

ALGEBRA SOLUTIONS  
2007 RICE MATH TOURNAMENT  
FEBRUARY 24, 2007

1. **Answer:**  $4^{1/9}, -1$

$f(x^{1/9}) = (x - 4)(x + 1)$  so  $f = 0$  means  $x = 4$  or  $x = -1$ , so  $f(4^{1/9}) = f(-1) = 0$ .

2. **Answer:**  $\frac{1}{4}$

Since  $(x - x_1)(x - 4x_1) = x^2 - 5x_1x + 4x_1^2$ , we know  $3 = 2(4x_1^2 + 5x_1)$  so  $x_1 = \frac{1}{4}$ . (The other root is negative.)

3. **Answer:**  $-37$

Simply note that  $(a + b)(a + c)(b + c) = (ab + ac + bc)(a + b + c) - abc = -6 \cdot 7 + 5 = -37$ .

4. **Answer:** 1339

Let  $k$  be a nonnegative integer. Let  $f(x) = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$ . If  $k \leq x < k + \frac{1}{3}$ , then  $f(x) = 6k$ . If  $k + \frac{1}{3} \leq x < k + \frac{1}{2}$ , then  $f(x) = 6k + 1$ . If  $k + \frac{1}{2} \leq x < k + \frac{2}{3}$ , then  $f(x) = 6k + 2$ . If  $k + \frac{2}{3} \leq x < k + 1$ , then  $f(x) = 6k + 3$ . There is therefore only a solution if  $n$  is 0, 1, 2, or 3 mod 6; there are  $2004 \cdot \frac{4}{6} + 3$  of these.

5. **Answer:**  $\frac{5}{144}$

There are clearly five correct guesses; counting the number of possible guesses is the difficult part. A possible guess  $q$  is  $\pm 1$  times a divisor of 90 divided by a divisor of 400. We count these by extending the idea of prime factorization: from the factorizations of 90 and 400: we have  $q = 2^i 3^j 5^k$  where  $-4 \leq i \leq 1$ ,  $0 \leq j \leq 2$ , and  $-2 \leq k \leq 1$ . There are thus  $6 \cdot 3 \cdot 4 = 72$  possible fractions making 144 possible guesses.

6. **Answer:** 881

We can factor  $4x^4 + y^4 = (4x^4 + 4x^2y^2 + y^4) - 4x^2y^2 = (2x^2 + y^2)^2 - (2xy)^2 = (2x^2 + 2xy + y^2)(2x^2 - 2xy + y^2)$ . Since  $4^9 + 9^4 = 4(16)^4 + 9^4$ , we plug in to obtain the factoring  $881 \cdot 305$ . Quick checking (up to 29) shows 881 to be prime.

7. **Answer:** 6

The expression is 6 times the arithmetic mean of the terms, which is always greater than or equal to the geometric mean, which is  $xy \cdot x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{xy} = 1$ . The minimum is achieved when all terms are equal, i.e.  $x = y = 1$ .

8. **Answer:** 4

$(r + s + t)^3 - 3(r + s + t)(r^2 + s^2 + t^2) + 2(r^3 + s^3 + t^3) = 6rst$  - just plug in!

9. **Answer:**  $\sqrt{5}$

Note that the equations reduce by substitution to  $a = b + \frac{1}{a+1/a}$  and  $b = a - \frac{1}{b+1/b}$ . Solving the second for  $a$ , substituting into the first, and reducing yields  $b^4 + b^2 - 1 = 0$ ; solving this as a quadratic in  $b^2$  yields only one positive value for  $b^2 = \frac{\sqrt{5}-1}{2}$ . Plugging back in and solving for  $a$  gives  $a^2 = \frac{\sqrt{5}+1}{2}$ .

10. **Answer:**  $-2015028$

Note that  $(x + 1)^2 - x^2 = 2x + 1$  so:

$$\begin{aligned}\sum_{k=1}^{2007} (-1)^k k^2 &= -2007^2 + \sum_{k=1}^{1003} (2(2k-1) + 1) \\ &= -2007^2 + 4 \frac{1003 \cdot 1003}{2} + 1003 \\ &= -2007^2 + 1003 \cdot 2007 = 2007(1003 - 2007)\end{aligned}$$