1. **Answer:** $\mathbf{\frac{11}{15}}$

   Let A be the event that the duck pays double and B be the event in which at least one duck egg hatches.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A)P(B) \text{ since A and B are independent.}$$
$$P(A \cup B) = \frac{2}{5} + \left(1 - \left(\frac{2}{3}\right)^2\right) - \frac{2}{5}\left(1 - \left(\frac{2}{3}\right)^2\right) = \frac{11}{15}$$

2. **Answer:** $\boldsymbol{\frac{44}{15} + \frac{4}{15}i}$.

   Consider:

$$\begin{aligned}
3.\overline{0123} &= 3 + \sum_{n=0}^{\infty} 0(2i)^{-(4n+1)} + \sum_{n=0}^{\infty} 1(2i)^{-(4n+2)} + \sum_{n=0}^{\infty} 2(2i)^{-(4n+3)} + \sum_{n=0}^{\infty} 3(2i)^{-(4n+4)} \\
&= 3 + \sum_{n=0}^{\infty}\left[-\frac{1}{4}\left(\frac{1}{16}\right)^n + \frac{i}{4}\left(\frac{1}{16}^n\right)^n + \frac{3}{16}\left(\frac{1}{16}\right)^n\right] \\
&= 3 + \sum_{n=0}^{\infty}\left(-\frac{1}{16} + \frac{1}{4}i\right)\left(\frac{1}{16}\right)^n \\
&= 3 + \left(-\frac{1}{16} + \frac{1}{4}i\right)\frac{1}{1 - \frac{1}{16}} \\
&= 3 + \frac{16}{15}\left(-\frac{1}{16} + \frac{1}{4}i\right) \\
&= \frac{44}{15} + \frac{4}{15}i.
\end{aligned}$$

3. **Answer: 6.**

   This is a trivial application of Ramsey Theory. Consider one of the people, P, in the group, and that he or she may or may not have met last year. Assume without loss of generality that P met at least three of them last year: A, B, and C. If any two of these met each other last year, then those two and P all met each other last year. Alternatively, none of A, B, and C met each other last year.

4. **Answer:** $\boldsymbol{\frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta}$

   Notice that $\cos(n\theta)+i\sin(n\theta) = (\cos\theta + i\sin\theta)^n = z^n$ and $\cos(-n\theta)+i\sin(-n\theta) = (\cos\theta + i\sin\theta)^{-n} = z^{-n}$. Adding these two equations, we get that $\cos(n\theta) = (z^n + z^{-n})/2$. Then $(\cos(\theta))^5 = (z+z^{-1})^5/32$. Expanding yields the binomial coefficients: $(z + z^{-1})^5 = z^5 + 5z^4(z^{-1}) + 10z^3(z^{-2}) + 10z^2(z^{-3}) + 5z(z^{-4}) + z^{-5}$. Then $(z + z^{-1})^5/32 = \frac{1}{16}(z^5 + z^{-5})/2 + \frac{5}{16}(z^3 + z^{-3})/2 + \frac{5}{8}(z + z^{-1})/2 = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$.

5. **Answer:** $\mathbf{2^{2011} - 1}$

   We evaluate the inner sum by the Hockey Stick Identity. This identity is

$$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1} \implies \sum_{i=j}^{2010}\binom{i}{j} = \binom{2011}{j+1},$$

   so that

$$\sum_{j=0}^{2010}\sum_{i=j}^{2010}\binom{i}{j} = \sum_{j=0}^{2010}\binom{2011}{j+1}.$$

Now, using the fact that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n,$$

we obtain

$$\sum_{j=0}^{2010} \binom{2011}{j+1} = \sum_{j=1}^{2011} \binom{2011}{j} = \sum_{j=0}^{2011} \binom{2011}{j} - \binom{2011}{0} = 2^{2011} - 1.$$

6. **Answer: 96**

   The number of blue cells is $n + m - 1$; the number of total cells is nm. So $2010(m + n - 1) = nm$, or $nm - 2010n - 2010m + 2010 = 0$. This factors as $(n - 2010)(m - 2010) - 2010^2 + 2010 = 0$, or $(n - 2010)(m - 2010) = 2010 * 2009$. Thus each of $n - 2010$ and $m - 2010$ must be one of the positive factors of $2010 * 2009$; for each positive factor, there is one ordered pair. Since $2010 * 2009 = 2 * 3 * 5 * 7^2 * 41 * 67$, there are $2 * 2 * 2 * 3 * 2 * 2 = 2^5 * 3 = 96$ solutions.

7. **Answer: $\frac{1}{p} - 1$**

   Let the probability that a bug's descendant's die out be x. There are two ways for the bugs to die out: either the initial bug dies (with probability $1 - p$), or the bug successfully splits (probability p) and both of its descendants die out (each with probability x). Therefore, $x = (1 - p) + px^2$. Solving this quadratic equation yields the two solutions $x = 1$ and $x = \frac{1}{p} - 1$. Which is correct?

   Define p_n to be the probability that the bug dies out within n generations. Then, by the same reasoning as before, $p_{n+1} = (1-p)+pp_n^2$. From the definition of p_n, we see that the sequence is always increasing. We will show that $p_n < \frac{1}{p} - 1$ for every n, which would imply that $x = \frac{1}{p} - 1$ is the correct solution. This can be done by induction. Notice that $p_0 = 0 < \frac{1}{p} - 1$. Now, suppose that $p_k < \frac{1}{p} - 1$ for some k. Then,

   $$p_{k+1} = (1 - p) + pp_k^2 < (1 - p) + p(\frac{1}{p} - 1)^2 = 1 - p + p(\frac{1}{p^2} - 2\frac{1}{p} + 1) = 1 - p + \frac{1}{p} - 2 + p = \frac{1}{p} - 1.$$

   This completes the induction, so we indeed have $p_n < \frac{1}{p} - 1$, and hence the correct answer is indeed $\frac{1}{p} - 1$.

8. **Answer: $3^{n+1} - 2^{n+1}$**

   We use the fact that if $P(x)$ is a polynomial of degree n, then $P(x + 1) - P(x)$ is a polynomial of degree $n - 1$. Define $\Delta P(x) = P(x + 1) - P(x)$. By induction on m, it can be easily proved that $\Delta^m P(x)$ is a polynomial of degree $n-m$ such that $\Delta^m P(k) = 2^m \cdot 3^k$ for $0 \le k \le n-m$. $(0 \le m \le n)$ Since a polynomial of degree 0 is constant, $\Delta^n P(k)$ should be $\Delta^n P(0) = 2^n$ for all k. Particularly $\Delta^n P(1) = 2^n$. Finally note that

$$\begin{aligned}
P(n+1) &= P(n) + (P(n+1) - P(n)) \\
&= P(n) + \Delta P(n) \\
&= P(n) + \Delta P(n-1) + (\Delta P(n) - \Delta P(n-1)) \\
&= P(n) + \Delta P(n-1) + \Delta^2 P(n-1) \\
&= P(n) + \Delta P(n-1) + \Delta^2 P(n-2) + (\Delta^2 P(n-1) - \Delta^2 P(n-2)) \\
&= P(n) + \Delta P(n-1) + \Delta^2 P(n-2) + \Delta^3 P(n-2) \\
&= \cdots \\
&= \sum_{i=0}^{n} \Delta^i P(n-i) + \Delta^{n+1} P(0) \\
&= \sum_{i=0}^{n} 2^i 3^{n-i} \\
&= 3^{n+1} - 2^{n+1}.
\end{aligned}$$

9. **Answer: 4**

   If x and y are solutions, then there is a quadratic equation $t^2 + at + b = (t-x)(t-y)0$ of which x and y are the roots. Then $-a = x + y$ and $b = xy$. We find that $x^2 + y^2 = a^2 - 2b = 9$ and $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = -\frac{a}{b} = 9$. So $a = -9b$. Substituting this in the first equation, we get $81b^2 - 2b - 9 = 0$. This has two roots for b, both of them real. Therefore there are two corresponding values of a, both real. In each case, the quadratic leads to two ordered pairs, which gives four total ordered pairs. It is easy to check that they are, indeed, distinct.

10. **Answer: −14400**

    Note that $n^2 \equiv 0, 1, 4 \mod 5$. We consider three cases.

    <u>Case 1:</u> $n^2 \equiv 0 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2}{5}$. In this case, $n \equiv 0 \mod 5$, so $n = 5a$ for some integer a. Then $\frac{n^2}{5} = 5a^2$, which is not prime unless $a = \pm 1$. Therefore, for this case, $n = \pm 5$ are the only values of n for which $\lfloor \frac{n^2}{5} \rfloor$ is prime.

    <u>Case 2:</u> $n^2 \equiv 1 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2-1}{5} = \frac{(n-1)(n+1)}{5}$. In this case, we have either $n = 5a + 1$ or $n = 5a - 1$ for some integer a. Then $\frac{n^2}{5} = a(n \pm 1)$, which cannot be prime if $a \neq \pm 1$. Therefore, for this case, $n = \pm 4, \pm 6$ are the only values of n for which $\lfloor \frac{n^2}{5} \rfloor$ might be prime. We can check that these values of n do indeed yield primes 3 and 7.

    <u>Case 3:</u> $n^2 \equiv 4 \mod 5$, so that $\lfloor \frac{n^2}{5} \rfloor = \frac{n^2-4}{5} = \frac{(n-2)(n+2)}{5}$. In this case, we have either $n = 5a + 2$ or $n = 5a - 2$ for some integer a. Then $\frac{n^2}{5} = a(n \pm 2)$, which cannot be prime if $a \neq \pm 1$. Therefore, for this case, $n = \pm 3, \pm 7$ are the only values of n for which $\lfloor \frac{n^2}{5} \rfloor$ might be prime. None of these values actually yield primes however, as they give $\lfloor \frac{n^2}{5} \rfloor = 1, 9$.

    Therefore, the only values of n for which $\lfloor \frac{n^2}{5} \rfloor$ is prime are $n = \pm 4, \pm 5, \pm 6$, and the product of these values of n is -14400.