1. **Answer: 28**

$$\begin{cases} r = 2g \\ g - 4 = 2(d-4) \\ r + 10 = 2(d+10) \end{cases}$$

So $r = 2g = 2(2(d-4)+4) = 4d - 8 = 4(r/2 - 5) - 8 = 2r - 28 \Rightarrow r = 28$.

2. **Answer: 5**

Divide the equation $P(x) = 0$ by x^3 to get $x^3 + ax^2 + bx + 1 + b\frac{1}{x} + a\frac{1}{x^2} + \frac{1}{x^3} = 0$. In this equation, replacing x by $\frac{1}{x}$ doesn't change anything, so anytime x is a root of $P(x) = 0$, $\frac{1}{x}$ is also a root. Any root other than ± 1 (since 0 isn't a root) must be paired with another, namely its reciprocal. And 1 is a root, while -1 is not. So the total number of real roots must be odd. Note that having an odd number of distinct real roots requires that 1 be a double root. This makes the maximum number of real roots 5.

3. **Answer: $\{i, -i, 1\}$**

First, notice that the three solutions are symmetric. We write our conditions as a system of equations:

$$\begin{cases} a + b + c = 1 & (1) \\ ab + bc + ca = 1 & (2) \\ abc = 1 & (3) \end{cases}$$

(3) can be rewritten $c = 1/ab$. Substituting that in (2), we get

$$\begin{aligned} ab + \tfrac{b}{ab} + \tfrac{a}{ab} &= 1 \\ ab + 1/a + 1/b &= 1 \\ a^2b + 1 + a/b &= 1 \\ a(ab + 1/b) &= 0 \end{aligned}$$

Because we know from (3) that $a = 0$ cannot be a solution, we throw it out:

$$\begin{aligned} ab + 1/b &= 0 \\ a &= -1/b^2 \end{aligned}$$

Substituting this as well as our expression for c in (1), we get:

$$\begin{aligned} \tfrac{-1}{b^2} + b + \tfrac{1}{-1/b^2(b)} &= 1 \\ \tfrac{-1}{b^2} + b - b &= 1 \\ \tfrac{-1}{b^2} &= 1 \\ b &= \pm i \end{aligned}$$

Letting any two variables be $-i$ and i, we easily find using any of our three equations that the third must equal 1.

4. **Answer: $\frac{1}{2}$**

$$\begin{aligned} 0 &= (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz) \\ \frac{-1}{2} &= xy + yz + xz \\ \frac{1}{4} &= (xy + yz + xz)^2 = x^2y^2 + x^2z^2 + y^2z^2 + 2(x^2yz + xy^2z + xyz^2) \\ &= x^2y^2 + x^2z^2 + y^2z^2 + 2xyz(x+y+z) = x^2y^2 + x^2z^2 + y^2z^2 \\ 1 &= (x^2 + y^2 + z^2)^2 = x^4 + y^4 + z^4 + 2(x^2y^+x^2z^2 + y^2z^2) \\ 1 &= x^4 + y^4 + z^4 + 2 \cdot \frac{1}{4} \end{aligned}$$

5. **Answer: $3x^2 + 61x + 2008$**

   The highest power of x that can occur in the determinant is x^2, so $D(x)$ must be quadratic; let it be $ax^2 + bx + c$. The constant term is $c = D(0) = 2008$, so we have $D(-1) - 2008 = -58 = a - b$ and $D(2) - 2008 = 134 = 4a + 2b$. Solving the pair of linear equations gives $a = 3$ and $b = 61$.

6. **Answer: 45**

   By the quadratic formula, the solutions to $x^2 - x - k = 0$ are precisely

$$\frac{1 \pm \sqrt{1+4k}}{2}.$$

   These solutions are integers precisely when $1 \pm \sqrt{1+4k}$ is an even integer, i.e. when $\sqrt{1+4k}$ is an odd integer. Since $1 + 4k$ is itself odd, $\sqrt{1+4k}$ is an odd integer precisely when $1 + 4k$ is a perfect square.

   Thus, we are interested in how many (nonnegative, to avoid double counting) integers a give an integer solution for k with $0 \le k \le 2008$ in $1 + 4k = a^2$, or equivalently to $4k = a^2 - 1$. Notice that $a^2 - 1$ is divisible by 4 precisely when a is odd. The only other restriction on a is that $4 \cdot 2008 \ge a^2 - 1$. Since $89 < \sqrt{4 \cdot 2008 + 1} < 90$, there are $\frac{90}{2} = 45$ values for a such that $4k = a^2 - 1$ has an integer solution for k with $0 \le k \le 2008$. Consequently, there are 45 values for k such that $x^2 - x - k = 0$ has integer solutions for k.

7. **Answer: $(16, 64)$**

   Since $2p^2 + q^2$ is even, q must be even, so we divide through by 2 to obtain $p^2 + 2\left(\frac{q}{2}\right)^2 = 2304$. Now, p must be even, so we divide through by 2 again. Repeating until the number on the right is no longer even, we find that $\left(\frac{p}{16}\right)^2 + 2\left(\frac{q}{32}\right)^2 = 9$, where $\frac{p}{16}$ and $\frac{q}{32}$ are integers. This has the obvious solution $1^2 + 2 \cdot 2^2 = 9$, which gives $(p, q) = (16, 64)$.

8. **Answer: 16**

$$\begin{aligned} P(x)Q(x) &= x^4 - 1 \\ &= (x^2 - 1)(x^2 + 1) \\ &= (x - 1)(x + 1)(x - i)(x + i) \end{aligned}$$

   We have four distinct monomial factors, so the number of possible $P(x)$ is $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 16$.

9. **Answer: 16**

   Solving the equation for y gives $y = \frac{x-43}{x-1}$. Essentially, we are now finding all $z = x - 1$ s.t. $z | z - 42 \Rightarrow z | 42$, since $z | z$. The possible values for z are $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42\}$. There is one pair (x, y) for each of these.

10. **Answer: $\frac{5}{16}$**

   Let $S = \sum_{k=1}^{\infty} \frac{k}{5^k}$. Then,

$$\begin{aligned} 5S &= 5\sum_{k=1}^{\infty} \frac{k}{5^k} = \sum_{k=1}^{\infty} \frac{k}{5^{k-1}} = \sum_{k=0}^{\infty} \frac{k+1}{5^k} = \sum_{k=0}^{\infty} \frac{k}{5^k} + \sum_{k=0}^{\infty} \frac{1}{5^k} \\ &= 0 + \sum_{k=1}^{\infty} \frac{k}{5^k} + \sum_{k=0}^{\infty} \frac{1}{5^k} = S + \sum_{k=0}^{\infty} \frac{1}{5^k} \\ 4S &= \sum_{k=0}^{\infty} \frac{1}{5^k} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} \\ S &= \frac{5}{16} \end{aligned}$$