1. **Answer: 2520**

$$\frac{8!}{(2!)^4} = \frac{7!}{2} = 2520$$

2. **Answer: $3(a-b)(b-c)(c-a)$**

   The expression is zero when any two of a, b, and c are equal. So it must have $(a-b)(b-c)(c-a)$ as a factor. But the original polynomial is degree 3, and so is this one, so the remaining factor must be a constant. The original polynomial contains a term $3ab^2$, but $(a-b)(b-c)(c-a)$ only contains a term ab^2, so the constant must be 3.

3. **Answer: 13**

   Consider the equation modulo 5. All fourth powers are either 0 or 1 mod 5. So one of x and y must be divisible by 5; suppose it's x. Then we must in fact have $x = 5$, since $x = 10$ is too large. This gives $y = 8$, and this is the only possible solution. So the answer is 13.

4. **Answer: 41**

   A recursion relationship describing this problem is
   $a_1 = 1, a_2 = 2, a_{2n} = a_n + a_{n-1}, a_{2n+1} = a_n$
   where a_n is the number of valid sums for n. Thus,
   $a_{657} = a_{328} = a_{164} + a_{163} = a_{82} + 2a_{81} = a_{41} + 3a_{40} = 3a_{19} + 4a_{20}$
   $= 4a_{10} + 7a_9 = 4a_5 + 11a_4 = 4a_2 + 11a_4 = 4 \cdot 2 + 11 \cdot 3 = 41.$

5. **Answer: 120**

   Define $\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$. Using integration by parts,

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty t^n e^{-t}dt \\
&= -t^n e^{-t}|_0^\infty + \int_0^\infty nt^{n-1}e^{-t}dt \\
&= 0 + n\int_0^\infty t^{n-1}e^{-t}dt \\
&= n\Gamma(n).
\end{aligned}$$

   Next we evaluate $\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}|_0^\infty = 0 - -1 = 1$. Thus, $\Gamma(n+1) = n\Gamma(n) = \ldots = n!\Gamma(1) = n!$. So for the problem, $\Gamma(6) = 5! = 120$.

6. **Answer: $\frac{2}{\pi}$**

$$\begin{aligned}
\text{Area of Rhombus } ABCD &= 4 * \frac{1}{2} * \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\
&= 2 * \cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}[\text{Rhombus } ABCD] &= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \sin\theta\, dx \\
&= \frac{2}{\pi} * 1 \\
&= \frac{2}{\pi}.
\end{aligned}$$

7. **Answer:** $\frac{3\sqrt{3}}{4}$

Let the angle between the longer base and the leg be θ.
The Area of the Trapezoid $\Delta(\theta) = \sin\theta + \sin\theta * \cos\theta = \sin\theta + \frac{1}{2}\sin 2\theta$
The area reaches extrema when its derivative is zero:
$\Delta' = \cos\theta + \cos 2\theta = 0$
We use the formula $\cos 2\theta = 2 * \cos^2\theta - 1$
$2 * \cos^2\theta + \cos\theta - 1 = 0$
$\cos\theta = \dfrac{-1 \pm \sqrt{9}}{4} = \dfrac{1}{2}$ or -1 (omitted)
$\sin\theta = \dfrac{\sqrt{3}}{2}$
$\Delta_{Max} = \boxed{\frac{3\sqrt{3}}{4}}$

8. **Answer:** $\frac{-n^2+1}{12n^2}$

$$\begin{aligned}
\sum_{k=1}^{n} \frac{k^2(k-n)}{n^4} &= \sum_{k=1}^{n} \frac{k^3 - k^2 n}{n^4} \\
&= \sum_{k=1}^{n} \frac{k^3}{n^4} - \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \frac{1}{n^4}\sum_{k=1}^{n} k^3 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \left(\frac{1}{n^4}\right)\left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{1}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{n^4 + 2n^3 + n^2}{4n^4} - \frac{2n^3 + 3n^2 + n}{6n^3} \\
&= \frac{-n^4 + n^2}{12n^4} \\
&= \frac{-n^2 + 1}{12n^2}.
\end{aligned}$$

9. **Answer:** $2\sqrt{17}$

Find the point on the parabola closest to the point (6,12). Call it (x, y) This point is where the normal line at x crosses the parabola. We find the derivative by:

$$\begin{aligned}
x &= y^2 \\
dx &= y\,dy \\
\frac{dy}{dx} &= \frac{1}{y}
\end{aligned}$$

The normal line will have slope of $-y$. It will contain $(6, 12)$. Its equation is:

$$\begin{aligned}
y - 12 &= -y(x - 6) \\
y &= -xy + 6y + 12 \\
y &= -\frac{y^3}{2} + 6y + 12 \\
2y &= -y^3 + 12y + 24 \\
0 &= y^3 - 10y - 24
\end{aligned}$$

The roots are 4 and two other imaginary answers, so 4 is the only one that works.

$$\begin{aligned} y - 12 &= -y(x-6) \\ -8 &= -4(x-6) \\ x &= 8 \end{aligned}$$

Find the distance between $(8,4)$ and $(6,12)$. The answer is $2\sqrt{17}$.

10. **Answer: 2**

More generally, define a function G by

$$G(m) = \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n}.$$

Thus we wish to evaluate $G(2009)$. Observe that for all $m \geq 1$:

$$\begin{aligned} G(m) &= \sum_{n=m}^{\infty} \frac{\binom{n}{m}}{2^n} \\ &= \sum_{n=m}^{\infty} \frac{\binom{n-1}{m-1} + \binom{n-1}{m}}{2^n} \\ &= \frac{1}{2} \sum_{n=m-1}^{\infty} \frac{\binom{n}{m-1}}{2^n} + \frac{1}{2} \sum_{n=m-1}^{\infty} \frac{\binom{n}{m}}{2^n} \\ &= \frac{1}{2}(G(m-1) + G(m)) \end{aligned}$$

And thus $G(m) = G(m-1)$. Thus it suffices to evaluate $G(0)$. However, this is simply a geometric series:

$$\begin{aligned} G(0) &= \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= 2. \end{aligned}$$

NOTE: By noticing that $\binom{n}{2009}$ is $\frac{1}{2009!}n^{2009}$ asymptotically, one can see this summation as a discrete analogue of the Euler Γ function, which is defined by $\Gamma(x) = \int_0^\infty \frac{t^{x-1}}{e^t} dt$. The solution above is similar to the proof that $\Gamma(n+1) = n\Gamma(n)$.

11. **Answer: 1266**

$$\begin{aligned} (1 + z_1^2 z_2)(1 + z_1 z_2^2) &= 1 + z_1^2 z_2 + z_1 z_2^2 + z_1^3 z_2^3 \\ &= 1 + z_1 z_2(z_1 + z_2) + (z_1 z_2)^3. \end{aligned}$$

Since $z_1 + z_2 = -6$ and $z_1 z_2 = 11$,

$$\begin{aligned} (1 + z_1^2 z_2)(1 + z_1 z_2^2) &= 1 + 11(-6) + 11^3 \\ &= 1266. \end{aligned}$$

12. **Answer: 13689**

$2009 = 7^2 \times 41$

We know for a number $n = a_1^{\alpha_1} \times a_2^{\alpha_2} \times \ldots \times a_n^{\alpha_n}$, it has $(\alpha_1 + 1) \times (\alpha_2 + 1) \times \ldots (\alpha_n + 1)$ factors.
Hence, for number N, we have the following options:
$\alpha_1 = 7 - 1 = 6$, $\alpha_2 = 7 \times 41 - 1 = 289 - 1 = 288$
$\alpha_1 = 7 - 1 = 6$, $\alpha_2 = 7 - 1 = 6$, $\alpha_3 = 41 - 1 - 40$
By the same fact mentioned above, N^2 has: $(2 * \alpha_1 + 1) \times (2 * \alpha_2 + 1) \times \ldots (2 * \alpha_n + 1)$ factors.
Calculating this number for both, we get the 2nd option gets us a bigger number: $13 \times 13 \times 81 = \boxed{13689}$

13. **Answer: 3**

$$17^{289} \equiv (14+3)^{289} \equiv \binom{289}{1} 14^{288} 3 + \ldots + \binom{289}{n} 13^{289-n} 3^n + \ldots$$
$$3^{289} \equiv 3^{289} (\mod 7)$$

Note that $3^3 \equiv 27 \equiv -1 (\mod 7)$. Then $3^{289} \equiv 3^{3\dot{\,}96} \dot{3}^1 \equiv (-1)^{96} \dot{3}^1 \equiv 3 \mod 7$. Thus, the remainder is 3.

14. **Answer: 17**

equation modulo 23, we get $-6(a-b) \equiv -10 \ (\mod 23)$. Since -4 is an inverse of -6 modulo 23, then we multiply to get $(a-b) \equiv 17 \ (\mod 23)$. Therefore, the smallest possible positive value for (a-b) is 17. This can be satisfied by $a = 5$, $b = -12$.

15. **Answer: 66**

$\lfloor \frac{2008}{31} \rfloor + \lfloor \frac{2008}{31^2} \rfloor + \lfloor \frac{2008}{31^3} \rfloor + \lfloor \frac{2008}{31^4} \rfloor + \cdots = 64 + 2 + 0 + 0 + \cdots = 66.$