Geometry Solutions
2005 Rice Math Tournament
February 26, 2005

1. **Answer: $800\pi ft^2$**

$$\frac{3}{4}\cdot 30^2\pi+\frac{1}{4}\cdot 10^2\pi+\frac{1}{4}\cdot 20^2\pi=800\pi$$

2. **Answer: $\frac{11}{12}$**

$$\begin{aligned}&1\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}+\cdots+\frac{1}{11}\cdot\frac{1}{12}\\&=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+\cdots+(\frac{1}{11}-\frac{1}{12})\\&=1-\frac{1}{12}\\&=\frac{11}{12}\end{aligned}$$

3. **Answer: 32**

Since all triangles are similar, $\overline{AE}:\overline{EB}=\overline{EB}:\overline{ED}$. Let $\frac{EB}{AE}=x$. Since $\overline{EB}^2=\overline{ED}^2+\overline{BD}^2$, $(16x)^2=8^2+(16x^2)^2$, $\therefore x=\frac{\sqrt{2}}{2}$. Since $\overline{AE}=16$, $\overline{BD}=16x^2$, the next vertical segment is $16x^2\times x^2$, and so on. $\therefore$ sum of all vertical segments is a geometric series $\frac{16}{1-x^2}=32$.

4. **Answer: $(0,\frac{5}{4})$**

Let the center have y-coordinate y_0.
The circle must have *exactly* two points of intersection with the parabola and $y_0>1$.
Thus $x^2+(x^2-y_0)^2=1$ (from the point (x_0,x_0^2) on the parabola) has exactly two solutions.
$x^4+(-2y_0+1)x^2+(y_0^2-1)=0$ has two double roots: $x+a$, $x-a$.

$$x^4+(1-2y_0)x^2+(y_0^2-1)=(x^2-a^2)^2$$

$$y_0^2-1=\left(\frac{1-2y_0}{2}\right)^2$$

$$y_0^2-1=y_0^2-y_0+\frac{1}{4}$$

$$y_0=\frac{5}{4}$$

5. **Answer: $\frac{1}{169}$**

Call the side length of the smaller hexagon a. Then

$$\left(\frac{a}{2}\right)^2+\left(\frac{r\sqrt{3}}{2}+a\sqrt{3}\right)^2=r^2$$

$$\Rightarrow 13a^2+12ar-r^2=0$$

$$\Rightarrow 13\left(\frac{a}{r}\right)^2+12\left(\frac{a}{r}\right)-1=0$$

$$\Rightarrow \frac{a}{r}=\frac{1}{13}$$

So the ratio of the areas is $\frac{1}{169}$.

6. **Answer: 0**

   By power of a point,
   $$a(c+b) = 17 \cdot 13$$
   and
   $$b(a+c) = 13 \cdot 17.$$
   So
   $$ac + ab = ab + bc$$
   $$ab = bc$$
   $$a = c$$
   Then $|MR - NS| = 0$.

7. **Answer: $\frac{2}{3}$**

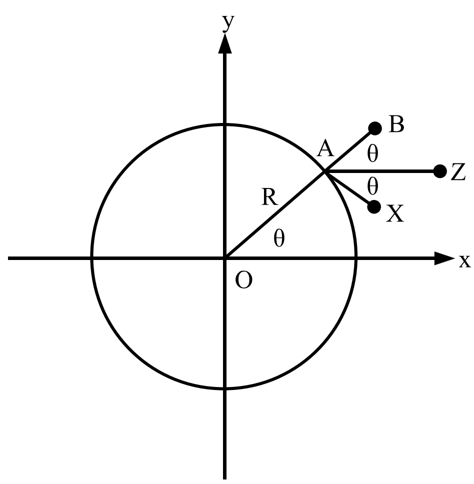
   Note that $\triangle AQR$ is similar to $\triangle ABC$. Also, the union of triangles $\triangle PQQ'$ and $\triangle RR'S$ is a triangle similar to $\triangle ABC$. The same is true for $\triangle BPP'$ and $\triangle SS'C$. So the total area enclosed by the triangles is $a^2 + b^2 + c^2$ where a, b, and c are the side lengths of AQ, PQ and BP respectively. The area enclosed by the two rectangles is maximized when that of the triangles is minimized. We know $a+b+c = 1$, and it is not hard to show that $a = b = c = \frac{1}{3}$ when this happens. It follows that the area enclosed by the triangles is $(\frac{1}{9} + \frac{1}{9} + \frac{1}{9})$ times the area of $\triangle ABC$. The maximum area of the rectangles is therefore $\frac{2}{3}$ that of $\triangle ABC$.

8. **Answer: 118**

   It is sufficient to calculate E. Consider the circle's curve only in the first quadrant starting in the bottom right corner, the path moves up through 20 squares and to the left through 20. Since the curve contains no lattice points ($x^2 + y^2 \neq 20.05^2$ for $x, y \in Z$) it passes through 41 $(20 + 20 + 1)$ squares in total.
   Hence $E = 4 \cdot 41 = 164$.
   So $I \approx 1260 - \frac{164}{2} = 1260 - 82 = 1188$
   So $\lfloor \frac{I}{10} \rfloor = 118$.

9. **Answer: 9π**

   Draw the horizontal line $\overline{AZ}$ (bisects $\angle BAX$). Find coordinates of x : $x = Rcos\theta + rcos(-\theta) = (R+r)cos\theta$; Find coordinates of y : $y = Rsin\theta + rsin(-\theta) = (R-r)cos\theta$. The polar equations of an elipse are of the form $x = acos\theta$, $y = bsin\theta$. And its area $A = ab\pi = (R+r)(R-r)\pi = \frac{1}{4}R^2\pi = 9\pi$



10. **Answer: $\frac{4}{27}$**

Center it at $(0,0,0),(a,0,0),(0,b,0),(a,b,c),(0,0,c)$, etc. Consider the line l to connect $(a,0,0)$ & $(0,b,c)$. Pick K on l to be (p,q,r). Then $\frac{b}{c}=\frac{r}{q}$ by projection onto the y-z axis. So $pqr=\left(pq^2\right)\left(\frac{b}{c}\right)$. Also, $(q-c)=\left(-\frac{c}{a}\right)(p)$; $q=c-\frac{c}{a}p \Rightarrow p=\frac{a}{c}(c-q)$; $p=a-\frac{a}{c}q$ (by projection onto x-y plane). So $V=pqr=pq^2\left(\frac{b}{c}\right)=\left(\frac{b}{c}\right)\left(q^2\right)(a)\left(1-\frac{1}{c}q\right)=(abc)\left(\frac{q}{c}\right)^2\left(1-\frac{q}{c}\right)=(abc)\,u^2\,(1-u)$ where $0<u<1$. To maximize u^2, try $u=\frac{2}{3}+d\left(d\in\left\{-\frac{2}{3},\frac{1}{3}\right\}\right)$

$$\begin{aligned}
\text{So } u^2(1-u) &= \left(d+\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}-d\right)\\
&= \left(d^2+\tfrac{4}{3}d+\tfrac{4}{9}\right)\left(\tfrac{1}{3}-d\right)\\
&= \tfrac{1}{3}\left(d^2+\tfrac{4}{3}d\right)+\tfrac{4}{27}-d^3-\tfrac{4}{3}d^2-\tfrac{4}{9}d\\
&= \tfrac{4}{27}+\left(-d^3-d^2\right)\leq\tfrac{4}{27}\text{ or } -\tfrac{2}{3}<d<\tfrac{1}{3}
\end{aligned}$$

Alternatively, Assume the vertices of the prism are $(0,0,0),(a,0,0),(0,b,0),(a,b,c),(0,0,c)$, etc. Let $C=(0,0,0)$ and l join $(a,b,0)$ to $(0,0,c)$. Then a point K on l has the form $(ta,tb,(1-t)c)$ for some $0\leq t\leq 1$. The resulting prism p' has volume $t^2(1-t)abc=t^2(1-t)$.