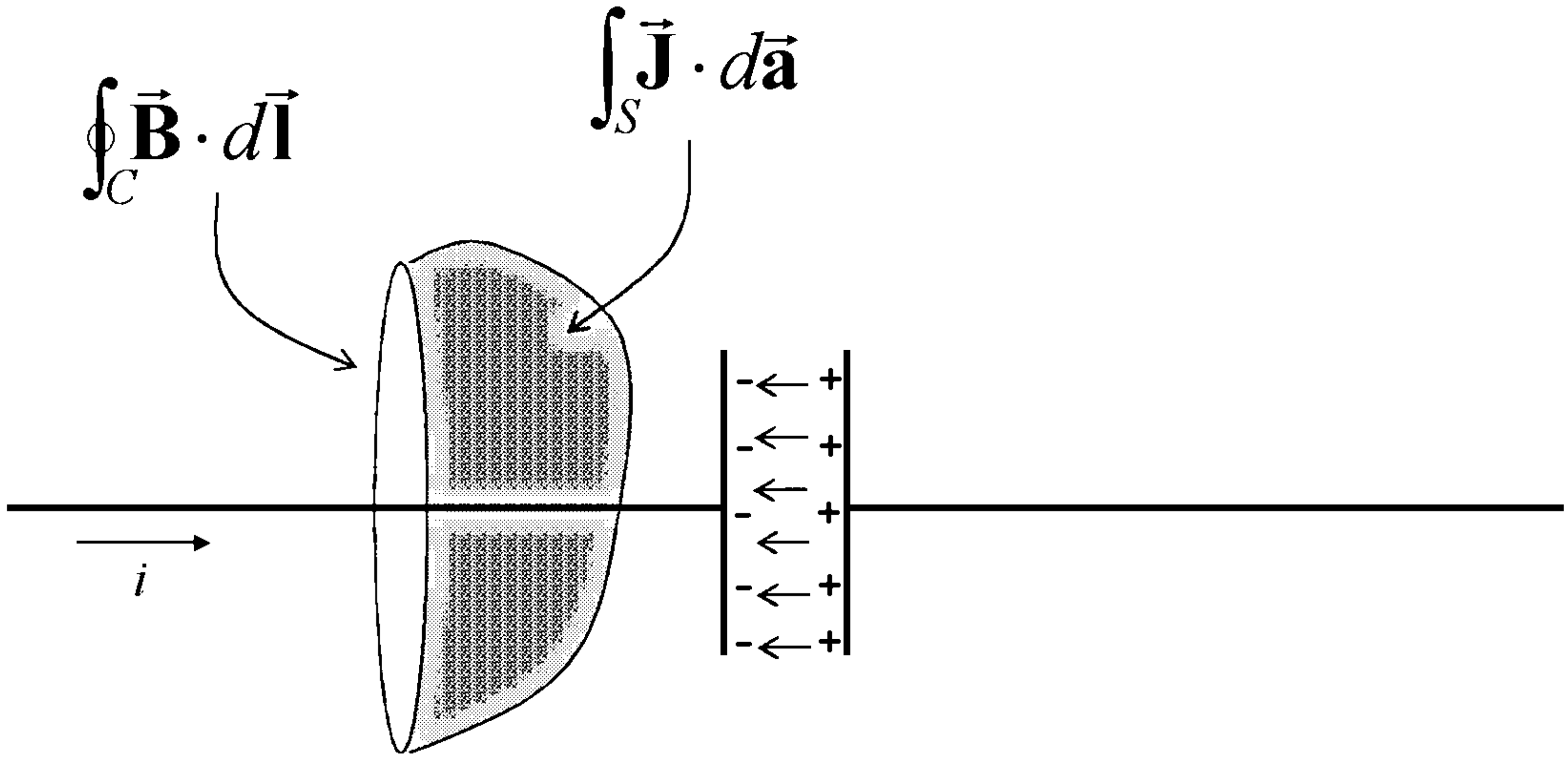
J = free current density

Ampere

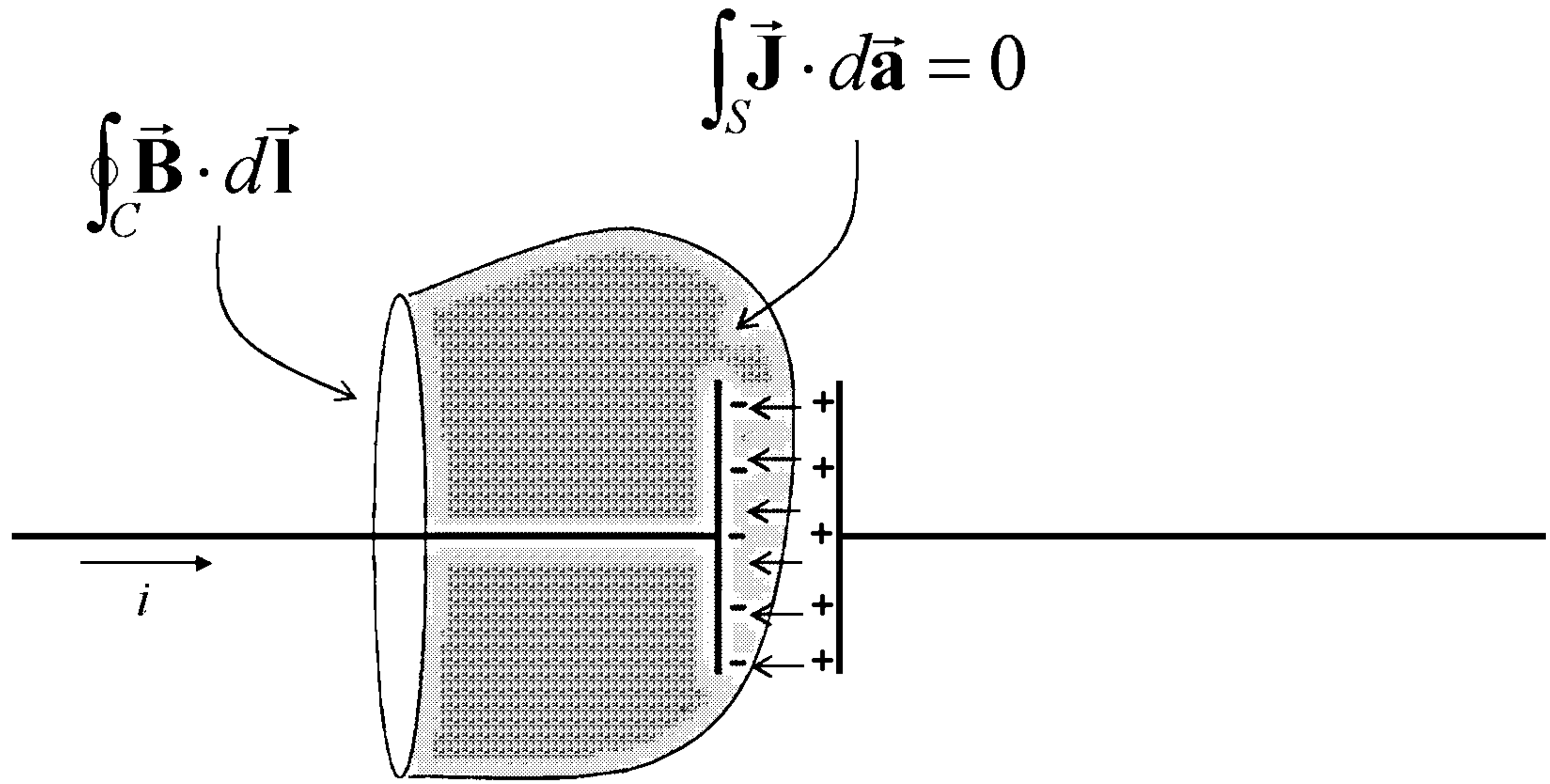
"Something is missing.."

Maxwell

Charging a capacitor



Charging a capacitor



Maxwell: "…the changing electric field in the capacitor is also a current."

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 \int_S \left(\vec{\mathbf{J}} + \frac{\partial}{\partial t} \varepsilon_0 \vec{\mathbf{E}} \right) \cdot d\vec{\mathbf{a}}$$

Ampere-Maxwell Eqn.
(Integral Form)

"Displacement current"

Get Stoked:

$$\int_S \left(\nabla \times \vec{\mathbf{B}} \right) \cdot d\vec{\mathbf{a}} = \mu_0 \int_S \left(\vec{\mathbf{J}} + \frac{\partial}{\partial t} \varepsilon_0 \vec{\mathbf{E}} \right) \cdot d\vec{\mathbf{a}}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \left(\vec{\mathbf{J}} + \frac{\partial}{\partial t} \varepsilon_0 \vec{\mathbf{E}} \right)$$

Ampere-Maxwell Eqn.
(differential form)

Maxwell's Equations in Free Space with no free charges or currents

$$\nabla \cdot \vec{\mathbf{E}} = 0$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{\mathbf{E}}$$