General Solutions
2004 Rice Math Tournament
February 28, 2004

1. **Answer: 24**

   Pretty straightforward. If a and b represent *this* and *that* respectively, $a + b = 3a^2$, and $\frac{b}{a} = 8$. It follows that $3a = a^2$, and $a = 0$ or 3. The former implies $b = 0$, contradicting $\frac{b}{a} = 8$ since $\frac{0}{0}$ is undefined. So $a = 3$, and $b = 24$, which is the answer.

2. **Answer: $\mathbf{\frac{8}{23}}$**

   The chances that either team wins are equal, so it's best to examine the probability that neither team wins. This occurs when there are three consecutive ties or one tie, one Rice victory and one Stanford victory. This yields 7 cases. There would be $3 \times 3 \times 3 = 27$ total possible cases, but a third game is not played if a team wins the first two. So there are only 23 total cases. A team victory occurs in $23 - 7 = 16$ of these cases, and Rice wins in half of them. The answer is $\frac{8}{23}$.

3. **Answer: 2**

   $2n - 2 = n!$ Try the first few values.

   (a) $2n - 2 = 0$, $1! = 1$

   (b) $2n - 2 = 2$, $2! = 2$

   (c) $2n - 2 = 4$, $3! = 6$

   Now as we increase n, $n!$ increases by much more than $2n - 2$. So the only solution is 2.

4. **Answer: $(\mathbf{2, 1, 5, 2})$**

   Since they're all positive and since $3 > \frac{37}{13} > 2$, it's a good idea to let $a = 2$. It follows that $b + \frac{1}{c+\frac{1}{d}}$ is between 1 and 2, so we should let $b = 1$. Likewise, one can confirm that $c = 5$ and $d = 2$. It is easy to see that this is the only solution.

5. **Answer: $\mathbf{\frac{\pi}{4+3\pi}}$**

   The area of the figure is the sum of the areas of the circles and that of the square less the overlap area. The overlap area is simply four congruent quarters of a circle. It is easy to see that the answer is independent of the radius of the circle. We get $\frac{\pi}{4+3\pi}$.

6. **Answer: 31**

   We can prove by induction on n that the following pattern holds for $0 \leq n \leq 499$: after $2n$ minutes, the first room contains $1000 - 2n$ people and the next n rooms each contain 2 people, and after $2n + 1$ minutes, the first room contains $1000 - (2n + 1)$ people, the next n rooms each contain 2 people, and the next room after that contains 1 person. So, after 60 minutes, we have one room with 940 people and 30 rooms with 2 people each.

7. **Answer: 9**

   Suppose the number n has $k + 1$ digits, the first of which is d. Then the number is at least $d \cdot 10^k$. On the other hand, each of the digits after the first is at most 9, so the product of the digits is at most $d \cdot 9^k$. Thus, if n equals the product of its digits, then

   $$d \cdot 10^k \leq n \leq d \cdot 9^k$$

   which forces $k = 0$, i.e., the number has only one digit. So $n = 9$ is clearly the largest possible value.

8. **Answer: 2**

   Plug in $x_1 = x_2 = x_3 = x_4 = x_5 = 0$. Then the equation reads $f(0) = 5f(0) - 8$, so $4f(0) = 8$, so $f(0) = 2$.

9. **Answer: 21600**

   In the top row, you can mark any of the 6 squares that is not a corner. In the bottom row, you can then mark any of the 5 squares that is not a corner and not in the same column as the square just marked. Then, in the second row, you have 6 choices for a square not in the same column as either of the two squares already marked; then there are 5 choices remaining for the third row, and so on down to 1 for the seventh row, in which you make the last mark. Thus, altogether, there are $6 \cdot 5 \cdot (6 \cdot 5 \cdot \ldots \cdot 1) = 30 \cdot 6! = 30 \cdot 720 = 21600$ possible sets of squares.

10. **Answer: 5**

    If the sides are x and y, we have $2x + 2y = 10$, so $x + y = 5$, and $\sqrt{x^2 + y^2} = \sqrt{15}$, so $x^2 + y^2 = 15$. Squaring the first equation gives $x^2+2xy+y^2 = 25$, and subtracting the second equation gives $2xy = 10$, so the area is $xy = 5$.

11. **Answer: $(7, 6, 4, 1)$**

    Since $D < A$, when A is subtracted from D in the ones' column, there will be a borrow from C in the tens' column. Thus, $D + 10 - A = C$. Next, consider the subtraction in the tens' column, $(C - 1) - B$. Since $C < B$, there will be a borrow from the hundreds' column, so $(C - 1 + 10) - B = A$. In the hundreds' column, $B - 1 \geq C$, so we do not need to borrow from A in the thousands' column. Thus, $(B - 1) - C = D$ and $A - D = B$. We thus have a system of four equations in four variables A, B, C, D, and solving by standard methods (e.g. substitution) produces $(A, B, C, D) = (7, 6, 4, 1)$.

12. **Answer: 39/25** Draw the altitude from C to AE, intersecting line BD at K and line AE at L. Then CK is the altitude of triangle BCD, so triangles CKX and CLY are similar. Since $\frac{CY}{CX} = \frac{8}{5}$, $\frac{CL}{CK} = \frac{8}{5}$. Also triangles CKB and CLA are similar, so that $\frac{CA}{CB} = \frac{8}{5}$, and triangles BCD and ACE are similar, so that $\frac{AE}{BD} = \frac{8}{5}$. The area of ACE is $\frac{1}{2}(AE)(CL)$, and the area of BCD is $\frac{1}{2}(BD)(CK)$, so the ratio of the area of ACE to the area of BCD is $\frac{64}{25}$. Therefore, the ratio of the area of $ABDE$ to the area of BCD is $\frac{39}{25}$.

13. **Answer: 10**

    The number in the ith row, jth column will receive the numbers $10(i - 1) + j$ and $10(j - 1) + i$, so the question is how many pairs (i, j) $(1 \leq i, j \leq 10)$ will have

    $$101 = [10(i - 1) + j] + [10(j - 1) + i] \quad \Leftrightarrow \quad 121 = 11i + 11j = 11(i + j).$$

    Now it is clear that this is achieved by the ten pairs $(1, 10), (2, 9), (3, 8), \ldots, (10, 1)$ and no others.

14. **Answer: 7/15**

    This is a case of conditional probability; the answer is the probability that the first ball is red and the second ball is black, divided by the probability that the second ball is black.

    First, we compute the numerator. If the first ball is drawn from Urn A, we have a probability of $\frac{2}{6}$ of getting a red ball, then a probability of $\frac{1}{2}$ of drawing the second ball from Urn B, and a further probability of $\frac{3}{6}$ of drawing a black ball. If the first ball is drawn from Urn B, we have probability $\frac{3}{6}$ of getting a red ball, then $\frac{1}{2}$ of drawing the second ball from Urn B, and $\frac{3}{5}$ of getting a black ball. So our numerator is

    $$\frac{1}{2}\left(\frac{2}{6} \cdot \frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{1}{2} \cdot \frac{3}{5}\right) = \frac{7}{60}.$$

    We similarly compute the denominator: if the first ball is drawn from Urn A, we have a probability of $\frac{1}{2}$ of drawing the second ball from Urn B, and $\frac{3}{6}$ of drawing a black ball. If the first ball is drawn from Urn B, then we have probability $\frac{3}{6}$ that it is red, in which case the second ball will be black with probability $\frac{1}{2} \cdot \frac{3}{5}$, and probability $\frac{3}{6}$ that the first ball is black, in which case the second is black with probability $\frac{1}{2} \cdot \frac{2}{5}$. So overall, our denominator is

$$\frac{1}{2}\left(\frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6}\left[\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5}\right]\right) = \frac{1}{4}.$$

Thus, the desired conditional probability is $\frac{\frac{7}{60}}{\frac{1}{4}} = \frac{7}{15}$.

15. **Answer: 8**

Let L be the union of all the lines of the tiling. Imagine walking from one end of the needle to the other. We enter a new triangle precisely when we cross one of the lines of the tiling. Therefore, the problem is equivalent to maximizing the number of times the needle crosses L. Now, the lines of the tiling each run in one of three directions. It is clear that the needle cannot cross more than three lines in any given direction, since the lines are a distance $\frac{\sqrt{3}}{2}$ apart and the needle would therefore have to be of length greater than $\frac{3\sqrt{3}}{2} > 2$. Moreover, it cannot cross three lines in each of two different directions. The closest two points of such opposite regions are at a distance of 2 (twice the length of a side of a triangle), so the needle cannot penetrate both regions.

Therefore, the needle can cross at most three lines in one direction and two lines in each of the other two directions, making for a maximum of $3 + 2 + 2 = 7$ crossings and $7 + 1 = 8$ triangles intersected.

16. **Answer: 6**

For positive integers a, b, we have

$$a! \mid b! \Leftrightarrow a! \leq b! \Leftrightarrow a \leq b.$$

Thus,

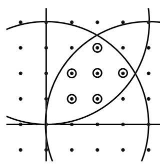
$$((n!)!)! \mid (2004!)! \Leftrightarrow (n!)! \leq 2004! \Leftrightarrow n! \leq 2004 \Leftrightarrow n \leq 6.$$

17. **Answer: $\frac{1}{4}$**

The only way that Andrea can ever flip HH is if she never flips T, in which case she must flip two heads immediately at the beginning. This happens with probability $\frac{1}{4}$.

18. **Answer: 6**

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0, 0), (4, 0), (0, 4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



19. **Answer: 0**

The horse must alternate white and black squares, and it ends on the same square where it started. Thus it lands on the same number of black squares (b) as white squares (w). Thus, its net earnings will be $2b - (b + w) = b - w = 0$ carrots, regardless of its path.

20. **Answer: 315**

Putting 8 people into 4 pairs and putting those 4 pairs into 2 pairs of pairs are independent. If the people are numbered from 1 to 8, there are 7 ways to choose the person to pair with person 1. Then there are 5 ways to choose the person to pair with the person who has the lowest remaining number, 3 ways to choose the next, and 1 way to choose the last (because there are only 2 people remaining). Thus, there are $7 \cdot 5 \cdot 3 \cdot 1$ ways to assign 8 people to pairs and similarly there are $3 \cdot 1$ ways to assign 4 pairs to 2 pairs of pairs, so there are $7 \cdot 5 \cdot 3 \cdot 3 = 315$ ways.

21. **Answer: 1**

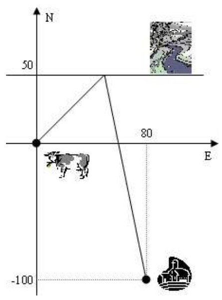
Consider the following addition:

$$\begin{array}{cr} & 111\ldots100\ldots01 \\ + & \underline{11\ldots11} \\ = & 1000\ldots\ldots\ldots00 \end{array}$$

By making the blocks of 1's and 0's appropriately long, we can ensure that the addends respectively contain 2004 and 2005 1's. (To be precise, we get $a = 2^{4008} - 2^{2005} + 1$ and $b = 2^{2005} - 1$.) Then the sum has only one 1. And clearly it is not possible to get any less than one 1.

22. **Answer: $40\sqrt{29}$**

Suppose we move the barn to its reflection across the river's edge. Then paths from the origin to the river and then to the old barn location correspond to paths from the origin to the river and then to the new barn location, by reflecting the second part of the path across the river, and corresponding paths have the same length. Now the shortest path from the origin to the river and then to the new barn location is a straight line. The new barn location is 200 feet north and 80 feet east of the origin, so the value of d is $\sqrt{200^2 + 80^2} = 40\sqrt{29}$.



23. **Answer: 62.5or$\mathbf{\frac{125}{2}}$**

Julie's distance is (10 mph) $\cdot$ ($\frac{6}{5}$ hrs) = 12 miles. Jim's walking distance, after falling off the train, is (3.5 mph) $\cdot$ (1 hr) = 3.5 miles at a right angle to the road. Therefore, Jim rode the train for $\sqrt{12^2 + 3.5^2} = \frac{1}{2}\sqrt{24^2 + 7^2} = \frac{25}{2}$ miles, and its speed is ($\frac{25}{2}$ mi)/($\frac{1}{5}$ hr) = 62.5 mph.

24. **Answer: $\mathbf{\frac{1}{2}}$**

Imagine placing the tetrahedron $ABCD$ flat on a table with vertex A at the top. By vectors or otherwise, we see that the center is $\frac{3}{4}$ of the way from A to the bottom face, so the reflection of this face lies in a horizontal plane halfway between A and BCD. In particular, it cuts off the smaller tetrahedron obtained by scaling the original tetrahedron by a factor of $\frac{1}{2}$ about A. Similarly, the reflections of the other three faces cut off tetrahedra obtained by scaling $ABCD$ by $\frac{1}{2}$ about B, C, and D. On the other hand, the octahedral piece remaining remaining after we remove these four smaller tetrahedra is in the intersection of $ABCD$ with its reflection, since the reflection sends this piece to itself. So the answer we seek is just the volume of this piece, which is

$$(\text{volume of } ABCD) - 4 \cdot (\text{volume of } ABCD \text{ scaled by a factor of } 1/2) = 1 - 4(1/2)^3 = 1/2.$$

25. **Answer:** $\mathbf{\frac{3}{4}}$

If Johann picks the point (a, b), the path will contain $\gcd(a, 2004 - b) + 1$ points. There will be an odd number of points in the path if $\gcd(a, 2004 - b)$ is even, which is true if and only if a and b are both even. Since there are 49^2 points with a, b both even and 98^2 total points, the probability that the path contains an even number of points is

$$\frac{98^2 - 49^2}{98^2} = \frac{49^2(2^2 - 1^2)}{49^2(2^2)} = \frac{3}{4}.$$