1. **Answer: 0**

   By L'Hopital's Rule,

$$\lim_{x\to 0}\frac{\tan x - x - \frac{x^3}{3}}{x} = \lim_{x\to 0}\frac{\sec^2 x - 1 - x^2}{1} = 1^2 - 1 - 0 = 0$$

2. **Answer: 3 values**

   If $\frac{n}{20-n} = k^2$ then $n = k^2(20-n)$ so $n = \frac{20k^2}{1+k^2}$. However, since $1+k^2$ is always coprime to k^2, $1+k^2$ must divide 20. Therefore, the only possible values of k are 1, 2, and 3, and the only possible values of n are the corresponding values 10, 16, and 18.

3. **Answer: $x_1 = x_2 = x_3 = \ldots = x_n = 2010$**

   Notice that

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(x_n + \frac{x_{n-1}^2}{x_n}\right) \Rightarrow 2x_1x_n = x_n^2 + x_{n-1}^2 \\
x_2 &= \frac{1}{2}\left(x_1 + \frac{x_n^2}{x_1}\right) \Rightarrow 2x_1x_2 = x_1^2 + x_n^2 \\
x_3 &= \frac{1}{2}\left(x_2 + \frac{x_1^2}{x_2}\right) \Rightarrow 2x_2x_3 = x_2^2 + x_1^2 \\
x_4 &= \frac{1}{2}\left(x_3 + \frac{x_2^2}{x_3}\right) \Rightarrow 2x_3x_4 = x_3^2 + x_2^2 \\
&\vdots \\
x_n &= \frac{1}{2}\left(x_{n-1} + \frac{x_{n-2}^2}{x_{n-1}}\right) \Rightarrow 2x_{n-1}x_n = x_{n-1}^2 + x_{n-2}^2.
\end{aligned}$$

   Adding all these equations up, subtracting the lefthand side to the righthand side of the equation, and factoring, we get

$$(x_1 - x_n)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots (x_{n-1} - x_n)^2 = 0.$$

   Since the sum of the squares is zero, each of the terms being added must be zero. Then it follows that $x_1 = x_2 = x_3 = \ldots = x_n = 2010$.

4. **Answer: 451 moles**

   The rate of change of spiders is the number of spiders, N_S in minus the consumption of spiders. Let $N_{S,0}$ be the initial number of spiders in the cauldron, $F_{S,0}$ the inlet stream of spiders and k the rate of consumption. Then,

$$\frac{\mathrm{d}N_S}{\mathrm{d}t} = F_{S,0} - kV; \quad N_S(t=0) = N_{S,0}$$

$$N_S = (F_S - kV)t + N_{s,0} = \left(1\frac{\text{mol}}{\text{min}} - \frac{1}{2}\frac{\text{mol}}{\text{min}}100\text{L}\right) + 500 = 451\text{moles}.$$

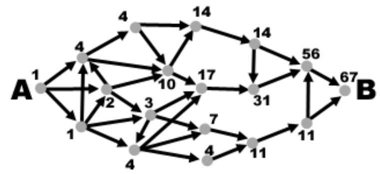
5. **Answer: $B > A > C > D$**

   It is easy to verify that

$$f(n) = \frac{1^n + 2^n + 3^n}{1^{n-1} + 2^{n-1} + 3^{n-1}}$$

   is a monotonically increasing function.

6. **Answer: 67**



Observe that if vertices $V_1, V_2, \ldots, V_n$ is a complete list of vertices that contain an edge leading into a vertex X and it is known that there are $p_1, p_2, \ldots p_n$ different paths into the vertices $V_1, V_2, \ldots, V_n$, respectively, then the number of paths into vertex X is $p_1 + p_2 + \ldots p_n$. Using this rule, we can start from the vertex A which trivially has 1 [empty] path into it and proceed inductively along the grid to get the following diagram, which ends with 67 paths on vertex B.

7. **Answer: $\mathbf{110 + 24\sqrt{6}}$**

   We can calculate the area of the middle triangle using Heron's formula. Hence, we can calculate the semiperimeter of the triangle, 9, and then calculate the area as $\sqrt{9 \times (9-5) \times (9-6) \times (9-7)} = 6\sqrt{6}$. Notice that $\angle BCA$ and $\angle ECD$ are supplementary. Hence, we can rotate $\triangle ECD$ about point C so that segments $\overline{AC}$ and $\overline{CD}$ overlap and the resulting figure BAE will be a triangle. In this position, we can see that $\triangle BAC$ and $\triangle DCE$ have the same altitude, and since they have the same base (length 7) they must have the same area. By the same reasoning, all the triangles must have the same area. Hence, the total area of the figure is simply the areas of the squares plus four times the area of the middle triangle $25 + 36 + 49 + 4 \times 6 \times \sqrt{6} = 110 + 24\sqrt{6}$.

8. **Answer: $\mathbf{-3439}$**

   The fact that the limit exists implies that

$$\lim_{x \to 0} (f(4x) + af(3x) + bf(2x) + cf(x) + df(0)) = (1 + a + b + c + d)f(0) = 0$$

therefore

$$a + b + c + d = -1.$$

Apply L'Hospital's Rule once, then we have

$$\lim_{x \to 0} \frac{f(4x) + af(3x) + bf(2x) + cf(x) + df(0)}{x^4} = \lim_{x \to 0} \frac{4f'(4x) + 3af'(3x) + 2bf'(2x) + cf'(x)}{4x^3}$$

and for the following limit to exist we also need

$$\lim_{x \to 0} (4f'(4x) + 3af'(3x) + 2bf'(2x) + cf'(x)) = (4 + 3a + 2b + c)f'(0) = 0,$$

therefore

$$3a + 2b + c = -4.$$

Repeat this process twice and get another two equations:

$$9a + 4b + c = -16$$

$$27a + 8b + c = -64$$

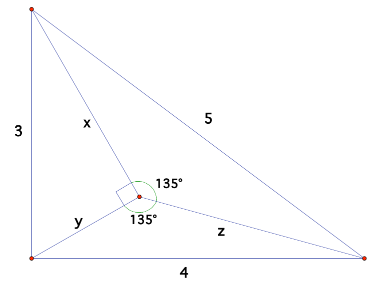
Solving these four equations one can get $(a, b, c, d) = (-4, 6, -4, 1)$, giving the answer $1000a + 100b + 10c + d = -3439$.

9. **Answer: $\boldsymbol{\frac{\pi}{3}}$**

   Let the edge length of the cone be r. Because PC is in the xy plane, $C = (\frac{r}{2}, \frac{r}{2}, 0)$ and thus $K = (\frac{r}{2}, \frac{r}{2}, \frac{r}{2})$. Therefore, the angle between the x axis and PK is $\text{Cos}^{-1}\left(\frac{x}{r}\right) = \text{Cos}^{-1}(1/2) = \frac{\pi}{3}$.

10. **Answer: $12\sqrt{2}$**

Note that these equations come from applying the Law of Cosines to the triangle in the figure. The desired value is simply $2\sqrt{2}A = 12\sqrt{2}$, where A is the area of the large triangle.



11. **Answer: $\frac{20}{3}$**

Let
$$F(x,y,z) = |x+y+z| + |x+y-z| + |x-y+z| + |-x+y+z|.$$
Note that
$$F(x,y,z) = F(-x,y,z) = F(x,-y,z) = F(x,y,-z)$$
so the region is symmetric with respect to all xy,yz,zx-planes. Thus one can only consider the first octant part(where $x,y,z \geq 0$) and reflect it to get a full figure.
Assume that x is the largest. Then $x-y+z, x+y-z \geq 0$. Since the equation of the form $A+|B| \leq C$ implies both $A+B \leq C$ and $A-B \leq C$, we have
$$(x+y+z)+(x+y-z)+(x-y+z)+(-x+y+z) = 2(x+y+z) \leq 4$$
and
$$(x+y+z)+(x+y-z)+(x-y+z)-(-x+y+z) = 4x \leq 4.$$
Thus $x+y+z \leq 2$, and $x,y,z \leq 1$. We can now see that the region is cube $0 \leq x,y,z \leq 1$ cut by the plane $x+y+z \leq 2$. Since the plane goes through three vertices $(1,1,0),(1,0,1),(0,1,1)$ of the cube, it cuts out a prism of volume $\frac{1}{6}$ out of the cube. So the volume is $1-\frac{1}{6} = \frac{5}{6}$. We multiply 8 to get the volume of whole region, so the answer is $\frac{5}{6}\cdot 8 = \frac{20}{3}$.

12. **Answer: 24**

Let v,e,t,q be the number of vertices, edges, triangular faces, and quadrilateral faces respectively. Note that each vertex is shared by exactly one quadrilateral, and a quadrilateral provides four vertices. By simple counting we get $v = 4q$. Apply the same thing to triangular face, then we have $4v = 3t$. Meanwhile from each vertex we have 5 edges coming out, so $5v = 2e$. Thus we have
$$q = 1/4v, t = 4/3v, e = 5/2v.$$
And from the Euler's formula $v - e + (t+q) = 2$, we have $(1-5/2+1/4+4/3)v = 1/12v = 2$, $v = 24$.

13. **Answer: $x^{2/3} + y^{2/3} = c^{2/3}$**

The equation of the ladder that makes an angle θ with the horizontal is $y = -(\tan\theta)x + c\sin\theta$. For any x-value, the point on f that intersects this ladder will have the maximum y for all possible θ. Hence, $\frac{dy}{d\theta} = 0 = -(\sec^2\theta)x + c\cos\theta \Rightarrow x = c\cos^3\theta \Rightarrow y = -(\tan\theta)c\cos^3\theta + c\sin\theta = c\sin^3\theta$. Using the identity $\sin^2\theta + \cos^2\theta = 1$, we see that $\left(\frac{x}{c}\right)^{2/3} + \left(\frac{y}{c}\right)^{2/3} = 1$.

14. **Answer: 6**

Notice that $ABCD$ is a cyclic quadrilateral. Set $AB = x$ and $AE = y$, so that $CE = 11-y$. We apply the Power of a Point Theorem at the point E to get $y(11-y) = 4\cdot 5$, so that $11y - y^2 = 20$.

We have two pairs of similar triangles: $\triangle ABE \sim \triangle DCE$ and $\triangle ADE \sim \triangle BCE$; these yield $AD = \frac{3}{2}y$ and $CD = \frac{5x}{y}$. Applying Ptolemy's Theorem to quadrilateral $ABCD$ then yields

$$x \cdot \frac{5x}{y} + 6 \cdot \frac{3}{2}y = 9 \cdot 11$$

Clearly the denominators and rearranging, we see that the y terms are precisely those given by the Power of a Point Theorem:

$$5x^2 = 99y - 9y^2 = 9(11 - y^2) = 9 \cdot 20 = 180,$$

and therefore $x = 6$.

15. **Answer: 7**

Let 1, 2, 3, 4, 5 be five balls.
Compare 1, 2, Without Loss of Generality (WLOG) $1 < 2$
Compare 3, 4, WLOG $3 < 4$
Compare 1, 3, WLOG $1 < 3$
Compare 3, 5.

(a) If $3 < 5$, there are eight remaining cases: compare 4, 5 (WLOG $4 < 5$), compare 2, 4
   - i. If $2 < 4$, compare 2, 3; done
   - ii. If $4 < 2$, compare 2, 5; done

(b) If $3 > 5$, there are seven remaining cases: compare 2, 3.
   - i. If $2 < 3$, compare 1, 5. If $5 < 1$ we are done, if $1 < 5$, compare 2, 5; done.
   - ii. If $2 > 3$, compare 1, 5, and then compare 2, 4; done.