The price of anarchy of serial cost sharing and other methods

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August 2005

Abstract

Users share an increasing marginal cost technology. A method charges non negative cost shares covering costs. We look at the worst surplus (relative to the efficient surplus) in a Nash equilibrium of the demand game, where the minimum is taken over all convex preferences quasi-linear in cost shares. We compare two budget-balanced methods, *average cost pricing* and *serial cost sharing*, and two methods collecting a budget surplus, *marginal cost pricing* and *incremental cost sharing*. In the latter case we count the budget surplus as a loss.

For any convex cost function, the average cost and serial methods guarantee a (relative) surplus no less than $\frac{1}{n}$, where n is the number of users. Neither marginal cost pricing, nor incremental cost sharing guarantees any positive gain.

With quadratic costs, the surplus guaranteed by serial cost sharing is $O(\frac{1}{\log n})$, and $O(\frac{1}{n})$ for the three other methods. This generalizes if the marginal cost is convex or concave, and its elasticity is bounded.

Key words: price of anarchy, cost sharing, average cost, serial cost, marginal cost, incremental cost.

Acknowledgments Support from the NSF, under grant SES-0414543, is gratefully acknowledged.The comments of seminar particiants at Microsoft Research have been very helpful.

1 Introduction

The "price of anarchy" is a performance index for resource allocation mechanisms recently introduced in the context of congestion games ([14]). Agents choose a route in a given network. These choices determine the individual delays, or congestion costs measured in some numeraire other than time. *Selfish routing* is the non cooperative game where agents minimize their own delay/cost. The price of anarchy is the ratio between total delay at the worst Nash equilibrium of this game, and the *efficient* delay, i.e., the minimum feasible total delay. A remarkable series of results computes this ratio for simple yet realistic specifications of the congestion game (see [7],[8],[25], and references therein) and offers some insights into the network design minimizing this index ([1],[24]).

The next step is to extend this index to *game forms*. A game form defines a set of messages/actions/strategies for each participant, and the outcome (allocation of resources) resulting from any profile of strategies. When we add to a game form a description of the agents' preferences over the outcomes, we have a bona fide game in strategic form. The crucial assumption on preferences is *quasi-linearity:* they are represented by utility functions linear in a certain commodity called the *numeraire.* With respect to a particular outcome prevailing before the game is played (the "status quo ex ante", or "disagreement" outcome), we can then speak of the *surplus (net utility gain,* measured in the numeraire) collected at a particular Nash equilibrium of the game, and compare it to the *efficient* surplus, namely the maximal feasible surplus at the utility profile in question, achievable by full cooperation of the players (or an omniscient manager). We call the ratio of these two numbers the *(relative) surplus* for this particular equilibrium in this particular game.

Now we evaluate a game form by the *worst* relative surplus (or its lower bound) where the minimization bears on all admissible (quasi-linear) preference profiles, and all Nash equilibrium outcomes in the resulting games. We call this ratio the *guaranteed surplus.* This seemingly crude computation eschews the equilibrium selection problem, and delivers a complete ordering of game forms that does not rely on the artificial choice of a probability distribution on individual types (i.e., utility functions). Beyond this ordinal information, the worst relative surplus also has cardinal meaning.

The first game forms to be evaluated in this way were the familiar *output-sharing methods* ([10],[12],[27],[36]) and *cost-sharing methods* ([3],[11]). This paper is a contribution to - and a critique of- this emerging literature. Below we describe these game forms and the scope of this paper, before formulating our critique.

A set of users share a one-input, one-output technology with increasing marginal costs. A *cost sharing* method is a simple game form to manage these commons while respecting the principle of *consumer sovereignty*: each user requests an arbitrary quantity of output/service (his strategy), then the method collects individual payments the sum of which is sufficient to cover the cost of meeting all individual demands. By a slight abuse of language, we call these payments *cost shares* even if their sum exceeds the actual cost of production. In the dual

output sharing model, users choose the amount of input (e.g., labor) they contribute to the production process, and the method allocates shares of output, the sum of which does not exceed the actual output produced.

Classic examples of cost sharing methods include the regulation of congestion by fees: peakload pricing [30], telephone billing [2], access to a network [20], capacity-sharing in networks [3],[11]. Here the assumption of quasi-linearity in money is plausible if income effects are negligible (see [16]). A more recent application is to *queuing games*: individual demands are the size of a single job, or the rate of a random flow of small jobs, and the cost is the resulting delay before completion of these jobs: [6],[18],[29]. Here the numeraire is time and quasi-linearity means linear waiting costs.

Output sharing methods are the traditional models of exploitation of exhaustible resources ([5],[9],[23]), and production cooperatives ([28],[35]). Other examples closer to home include congestion games on a network, where agents bid (input) for capacity (output): [10],[12],[27],[36]. The design issue is precisely the choice of an output sharing method.

We focus here on cost sharing methods. A parallel discussion of output sharing methods is not conceptually problematic, yet the computations are different enough that further work is needed to systematically compare the two models: see the discussion in section 6.2.

The ultimate goal of our approach is to find, for a given cost function and number of users, the cost sharing method(s) with the highest guaranteed surplus. Our results throw some light on this fascinating question, but fall very short of a complete answer. We only compare the surplus guaranteed by four very important methods.

Two of these methods are *budget balanced*, while the other two generate a *budget surplus*, i.e., they charge more than the actual cost of production. In the game theoretical literature on cost sharing, most of the attention is on budget balanced methods, which is only natural when the focus is fairness[1], less so when incentives matter. For instance under the *incremental* method (one of the two methods overcharging discussed here), the efficient quantity of output is always produced in any Nash equilibrium; this property is out of reach for budget balanced methods.

The two budget balanced methods, *Average cost pricing (ACP)* and *Serial cost sharing (SER)*, are the methods most often discussed in the literature, both for their axiomatic properties of fairness ([19],[21],[31]) and for the strategic properties of the associated demand games ([18],[22],[34]). One of our two non budget balanced methods, *Marginal cost pricing (MCP),* is often discussed both for its axiomatic ([26],[33]) and incentives ([3],[11]) properties. Finally *Incremental cost sharing (INC)*, though less familiar, is equally natural: each agent pays the difference between total cost and the cost of serving the demand of all other agents.

We are ready to criticize the recent literature ([3],[10],[11],[12],[27],[36]) from which this paper is born. The key is the accounting of the overcharge. We count

[1] Two surveys on the axiomatic discussion of cost sharing are [17] and [33].

this excess payment as a surplus loss, namely an additional cost that never returns into the agents' pockets. These authors take the opposite viewpoint, they ignore the overcharge in the computation of the equilibrium surplus. In other words, the latter is computed as the sum of individual utilities for the output requested by each agent, minus the *actual* cost of total demand, even though the agents are charged *more* than this cost.

Call our computation of the surplus *parsimonious*, and the alternative one *optimistic*. Firstly the claim that the budget surplus can be redistributed among users without affecting the incentives (see [3]), is logically untenable: a rational agent anticipates the corresponding cash rebate and takes it into account to choose her optimal level of demand. The most sensible justification of the optimistic computation is that the excess payment goes to an outside agent (the manager of the method, or some entity using the money in the "public interest") whose welfare matters as much as that of the actual users of the technology. However this interpretation makes sense only if the users are not controlling the choice of the cost sharing method. We adopt instead the standard viewpoint of mechanism design: users jointly choose the c.s. method and compare different methods by the net utility they deliver.

Under the optimistic evaluation of its equilibrium surplus, the marginal cost pricing rule comes out in a much more favorable light than in the results reported below. For instance with quadratic costs, MCP guarantees a relative surplus of more than $\frac{2}{3}$ ([3],[11]), irrespective of the number of users, whereas in the parsimonious computation, MCP only guarantees $\frac{2}{n+3}$ (Proposition 3). The contrast is even more striking for the incremental cost sharing method. In the parsimonious evaluation, INC guarantees a relative surplus $\frac{1}{n}$ (again with quadratic costs), but in the optimistic computation, INC loses no surplus at all (recall that it yields the efficient profile of output demands in equilibrium), so its price of anarchy is zero! This holds for *any* convex cost function, therefore this simple cost sharing method, and the corresponding *output sharing* method (see section 6.2) are first best optimal in the optimistic evaluation, which all but puts this approach to rest.

Summary of the results

We find that for a given cost function C, the (relative) surplus guaranteed by our four methods goes to zero as n, the number of users, goes to infinity. But the asymptotic behavior of the serial method is considerably superior to that of the other three. For a large class of cost functions, the surplus guaranteed by the serial method is $O(\frac{1}{\log n})$, whereas that of the three other methods is $O(\frac{1}{n})$ (Theorem 1 in section 5.3). The cost function must be concave with elasticity bounded from below ($\inf_z \frac{zC''(z)}{C'(z)} > 0$), or convex with elasticity bounded from above ($\sup_z \frac{zC''(z)}{C'(z)} < \infty$). This result is reminiscent of the findings in [4] and [13], in the simpler model where each user demands at most one indivisible unit of output, and service can be random[2].

[2] [4] compares Average cost sharing and the Random Priority method, akin to serial cost sharing, and find that RP unambiguously dominates ACP when the commons is increasingly crowded, which is modelled as a replication process. [13] shows that the worst *absolute* surplus

We also compute the guaranteed surplus of our four methods for the natural "power" cost functions $C(z) = z^q, 1 < q < \infty$. The ranking of the methods depends much on the degree of convexity of C, namely q. With two agents, $n = 2$, the serial method wins for $1 < q < 1.36$, average cost wins for $1.36 < q < 6.4$, and the incremental method wins for $6.4 < q$ (section 5.2).

Finally we show (section 4) that the average cost and serial methods offer a "universal" guarantee of $\frac{1}{n}$: irrespective of the convex cost function C, the (relative) surplus of either method is no less than $\frac{1}{n}$. Neither one of the other two methods offers any positive guarantee independent of C.

Our results open many more questions than they answer. The main challenge is to find, for a given C and n, the cost sharing method(s) with the highest guaranteed surplus gain $\gamma(n, C)$. Short of a complete answer to this question, we want to know if the best universal guarantee, namely the infimum of $\gamma(n, C)$ over all increasing convex cost functions, is greater than $\frac{1}{n}$; and how the asymptotic behavior of $\gamma(n, C)$ when n goes to infinity compares to $O(\frac{1}{\log n})$. We comment on this and other open questions in section 6.

2 The model

We make the following assumptions on the cost function

$$C(0) = 0; C \text{ is weakly increasing, convex, and } \lim_{\infty} \frac{C(z)}{z} = \infty$$

We write the set of potential users of the technology as $N = \{1, 2, .., n\}$.

Definition 1
A cost sharing method for n agents and the cost function C is a mapping $\xi : x \in \mathbb{R}^n_+ \to y \in \mathbb{R}^n_+$ such that

- *$y_N = \sum_N \xi_i(x) \geq C(x_N)$ for all x: costs are covered (possibly in excess),*
- *ξ is symmetric in all variables: no discrimination among agents,*
- *for all i and x_{-i}, the cost share function $x_i \to y_i = \xi_i(x)$ is convex, increasing, and $\xi_i(0, x_{-i}) = 0$ (no charge for zero demand).*

The first inequality is the technological constraint. The symmetry assumption is for simplicity (the four methods we discuss are symmetric). The monotonicity and convexity assumptions on cost shares are crucial to the existence of an equilibrium (see below). Finally "zero charge for zero demand" is without loss of generality: indeed, if a zero demand agent can be charged (as in the natural *equal split* method $\xi_i(x) = \frac{1}{n}C(x_N)$), it is easy to pick a utility profile u for which the corresponding Nash equilibrium collects "negative" surplus. Such a method can therefore be discarded in the present discussion.

loss is always higher under ACP than under RP.

Agent i's preferences are quasi-linear in cost shares, hence represented by a utility function $U_i(x_i) - y_i$, where we assume

$$U_i(0) = 0,\ U_i \text{ is weakly increasing and concave}$$

We write U for a profile of utility functions $U = (U_i; i \in N)$.

The status quo ex ante is that no one demands any service, $x_i = 0$, and no one is charged $y_i = 0$. The **surplus** at the demand profile x and utility profile U is $s(n, C, x, U) = \{\sum_N U_i(x_i)\} - C(x_N)$, and the **efficient surplus** given C and U is

$$es(n, C, U) = \sup_{x \in \mathbb{R}^n_+} s(n, C, x, U) \tag{1}$$

Note that the efficient surplus is always non negative, because our normalizations $U_i(0) = 0$ and $C(0) = 0$ imply $s(n, C, 0, U) = 0$. We can assume $es(n, C, U) > 0$, because the case of a null efficient surplus is trivial.

For any utility profile U, and a given c.s. method ξ, in the **demand game** (n, C, ξ, U) agents choose their demands $x_j, j \in N$, and the resulting payoff to agent i is $U_i(x_i) - \xi_i(x)$. Concavity of U_i, convexity of $x_i \to y_i = \xi_i(x)$, and $\lim_\infty \frac{C(z)}{z} = \infty$, imply that the set $\mathcal{NE}(n, C, \xi, U)$ of Nash equilibrium demand profiles is non empty.

Zero charge for zero demand allows any agent i to guarantee zero net utility by demanding $x_i^* = 0$. Thus at any Nash equilibrium outcome x, the (absolute) surplus $\sum_N U_i(x_i) - y_N$ is non negative. Note that this surplus is $s(n, C, x, U)$ if the method ξ is *budget balanced* ($\sum_N \xi_i(x) = C(x_N)$ for all x), otherwise it is smaller. Our performance index is the worst relative surplus at any Nash equilibrium of the demand game, over all quasi-linear utility profiles.

Definition 2

*Given n and C, the **guaranteed (relative) surplus** of the method ξ is:*

$$\gamma(n, C, \xi) = \inf_{U, x \in \mathcal{NE}(n, C, \xi, U)} \frac{\sum_N U_i(x_i) - \sum_N \xi_i(x)}{es(n, C, U)} \tag{2}$$

where the infimum bears over all profiles U as defined above.

By construction, $0 \leq \gamma(n, C, \xi) \leq 1$. As no confusion may arise, we speak below of *surplus*, in lieu of *relative surplus*.We conclude this section with a simple result, due to Johari ([10]), that is key to the entire exercise.

Linearity Lemma

When computing the guaranteed surplus $\gamma(n, C, \xi)$, it is enough to assume for all i a linear utility function $U_i(x_i) = u_i \cdot x_i$, $u_i \in \mathbb{R}_+$.

Proof.

Fix N, a utility profile U, a cost function C, and a c.s. method ξ. Below we omit the variables n, C, and ξ, that remain fixed, so we write $\mathcal{NE}(U), es(U)$ instead of $\mathcal{NE}(n, C, \xi, U)$, $es(n, C, U)$ and so on.

Let z be an efficient demand profile, so that $es(U) = \{\sum_N U_i(z_i)\} - C(z_N)$, and let $x \in \mathcal{NE}(U)$ be a Nash equilibrium demand. Set $u_i = U_i'(x_i)$[3], and

[3]If U_i has a kink at x_i, use any of the supporting slopes.

observe that $x \in \mathcal{NE}(u)$ as well, because $y_i = \xi_i(x)$ is convex in x_i. Write $\sigma(x,U) = \sum_N U_i(x_i) - \sum_N \xi_i(x)$, and define $\sigma(x,u)$ similarly. We have

$$\sigma(x,u) = \sigma(x,U) - \sum_N \{U_i(x_i) - u_i \cdot x_i\} \tag{3}$$

$$U_i(x_i) - u_i \cdot x_i = \max_{a \geq 0}\{U_i(a) - u_i \cdot a\} \geq U_i(z_i) - u_i \cdot z_i$$

Denote $\delta(w) = \sum_N \{U_i(w_i) - u_i \cdot w_i\}$. By (3), $\delta(x) \leq \sigma(x,U) \leq es(U)$. If $\delta(x) = es(U)$, then $\sigma(x,u) = 0$ (by (3) again) and we are done. Now assume $\delta(x) < es(U)$ and compute

$$es(u) \geq s(z,u) = es(U) - \delta(z) \geq es(U) - \delta(x)$$

$$\Rightarrow \frac{\sigma(x,U)}{es(U)} \geq \frac{\sigma(x,U) - \delta(x)}{es(U) - \delta(x)} \geq \frac{\sigma(x,u)}{es(u)}$$

■

A first consequence of this result: when computing $\gamma(n,C,\xi)$, we can assume without loss of generality $C'(0) = 0$. For any C, set $\widetilde{C}(z) = C(z) - C'(0) \cdot z$. For any profile u of linear utilities, set $\widetilde{u_i} = (u_i - C'(0))_+$. Finally define $\widetilde{\xi_i}(x) = \xi_i(x) - C'(0) \cdot x_i$. Then $\sigma(n,C,\xi,x,u) = \sigma(n,\widetilde{C},\widetilde{\xi},x,\widetilde{u})$ and $es(C,u) = es(\widetilde{C},\widetilde{u})$ implying $\gamma(n,C,\xi) = \gamma(n,\widetilde{C},\widetilde{\xi})$.

3 Four cost sharing methods

We focus on the following four cost sharing methods:

- **Average cost pricing (ACP):** $y_i = \xi_i(x) = \frac{x_i}{x_N} \cdot C(x_N) = x_i \cdot AC(x_N)$,
- **Marginal cost pricing (MCP):** $y_i = \xi_i(x) = x_i \cdot C'(x_N)$,
- **Incremental cost sharing (INC):** $y_i = \xi_i(x) = C(x_N) - C(x_{N \setminus i})$,
- **Serial cost sharing (SER):**

  if $x_1 \leq x_2 \leq \cdot\cdot \leq x_n$, and $x^i = (n-i+1)x_i + \sum_{j=1}^{i-1} x_j$ for all i

$$y_i = \xi_i(x) = \frac{1}{n-i+1} C(x^i) - \sum_{j=1}^{i-1} \frac{1}{(n-j+1)(n-j)} C(x^j) \tag{4}$$

(other cases deduced from the symmetry of ξ).

All four methods are well defined and meet the three requirements in Definition 1, with one exception. The definition of MCP requires some fine tuning if C has a kink at x_N: then we interpret $C'(x_N)$ as the left derivative (the smallest supporting slope). Moreover agent i's cost share is convex in x_i if and only if

$z \to z \cdot C'(z)$ is convex, which does not necessarily follows from the convexity of C. We keep this additional assumption in mind when we discuss MCP.

Both ACP and SER are *budget-balanced* methods, $y_N = C(x_N)$. Recall that SER stands out uniquely for the strategic properties of the demand game, in which a single Nash equilibrium exists for any profile of convex preferences, not necessarily quasi-linear ([18]). When preferences are quasi-linear (and C is strictly convex), ACP also has a unique Nash equilibrium ([34]). In both demand games, the equilibrium involves *over*production (greater than the efficient level of production).

The equilibrium properties of MCP and INC are quite different: MCP *under*produces in equilibrium, while INC produces exactly the efficient quantity[4].

The discussion of sections 5 and 6 contrasts the behavior of the three methods ACP, MCP, INC on one hand, with that of SER on the other hand: see Proposition 2 and Theorem 1. A more elementary observation is that at any demand profile x, ACP charges less than INC, and the latter less than MCP:

$$\xi_i^{ACP}(x) \leq \xi_i^{INC}(x) \leq \xi_i^{MCP}(x) \text{ for any } i \in N \text{ and any } x$$

(this follows from convexity of C and $C(0) = 0$). Of course, these inequalities have no direct implicationf for ranking the surplus collected by these three methods in equilibrium. We now give examples of problems (n, C, u) where in turn, each one of the four methods collects the largest surplus in equilibrium[5].

In all examples, the utilities are linear. This simplifies computations: the efficient demand is $\max_i u_i$, and in an efficient outcome, only the highest utility agents are active ($x_i > 0$).

Consider first a *unanimous* profile, $u_i = u_0$ for all i. It is well known, and easy to check, that the (unique) equilibrium of SER is then efficient. On the other hand, both MCP and ACP have a unique and inefficient equilibrium. One checks easily that their surplus (both relative and absolute) is $O(\frac{1}{n})$ when C, u_0 are fixed. As for INC, its demand game has a continuum of equilibria, namely all profiles x such that $x_N = u_0$. The worst of these equilibria, surpluswise, is $x_i = \frac{1}{n}u_0$, its surplus is also $O(\frac{1}{n})$. Note that for a unanimous profile of strictly concave utilities, $U_i = U_j$ for all i, j, all statements above are preserved, with the only difference that the INC demand game now has a unique and symmetric equilibrium.

Next consider a profile u where $u_1 > \max_{i \geq 2} u_i$. Here the equilibrium of INC is unique and efficient, namely player 1 demands the efficient level u_1, all others are shut out (this does *not* hold if utilities are not linear). The equilibrium of ACP and MCP is the same only if u_1 is sufficiently larger than $\max_{i \geq 2} u_i$. For instance with quadratic costs, $C(z) = \frac{1}{2}z^2$ we need $u_1 \geq 2 \max_{i \geq 2} u_i$. If $u_1 < 2u_2$, at least agents 1 and 2 are active in equilibrium, and the latter is

[4] The F.O.C. in any Nash equilibrium x of INC are $U_i'(x_i) = C'(x_N)$, precisely as for the surplus maximizing demand profile. To prove the claim for MCP, note that the F.O.C. imply there $U_i'(x_i) \geq C'(x_N)$ for all i.

[5] At this point we are not computing the *guaranteed* surplus gain given n, C, only the surplus gain at a particular utility profile.

inefficient. In the equilibrium of SER, every agent is active implying inefficiency again.

Finally we provide two less general examples where successively ACP then MCP collects the largest surplus in equilibrium. We use the following notation for the relative surplus in a specific equilibrium $r = \frac{\sum_N u_i x_i - \sum_N \xi_i(x)}{es(C,u)}$.

Example 1: $n = 3; C(z) = \frac{1}{2}z^2; u_1 = u_2 = 15, u_3 = 8$. *ACP wins*

In the equilibria of ACP, MCP and INC, only players 1 and 2 are active. We have

ACP: $x_1 = x_2 = 10 \Rightarrow r = 89\%$

MCP: $x'_1 = x'_2 = 5 \Rightarrow r' = 44\%$

INC: $x''_1 = x''_2 = 7.5$ (worst equilibrium) $\Rightarrow r'' = 50\%$

The equilibrium demand profile of SER is $x'''_1 = x'''_2 = \frac{37}{6}$, $x'''_3 = \frac{8}{3}$ with $r''' = 83\%$.

Example 2: $n = 4; C(z) = \frac{1}{2}(z-3)^2$ if $z \geq 3, = 0$ if $z \leq 3; u_1 = u_2 = u_3 = 1, u_4 = 0.1$. *MCP wins*

The efficient total demand is $x = 4$, and $es(4, C, u) = 3.5$. In the equilibrium of MCP, the first three players each demand $x_i = 0.98$, and $x_4 = 0.08$. Then $r = 89\%$. Player 4 is shut out in ACP and INC. In the former, we find $x'_i = 1.87$ for $i = 1, 2, 3$ and $r' = 64\%$. In the latter, $x''_i = \frac{4}{3}$ and $r'' = 71\%$. Finally the equilibrium demand profile of SER is $x'''_1 = x'''_2 = x'''_3 = 1.075$, $x'''_4 = 0.775$, and $r''' = 80\%$.

4 Universal guarantees

Average cost pricing and serial cost sharing both guarantee a positive share of surplus, irrespective of the cost function. Neither the marginal cost nor the incremental method does.

Proposition 1

i) for any convex increasing cost function C and any n

$$\gamma(n, C, ACP) \geq \frac{1}{n}; \quad \gamma(n, C, SER) \geq \frac{1}{n} \tag{5}$$

ii) for any convex two-piece linear cost function C and any n

$$\gamma(n, C, ACP) = \gamma(n, C, SER) = \frac{1}{n}; \gamma(n, C, MCP) = \gamma(n, C, INC) = 0$$

Note our abuse of language: a two-pieces linear function fails $\lim_\infty \frac{C(z)}{z} = \infty$. Strictly speaking, statement *ii)* applies to any function in our domain that is two-pieces linear on some interval $[0, \overline{z}]$.

Proof

Statement i. To prove (5) for the serial method, we establish first an inequality interesting in its own right. In any Nash equilibrium of this method, each player is guaranteed at least $\frac{1}{n}$ of her *stand alone* surplus, namely the surplus she achieves by using the facility all by herself.

Formally, set $sa(C,U_i) = \max_{z\geq 0} U_i(z) - C(z)$. By concavity of U_i and $U_i(0)=0$, we have for all z

$$U_i(z) - C(z) \leq nU_i(\frac{z}{n}) - C(n\cdot\frac{z}{n}) \Rightarrow sa(C,U_i) \leq n\cdot\max_{z\geq 0} U_i(z) - \frac{1}{n}C(nz)$$

Because C is convex, the serial cost share is bounded by $\xi_i(x) \leq \frac{1}{n}C(nx)$. This is easy to check from (4). Fix U and $x \in \mathcal{NE}(n,C,SER,U)$, then compute

$$U_i(x_i) - \xi_i(x) = \max_{z_i\geq 0} U_i(z_i) - \xi_i(z_i,x_{-i}) \geq \max_{z\geq 0} U_i(z) - \frac{1}{n}C(nz) \geq \frac{1}{n}sa(C,U_i)$$

as claimed. Next

$$n\cdot s(n,C,x,U) \geq \sum_N sa(C,U_i) \geq es(n,C,U)$$

where the last inequality is a simple consequence of decreasing returns. This proves (5) for SER.

To prove (5) for ACP, and a given cost function C, it is enough (by the linearity Lemma) to consider a profile u of linear utilities, and $x \in \mathcal{NE}(n,C,ACP,u)$. Rename the players so that $1,2,..,m$ are active, $x_i > 0$, while $m+1,..,n$ are inactive, $x_j = 0$, and $u_1 = \max u_i$. The F.O.C. for our equilibrium read

$$u_i = \lambda_i C'(x_N) + (1-\lambda_i)AC(x_N) \text{ for } i = 1,..,m, \text{ with } \lambda_i = \frac{x_i}{x_N} \qquad (6)$$

Therefore

$$s(n,C,x,u) = (x_N\cdot C'(x_N) - C(x_N))\cdot(\sum_{i=1}^m \lambda_i^2) \qquad (7)$$

Setting $\Gamma(z) = z\cdot C'(z) - C(z)$, observe that Γ is increasing and that $es(n,C,u) = \Gamma(\{C'\}^{-1}(u_1))$. Equation (6) implies $u_1 \leq C'(x_N)$, therefore

$$es(n,C,u) \leq \Gamma(x_N)$$

Combine this inequality with (7), and with $\sum_{i=1}^m \lambda_i^2 \geq \frac{1}{m} \geq \frac{1}{n}$, to conclude the proof of statement i.

Statement ii. We can choose without loss of generality the cost function $C(z) = (z-1)_+$. Fix n and consider the linear utility profile $u_1 = 1, u_i = \varepsilon$ for $i \geq 2$. The unique equilibrium of SER is $x_i = \frac{1}{n}$ for all i, and $r = \frac{1}{n} + \frac{(n-1)}{n}\varepsilon$. In view of statement i), this implies $\gamma(n,C,SER) = \frac{1}{n}$.

Next, the demand profile $x'_1 = 0, x'_i = \frac{1}{n-1}$ for $i \geq 2$ is *one* of the equilibria in the INC and in the MCP game; its relative surplus is $r' = \varepsilon$. Finally consider the unanimity profile $u_i = 1-\varepsilon$ for all i, and check that the unique equilibrium of ACP is $x''_i = \frac{1}{\varepsilon}\frac{(n-1)}{n^2}$ for all i, with corresponding surplus $r'' = \frac{1}{n(1-\varepsilon)}$.■

5 Cost dependent guarantees

5.1 Two players and $C'(z)=z^p$

For the simple family of "power" cost functions, we compute explicitely the guaranteed surplus in the case of two agents.

Proposition 2

Fix p, $0 < p < \infty$, and define the cost function $C_p(z) = \frac{1}{p+1} z^{p+1} \Leftrightarrow C'(z) = z^p$. We have

$$\gamma(2, C_p, SER) = 1 - \frac{1}{2(p+1)^{\frac{1}{p}}}$$

$$\gamma(2, C_p, INC) = 1 - \frac{1}{p}(1 - \frac{1}{2^p})$$

$$\gamma(2, C_p, ACP) = (p+1)^{\frac{p+1}{p}} \frac{(2\lambda^2 - 2\lambda + 1)}{(1+p\lambda)^{\frac{p+1}{p}}}, \text{ where } \lambda = \frac{p+3}{3 + \sqrt[2]{2p^2+4p+3}}$$

$$\gamma(2, C_p, MCP) = \frac{1}{(p+1)^{\frac{1}{p}}} \gamma(2, C_p, ACP)$$

The proofs of Proposition 2, as well as Proposition 3 and Theorem 1 below, are gathered in section 7.

The above formulas are not very intuitive, and the reader should consult Figures 1 and 2, where the four functions of the variable p are drawn with two different scales.

The first observation is that the guaranteed surplus of SER *decreases* with p, from $1 - \frac{1}{2e} = 82\%$ for p near zero to 50% for p very large, whereas that of the other three methods *increases* throughout. The more convex the marginal cost, the lower the surplus guaranteed by SER, but the higher that guaranteed by the others. Among the latter three methods, ACP offers a remarkably flat guarantee, increasing from 77% for p near zero, to 83% for p very large. Contrast this with INC, starting as low as 31% , and reaching 100% when p goes to infinity.

For a concave marginal cost function, $0 < p < 1$, ACP and SER dominate INC, as they guarantee 50% more surplus than INC. ACP has the highest guarantee for $0.36 < p < 5.41$, where this ratio remains within 1% of 80%; for $0 < p < 0.36$ SER offers a barely higher guarantee than ACP; INC guarantees the highest surplus for $p > 5.41$, and this surplus is $1 - O(\frac{1}{p})$ for large p. Finally MCP is dominated by both ACP and INC on the entire range of p.

5.2 Quadratic costs and many players

Our main results, in this subsection and the next one, describe the asymptotic behavior of our performance index in the number n of users. Note that the equations giving the guaranteed surplus for ACP, MCP, and INC in Proposition 2 generalize easily to arbitrary n: see sections 7.2 to 7.4. This is not the case for SER, however, except in the simple and much studied case ([3],[9]) of a quadratic cost function.

Recall that $C(0) = 0$, and that the assumption $C'(0) = 0$ is without loss of generality. Thus our index is the same for all quadratic cost functions, such as C_1.

Proposition 3

In the following we write $K_n = 1 + \frac{1}{3} + .. + \frac{1}{2n-1} \simeq 1 + \frac{\log n}{2}$. *For all* n *we have*

$$\gamma(n, C_1, SER) = \frac{1}{K_n}$$

$$\gamma(n, C_1, ACP) = \frac{4}{n+3}, \gamma(n, C_1, MCP) = \frac{2}{n+3}, \gamma(n, C_1, INC) = \frac{1}{n}$$

For quadratic costs, ACP performs (exactly) twice better than MCP, and the latter (approximately) twice better than INC. But SER outperforms them all for $n \geq 5$

n	2	3	5	10	20	100	1000
$100\gamma(ACP)$	80	67	50	31	18	4	0.4
$100\gamma(SER)$	75	65	56	47	40	30	22

5.3 Main result

The asymptotic superiority of the serial method is now extended to a large class of cost functions.

Theorem 1

i) Suppose the marginal cost function C' *is concave, and*

$$\inf_z \frac{zC''(z)}{C'(z) - C'(0)} = p > 0 \tag{8}$$

Then we have for all n

$$\gamma(n, C, SER) \geq \frac{p}{K_n}$$

$$\gamma(n, C, ACP) \leq \frac{4}{n+3}, \gamma(n, C, MCP) \leq \frac{2}{n}, \gamma(n, C, INC) \leq \frac{4}{3n}$$

ii) Suppose the marginal cost function C' *is convex, and*

$$\sup_z \frac{zC''(z)}{C'(z) - C'(0)} = p < \infty \tag{9}$$

Note that $p \geq 1$. Then we have for all n

$$\gamma(n, C, SER) \geq \frac{1}{2p-1}\frac{1}{K_n}$$

$$\begin{aligned} \frac{4}{n+3} &\leq \gamma(n, C, ACP) \leq \frac{4(2p-1)}{n}, \\ \gamma(n, C, MCP) &\leq \frac{2p}{n}, \gamma(n, C, INC) \leq \frac{p}{n} \end{aligned}$$

The bounds on the *elasticity* $\frac{zC''}{C'}$ mean that the marginal cost is either "not too concave" or "not too convex" . Formally it amounts to assume that this function can be written $C'(z) = z^p \cdot \varphi(z)$, where φ is non decreasing (case C' concave), or non increasing (case C' convex). If the elasticity is unbounded, we do not claim any asymptotic superiority of the serial method.

From Proposition 2 and the Theorem, we see that convexity (resp. concavity) of the marginal cost increases (resp. decreases) the surplus guaranteed by ACP. I conjecture that the opposite statement holds for SER: if C' is concave, $\gamma(n, C, SER) \geq \frac{1}{K_n}$, and the opposite inequality if C' is convex.

A consequence of the Theorem is that for any n we can find cost functions C for which $\gamma(n, C, SER)$ is the best of all four methods. For $n \geq 4$, any C with convex marginal cost will do (statement i)); for $n = 2, 3$, we can take C_p for p small enough.

From the simple formula $\gamma(n, C_p, INC) = 1 - \frac{n-1}{p}(1 - (\frac{n-1}{n})^p)$ (section 7.4), we see that for any n we can also find cost functions such that $\gamma(n, C, INC)$ is largest. It is not clear whether the same statement holds for ACP (Proposition 2 proves this for $n = 2$) or for MCP.

6 Open questions and variants

6.1 The main challenge

As discussed in the Introduction, the biggest challenge is to solve (even approximately) the following problem for all n and C:

$$\gamma^*(n, C) = \sup_{\xi} \gamma(n, C, \xi) \tag{10}$$

where ξ is a c.s. method from Definition 1. Going beyond that Definition, we could include non symmetric c.s. methods as well, but it is unclear whether this would improve upon the optimal value of (10).

If a complete solution of (10) is not tractable, two less ambitious yet particularly interesting questions are a) the asymptotic behavior of $\gamma^*(n, C)$ when n is large but C is fixed, and b) the universal bound $\gamma^{**}(n) = \inf_C \gamma^*(n, C)$. On a) we only know that for many cost functions, $\gamma^*(n, C)$ is at least $O(\frac{1}{\log n})$; on b), that $\gamma^{**}(n) \geq \frac{1}{n}$.

6.2 Output sharing games

As discussed in the introduction, output sharing games, where players contribute input and the method distributes the resulting output, are the "dual" of cost sharing games, and equally important in the applications. All definitions are easily transposed, but only some of our results are preserved identically in this context.

Consider serial cost sharing. If F is an increasing production function, the serial output sharing formula is given precisely by the same equations (4) where F replaces C, and we exchange x and y. With the notation $y^i = (n-i+1)y_i + \sum_{j=1}^{i-1} y_j$ we can write the cost and output sharing versions of (4) in the more compact way:

$$\begin{array}{c} \text{cost sharing: } y^i = C(x^i) \text{ for all } i \\ \text{output sharing: } x^i = F(y^i) \text{ for all } i \end{array}$$

Consider now $C$and F, both increasing and inverse of each other: $t = C(z) \Leftrightarrow z = F(t)$ (implying $C(0) = F(0) = 0$). The two systems above are then equivalent (monotonicity of C and F ensures $\{x_1 \leq x_2 \leq \cdots\} \Leftrightarrow \{y_1 \leq y_2 \leq \cdots\}$), thus the output sharing method $\varsigma^{SER^*} : y \to x$ is the inverse of the cost sharing method $\xi^{SER} : x \to y$

$$x = \varsigma^{SER^*}(y) \Leftrightarrow y = \xi^{SER}(x) \tag{11}$$

Moreover, if $\{C \text{ is convex}\} \Leftrightarrow \{F \text{ is concave}\}$ and U is any utility profile, we claim that the (unique) Nash equilibrium x of the demand game (n, C, SER, U) is "the same" as the Nash equilibrium y of the supply game (n, F, SER^*, U), in the sense that the resulting outcomes (x, y) coincide. Indeed in the cost sharing version we have $\frac{\partial \xi_i}{\partial x_i}(x) = C'(x^i)$, and in the output sharing version $\frac{\partial \varsigma_i}{\partial y_i}(x) = F'(y^i)$, therefore the F.O.C. of the equilibria x and y of the two respective games are

$$U_i'(x_i) = C'(x^i) \Leftrightarrow U_i'(x_i) \cdot F'(y^i) = 1 \tag{12}$$

Hence the equilibrium surplus is the same in the two games and $\gamma(n, C, SER) = \gamma(n, F, SER^*)$. All results pertaining to serial cost sharing extend word for word to the serial output sharing, in particular the universal guarantee of $\frac{1}{n}$, and the asymptotic guarantee of $O(\frac{1}{\log n})$.

The situation is quite different for *average return* (AR), namely the output sharing method $x_i = \frac{y_i}{y_N} \cdot F(y_N) = y_i \cdot AF(y_N)$. It is the dual of average cost pricing, in the sense that if C and F are inverse of each other, then ξ^{ACP} and ς^{AR} are inverse of each other in the sense of (11). However, the F.O.C. of the two games are no longer the same system of equations, so that the equilibria of the two games (n, C, ACP, U) and (n, F, AR, U) are two different outcomes. In view of the results of [22], we expect a worse surplus performance from AR than ACP. If C is strictly convex (F strictly concave) they both have a unique Nash equilibrium, and the overproduction in the AR equilibrium is larger than

in the ACP equilibrium. Moreover for a unanimous utility profile the (unique) equilibrium of ACP is Pareto superior to that of AR. Thus a plausible conjecture is $\gamma(n, C, ACP) > \gamma(n, F, AR)$. Another open question is whether AR offers any positive universal guarantee?

Several more output sharing methods are worthy of our interest. The analogs of marginal cost pricing and incremental cost sharing are

- **Marginal return (MR):** $x_i = y_i \cdot F'(y_N)$
- **Incremental output sharing (ICO):** $x_i = F(y_N) - F(y_{N \setminus i})$

(we do not speak of "dual" methods here because the equivalence $y = \xi(x) \Leftrightarrow x = \varsigma(y)$ does not hold). These two methods distribute less output than actually produced by the total input.

Finally we mention two methods that have only been discussed as output-sharing methods. The *decreasing serial* method ([15],[32]) is defined by a formula similar to that of ς^{SER^*}, but where y_1 is the largest contribution instead of the smallest, and the output share of any agent does not depend upon the demands smaller than her own. And [12] introduces yet another method, intermediate between MR and AR. Further work is needed to understand the price of anarchy of these various methods under our parsimonious interpretation of the equilibrium surplus.

6.3 Output as numeraire

Another variant of our model picks for numeraire the output commodity, with respect to which preferences are quasi-linear.

The choice of the input as numeraire is natural, in particular when it is money. The alternative choice of the output is equally plausible in certain contexts.Consider a queuing game where user i requests an amount x_i of service and suffers a delay y_i. The linearity of utilities in output/service makes sense if the "service" can be measured by its market value. Then the utility takes the form $x_i - V_i(y_i)$, where the cost of delay V_i is increasing and convex. Under this alternative choice of the numeraire, it appears that the general thrust of our results is preserved, but the actual computations are interestingly different, and all formulas in Propositions 2 and 3 need to be reevaluated anew.

7 Proof of Propositions 2,3 and Theorem 1

Given n, C, a c.s. method ξ, and a demand profile x such that $x_i > 0$ for all i, there is a unique linear utility profile u such that x is an equilibrium of the corresponding demand game, namely $u_i = \frac{\partial \xi_i}{\partial x_i}(x)$. This equilibrium yields the surplus $\sigma(x, u) = \sum_N x_i \cdot \frac{\partial \xi_i}{\partial x_i}(x) - \xi_N(x)$. The efficient surplus in this economy is $es(u) = \Gamma(\{C'\}^{-1}(\frac{\partial \xi_{i^*}}{\partial x_{i^*}}(x)))$, where $\frac{\partial \xi_{i^*}}{\partial x_{i^*}}(x) = \max_i \frac{\partial \xi_i}{\partial x_i}(x)$, and $\Gamma(z) = z \cdot C'(z) - C(z)$ is the stand alone surplus of a linear utility $u_i = C'(z)$.

For all four methods discussed below, an equilibrium x such that $x_i = 0$ for some i, is also an equilibrium in the game with $n-1$ players, therefore $\gamma(n, C, \xi)$ is non increasing in n. Moroever, the computations below make it clear that it is strictly decreasing in all cases, so that there is no loss of generality in considering only strictly positive demand profiles x. We now rewrite the minimization problem leading to $\gamma(n, C, \xi)$, with the following notations

$$\begin{aligned} \gamma(n, C, \xi) &= \inf_{x \gg 0} \frac{\sigma(x)}{es(x)}; \, \sigma(x) = \sum_N x_i \cdot \frac{\partial \xi_i}{\partial x_i}(x) - \xi_N(x) \\ es(x) &= \Gamma(\{C'\}^{-1}(\max_i \frac{\partial \xi_i}{\partial x_i}(x))) \end{aligned} \tag{13}$$

7.1 Results for the serial method

We fix x such that $0 < x_1 \leq x_2 \leq \cdot\cdot \leq x_n$, and recall the notation $x^i = (n-i+1)x_i + \sum_{j=1}^{i-1} x_j$ for all $i \geq 1, x^0 = 0$. The key observation (see section 6.2) is $\frac{\partial \xi_i}{\partial x_i}(x) = C'(x^i)$ for all $i \geq 1$. In particular the total output $x^n = x_N$ is the efficient amount for the utility profile u where $u_i = \frac{\partial \xi_i}{\partial x_i}(x)$ for all i, because $\max u_i = u_n = C'(x_N)$. Note also that the efficient surplus is $\Gamma(x_N)$. Now we set for all i

$$a_i = x^i - x^{i-1} \Rightarrow x_i = \sum_1^i \frac{a_j}{n-j+1} \text{ and } b_i = C'(x^i) - C'(x^{i-1})$$

Routine computations give

$$\begin{aligned} es(x) &= \Gamma(x_N) = b_N \cdot a_N - C(a_N) \\ \sigma(x) &= es(x) - \sum_{i<j} \frac{j-i}{n-i+1} a_i b_j \end{aligned} \tag{14}$$

where the summation bears on all pairs (i, j) with $i < j$.

7.1.1 Proposition 2

For C_p we have $\Gamma(z) = \frac{p}{p+1} z^{p+1}$, and for $n = 2$, (14) gives

$$\frac{\sigma(x)}{es(x)} = 1 - \frac{1}{2} \frac{a_1 b_2}{\Gamma(a_{12})} = 1 - \frac{p+1}{2p} \lambda(1 - \lambda^p), \text{ where } \lambda = \frac{a_1}{a_{12}}$$

Maximizing $\lambda(1-\lambda^p)$ with respect to $\lambda, 0 \leq \lambda \leq 1$, leads to $\lambda^* = \frac{1}{(p+1)^{\frac{1}{p}}}$, and the desired formula for $\gamma(2, C_p, SER)$.

7.1.2 Proposition 3

With quadratic costs, $C'(z) = z$, we get $a_i = b_i$ and (14) becomes

$$\sigma(x) = \frac{1}{2}a_N^2 - \sum_{i<j} \frac{j-i}{n-i+1} a_i a_j = \frac{1}{2}\{\sum_N a_i^2 + 2\sum_{i<j} \frac{n-j+1}{n-i+1} a_i a_j\}$$

Because $es(x) = \frac{1}{2}a_N^2$, we see that $\gamma(n, C_1, SER) = 2\min\sigma(x)$, where $\sigma(x)$ is the above quadratic form in a, and the minimum is taken over $a \geq 0, a_N = 1$. Change variables to a', with $a'_i = \frac{1}{i}a_{n-i+1}$, so that ${}^t a' M a' = 2\sigma(x)$, with $M_{ij} = \min\{i, j\}$. Check that M is a positive definite quadratic form, because its k-th principal minor is $1 \cdot 3 \cdot \cdots \cdot (2k-1) > 0$.

Define the vector e, $e_i = i, i = 1, \cdots, n$. We are minimizing ${}^t a' M a'$ under the constraint $e \cdot a' = 1$. The optimal solution $a'_*$and optimal value are

$$a'_* = \frac{1}{{}^t e M^{-1} e} M^{-1} e;\ \gamma(n, C_1, SER) = \frac{1}{{}^t e M^{-1} e}$$

Finally compute

$$\begin{aligned} (M^{-1}e)_i &= \frac{1}{2i-1} - \frac{1}{2i+1} \text{ for } i \leq n-1,\ (M^{-1}e)_n = \frac{1}{2n-1} \\ &\Rightarrow\ {}^t e M^{-1} e = K_n \end{aligned}$$

Remark The vector x achieving the minimum of $\sigma(x)$ is simply $x_i = \frac{1}{2(n-i)+1}$. The corresponding linear utilities are

$$u_i = x^i = \frac{n-i+1}{2(n-i)+1} + \sum_1^{i-1} \frac{1}{2(n-j)+1}$$

7.1.3 Theorem 1, statement i)

Given C as in statement i) and x as in the beginning of section 7, define $AC' = \frac{C'(x_N)}{x_N}$ and consider the quadratic cost function $\widetilde{C}$: $\widetilde{C}'(z) = AC' \cdot z$. We compare the games (n, C, u) and $(n, \widetilde{C}, \widetilde{u})$ where $u_i = C'(x^i)$ and $\widetilde{u_i} = AC' \cdot x^i = \widetilde{C}'(x^i)$. By construction x is the unique equilibrium in both games, so that the demand increments a_i also coincide, but the utility increments b_i and $\widetilde{b_i}$, differ. Their comparison below is key to the proof.

Because $\widetilde{C}$ is quadratic $\gamma(n, \widetilde{C}, SER) = \frac{1}{K_n}$. In view of (13), it will be enough to prove

$$\frac{\sigma(x, C)}{es(x, C)} \geq p \frac{\sigma(x, \widetilde{C})}{es(x, \widetilde{C})}$$

Concavity of C implies $\widetilde{C}(z) \leq C(z)$ for $z \leq x_N$, therefore $es(x, C) \leq es(x, \widetilde{C})$. To prove $\sigma(x, C) \geq p \cdot \sigma(x, \widetilde{C})$, we first compare $b_i = C'(x^i) - C'(x^{i-1})$ to $\widetilde{b_i} = \widetilde{C}'(x^i) - \widetilde{C}'(x^{i-1}) = AC' \cdot a_i$. From the concavity of C' and the bound on its elasticity we have

$$b_i \geq C''(x^i) a_i \geq C''(x_N) a_i \geq pAC' a_i = p\widetilde{b_i}$$

Next we rearrange equations (14) as follows

$$\begin{aligned}\sigma(x,C) &= \sum_N (\sum_{j=1}^{i} b_j)a_i + \sum_{i<j} \frac{n-j+1}{n-i+1} a_i b_j - C(x_N) \qquad (15)\\ &= \sum_N \{C'(x^i)(x^i - x^{i-1}) - (C(x^i) - C(x^{i-1}))\} + \sum_{i<j} \frac{n-j+1}{n-i+1} a_i b_j\end{aligned}$$

The same computation for the equilibrium surplus at $(n, \widetilde{C}, \widetilde{u})$ gives

$$\sigma(x,\widetilde{C}) = \sum_N \frac{1}{2} a_i \widetilde{b_i} + \sum_{i<j} \frac{n-j+1}{n-i+1} a_i \widetilde{b_j} \qquad (16)$$

To compare these two surplus, it remains to compare the terms in the two left hand summations. Concavity of C' implies

$$\begin{aligned}C'(x^i)(x^i - x^{i-1}) - (C(x^i) - C(x^{i-1})) &\geq \frac{1}{2} C''(x^i)(x^i - x^{i-1})^2\\ &= \frac{1}{2} C''(x^i) a_i^2 \geq \frac{1}{2} p a_i \widetilde{b_i}\end{aligned}$$

To show this, set $f(z) = C'(x^i)(x^i - z) - (C(x^i) - C(z))$, check that f is convex, reaches its minimum zero at $z = x^i$, and satisfies $f'(z) \leq C''(x^i)(z - x)$. Gathering the inequalities above, we obtain $\sigma(x,C) \geq p \cdot \sigma(x,\widetilde{C})$, as desired.

7.1.4 Theorem 1, statement ii)

The proof is similar. We are given x and C, and assume without loss $C'(0) = 0$. We use the same auxiliary quadratic cost function $\widetilde{C}'(z) = AC' \cdot z$. Unlike in the above proof, we keep the same utility profile $\widetilde{u} = u$, but adjust the demand profile to $\widetilde{x}$ such that $\widetilde{x}^i = \frac{1}{AC'} C'(x^i) \Leftrightarrow \widetilde{C}'(\widetilde{x}^i) = C'(x^i)$. Thus the utility increments b_i are the same in the equilibrium x of the game (n, C, u) and $\widetilde{x}$ of the game $(n, \widetilde{C}, u)$, but the demand increments a_i and $\widetilde{a_i}$ differ:

$$a_i = x^i - x^{i-1};\ \widetilde{a_i} = \frac{1}{AC'}(C'(x^i) - C'(x^{i-1})) = \frac{1}{AC'} b_i$$

We proceed to show

$$\frac{\sigma(x,C)}{es(x,C)} \geq \frac{1}{2p-1} \frac{\sigma(\widetilde{x},\widetilde{C})}{es(\widetilde{x},\widetilde{C})} \qquad (17)$$

Convexity of C' and the upper bound on its elasticity give

$$b_i \leq C''(x^i) a_i \leq C''(x_N) a_i \Rightarrow \widetilde{a_i} \leq p a_i$$

Equation (15) for $\sigma(x,C)$ is unchanged, and (16) becomes

$$\sigma(\widetilde{x},\widetilde{C}) = \sum_N \frac{1}{2} \widetilde{a_i} b_i + \sum_{i<j} \frac{n-j+1}{n-i+1} \widetilde{a_i} \cdot b_j$$

We show next

$$C'(x^i)(x^i - x^{i-1}) - (C(x^i) - C(x^{i-1})) \geq \frac{1}{2} a_i b_i \geq \frac{1}{2p} \widetilde{a}_i b_i$$

The left hand inequality is

$$C(x^i) - C(x^{i-1}) \leq (x^i - x^{i-1}) \frac{C'(x^i) + C'(x^{i-1})}{2}$$

which follows easily from the convexity of C'. Gathering the inequalities above we have shown $\sigma(x, C) \geq \frac{1}{p} \sigma(\widetilde{x}, \widetilde{C})$.

We compare now the efficient surplus in our two games. Convexity of C' implies

$$C'(z) \geq \max\{C'(x_N) - C''(x_N)(x_N - z), 0\}$$

Set $z^* = x_N - \frac{C'(x_N)}{C''(x_N)}$. From $C'(0) = 0$, the upper bound on elasticity, and the inequality above, we have

$$\begin{aligned} C'(x_N) &\geq \int_0^{x_N} \max\{C'(x_N) - C''(x_N)(x_N - z), 0\} dz \\ &= \int_{z^*}^{x_N} (C'(x_N) - C''(x_N)(x_N - z)) dz = \frac{C'(x_N)^2}{2C''(x_N)} \geq \frac{1}{2p} x_N C'(x_N) \end{aligned}$$

As $es(x, C) = \Gamma(x_N) = x_N C'(x_N) - C(x_N)$, we have $es(x, C) \leq (1 - \frac{1}{2p}) x_N C'(x_N)$. On the other hand $es(\widetilde{x}, \widetilde{C}) = \frac{AC'}{2} x_N^2 = \frac{1}{2} x_N C'(x_N)$. Therefore $es(x, C) \leq (2 - \frac{1}{p}) es(\widetilde{x}, \widetilde{C})$. This concludes the proof of inequality (17).

7.2 Results for average cost pricing

From the proof of Proposition 1 (statement i) we derive a general formula for $\gamma(n, C, ACP)$. Fix a strictly positive demand vector x with $x_1 = \max_i x_i$, and u the associated utility profile (see at the beginning of section 7), where by equation (6), $u_1 = \max_i u_i$ as well. Then equation (7) implies

$$\frac{\sigma(x)}{es(x)} = \frac{\Gamma(x_N)}{\Gamma(\{C'\}^{-1}(u_1))} (\sum_N \lambda_i^2) \text{ where } \lambda_i = \frac{x_i}{x_N} \text{ , } u_1 = \lambda_1 C'(x_N) + (1 - \lambda_1) AC(x_N)$$

Fixing $z = x_N$ and λ_1, this ratio is minimal when for all $i \geq 2$, $\lambda_i = \frac{1 - \lambda_1}{n-1}$. Thus

$$\gamma(n, C, ACP) = \inf_{z \geq 0, \frac{1}{n} \leq \lambda \leq 1} \frac{\Gamma(z)}{\Gamma(\{C'\}^{-1}(\lambda C'(z) + (1 - \lambda) AC(z)))} \frac{n\lambda^2 - 2\lambda + 1}{n - 1} \tag{18}$$

Note that the same computation applies to the case of a demand x where only m agents are active ($x_i > 0$): simply replace n by m in (18). As $\frac{m\lambda^2 - 2\lambda + 1}{m-1}$ is decreasing in m, we see that the restriction to strictly positive demands is without loss.

7.2.1 Propositions 2 and 3

From (18), the computation of $\gamma(n, C_p, ACP)$ is straightforward, taking into account $\Gamma(z) = \frac{p}{p+1} z^{p+1}$:

$$\gamma(n, C_p, ACP) = \frac{(p+1)^{\frac{p+1}{p}}}{n-1} \inf_{\frac{1}{n} \leq \lambda \leq 1} \frac{n\lambda^2 - 2\lambda + 1}{(1+p\lambda)^{\frac{p+1}{p}}}$$

The solution of this optimization problem is

$$\lambda^* = \frac{p+3}{n+1+\sqrt[2]{(n+1)^2 + n(p-1)(p+3)}}$$

We omit the tedious yet routine check that $\frac{1}{n} \leq \lambda^* \leq 1$. Hence $\gamma(2, C_p, ACP)$ in Proposition 2, and $\gamma(n, C_1, ACP)$ in Proposition 3.

7.2.2 Theorem 1, statement i)

For the quadratic cost function C_1 we have $\frac{\Gamma(z)}{\Gamma(\{C'\}^{-1}(\lambda C'(z) + (1-\lambda)AC(z)))} = \frac{4}{(1+\lambda)^2}$. Thus by (18) it will be enough to pick any C such that C' is concave, $C'(0) = 0$, and show the following inequality for any $z \geq 0$,and $\lambda, 0 \leq \lambda \leq 1$:

$$\Gamma(\{C'\}^{-1}(\lambda C'(z) + (1-\lambda)AC(z))) \geq \frac{4}{(1+\lambda)^2}\Gamma(z) \tag{19}$$

First we claim $C'(\frac{z}{2}) \geq AC(z)$. Suppose not. Let α be the slope of a supporting line of C' at $\frac{z}{2}$, we have

$$C'(t) \leq \min\{\alpha(t - \frac{z}{2}) + AC(z), C'(z)\} \text{ for all } t, 0 \leq t \leq z \tag{20}$$

and the inequality is strict at least in a neighborhood of $\frac{z}{2}$. In particular $C'(z) \leq \alpha\frac{z}{2} + AC(z)$, hence $z^* = \frac{z}{2} + \frac{C'(z) - AC(z)}{\alpha} \leq z$. Tthe right hand side in (20) equals $\alpha(t - \frac{z}{2}) + AC(z)$ for $0 \leq t \leq z^*$, and $C'(z)$ between z^* and z. Because $C'(0) = 0$, we integrate inequality (20) as

$$C(z) < \int_0^{z^*} \{\alpha(t - \frac{z}{2}) + AC(z)\}dt + (z - z^*)C'(z)$$

With the notation $\Delta = C'(z) - AC(z)$, straightforward algebra yields

$$0 < -\frac{z\Delta}{2} - \frac{\Delta^2}{2\alpha} - \frac{\alpha z^2}{8}$$

a contradiction.

Now $C'(\frac{z}{2}) \geq AC(z)$ implies that for $\frac{z}{2} \leq t \leq z$, the graph of C' remains above the line joining $(\frac{z}{2}, AC(z))$ to $(z, C'(z))$, namely

$$C'(t) \geq AC(z) + \frac{2\Delta}{z}(t - \frac{z}{2}) \tag{21}$$

Let $\widetilde{z}$ and u_1 be defined by

$$C'(\widetilde{z}) = u_1 = \lambda C'(z) + (1-\lambda)AC(z) \Rightarrow \Gamma(\{C'\}^{-1}(\lambda C'(z) + (1-\lambda)AC(z)) = \Gamma(\widetilde{z})$$

Check first $\widetilde{z} \leq \frac{1+\lambda}{2}z$. If $\widetilde{z} \leq \frac{z}{2}$ this is clear, otherwise $\frac{z}{2} \leq \widetilde{z} \leq z$ and we apply (21) at $t = \widetilde{z}$. Next for any $t, \frac{z}{2} \leq t \leq z$, (21) is equivalently written as

$$C'(t) - u_1 \geq \Delta(\frac{2}{z}t - (\lambda + 1)) \tag{22}$$

Integrating (22) between $\widetilde{z}$ and z, we get

$$C(z) - C(\widetilde{z}) - u_1(z - \widetilde{z}) \geq \int_{\frac{1+\lambda}{2}z}^{z} (C'(t) - u_1)dt \geq \Delta\frac{(1-\lambda)^2}{4}z$$

$$\Rightarrow \Gamma(\widetilde{z}) = u_1\widetilde{z} - C(\widetilde{z}) \geq \Delta\frac{(1-\lambda)^2}{4}z + (u_1 z - C(z)) = \Delta z\frac{(1+\lambda)^2}{4} = \frac{(1+\lambda)^2}{4}\Gamma(z)$$

concluding the proof.

Note that we only used the concavity of C', not the bound on its elasticity.

7.2.3 Theorem 1, statement ii)

The proof of $\frac{4}{n+3} \leq \gamma(n, C, ACP)$ is the mirror image of the above argument, so we only sketch it. Convexity of C' implies first $C'(\frac{z}{2}) \leq AC(z)$, so that the graph of C' is below the line joining $(\frac{z}{2}, AC(z))$ to $(z, C'(z))$, namely the opposite of inequality (21). Defining $\widetilde{z}$ and u_1 in the same way, we get the opposite of inequality (22), and of all the subsequent inequalities, concluding the proof.

We now use the upper bound on the elasticity of C' to derive the upper bound on $\gamma(n, C, ACP)$. Taking $\lambda = \frac{1}{n}$ in the right hand term of (18), and recalling that $\Gamma \circ \{C'\}^{-1}$ is non decreasing, we have for any z

$$\gamma(n, C, ACP) \leq \frac{\Gamma(z)}{\Gamma(\{C'\}^{-1}(AC(z)))}\frac{1}{n}$$

Convexity of C', and the upper bound on its elasticity give for all $t, 0 \leq t \leq z$:

$$C'(t) \geq C'(z) - C''(z)(z - t) \geq C'(z)(p\frac{t}{z} - (p-1)) \tag{23}$$

The right hand term in (23) is non negative for $t \geq z^* = (1 - \frac{1}{p})z$, therefore

$$C(z) \geq \int_{z^*}^{z} C'(z)(p\frac{t}{z} - (p-1))dt = \frac{1}{2p}zC'(z) \Rightarrow \Gamma(z) \leq (1 - \frac{1}{2p})zC'(z) \tag{24}$$

Next define $\widetilde{z}$ by $C'(\widetilde{z}) = AC(z)$, and check

$$\Gamma(\{C'\}^{-1}(AC(z))) = \Gamma(\widetilde{z}) = \int_{\widetilde{z}}^{z} (C'(t) - AC(z))dt \tag{25}$$

Finally we define z^0 by $C'(z)(p\frac{z^0}{z} - (p-1)) = AC(z)$; applying (23) at $t = z^0$ we get $C'(z^0) \geq AC(z) = C'(\widetilde{z}) \Rightarrow z^0 \geq \widetilde{z}$. On the other hand, convexity of C' implies $AC(z) \leq \frac{C'(z)}{2}$ (we omit the easy argument), therefore $z^0 \leq (1 - \frac{1}{2p})z$. We evaluate now $\Gamma(\widetilde{z})$ in (25) with the help of (23) and $\widetilde{z} \leq (1 - \frac{1}{2p})z$:

$$\begin{aligned} \Gamma(\widetilde{z}) &\geq \int_{(1-\frac{1}{2p})z}^{z} (C'(t) - AC(z))dt \geq \int_{(1-\frac{1}{2p})z}^{z} (C'(t) - \frac{C'(z)}{2})dt \\ &\geq C'(z) \int_{(1-\frac{1}{2p})z}^{z} (p\frac{t}{z} - (p - \frac{1}{2}))dt = \frac{1}{8p} z C'(z) \end{aligned} \tag{26}$$

Combining (24) and (26) gives $\frac{\Gamma(z)}{\Gamma(\{C'\}^{-1}(AC(z)))} \leq 4(2p-1)$, as desired.

7.3 Results for marginal cost pricing

We derive a general formula for $\gamma(n, C, MCP)$ very similar to (18) for average cost pricing. Fix a strictly positive demand vector x with $x_1 = \max_i x_i$, and u the associated utility profile (see at the beginning of section 7):

$$u_i = C'(x_N) + \lambda_i x_N C''(x_N) \text{ for all } i, \text{ with } \lambda_i = \frac{x_i}{x_N}$$

Thus $u_1 = \max_i u_i$ as well, and $es(x) = \Gamma(\{C'\}^{-1}(u_1))$. Moreover

$$\sigma(x) = x_N^2 C''(x_N)(\sum_N \lambda_i)^2$$

Fixing $z = x_N$, and λ_1, this ratio is minimal when for all $i \geq 2$, $\lambda_i = \frac{1-\lambda_1}{n-1}$. Thus

$$\gamma(n, C, MCP) = \inf_{z \geq 0, \frac{1}{n} \leq \lambda \leq 1} \frac{z^2 C''(z)}{\Gamma(\{C'\}^{-1}(C'(z) + \lambda z C''(z)))} \frac{n\lambda^2 - 2\lambda + 1}{n-1} \tag{27}$$

7.3.1 Propositions 2 and 3

Comparing equations (18) and (27) implies for all n and p :

$$\gamma(n, C_p, MCP) = \frac{1}{(p+1)^{\frac{1}{p}}} \gamma(n, C_p, ACP)$$

a slightly more general statement than in Proposition 2.

7.3.2 Theorem 1

Taking $\lambda = \frac{1}{n}$ in (27), and using the monotonicity of $\Gamma \circ \{C'\}^{-1}$, we have for *any* C and any z

$$\gamma(n, C, MCP) \leq \frac{z^2 C''(z)}{\Gamma(z)} \frac{1}{n}$$

We let the reader check the following inequalities

$$\begin{aligned} C' \text{ concave} \quad &\Rightarrow \quad \Gamma(z) \geq \frac{1}{2} z^2 C''(z) \\ C' \text{ convex} \quad &\Rightarrow \quad \Gamma(z) \geq \frac{1}{2} z C'(z) \end{aligned}$$

(the second inequality is the same as $AC(z) \leq \frac{C'(z)}{2}$, already used in subsection 7.2.2). The two statements in Theorem 1 follow at once.

7.4 Results for the incremental method

We derive a simple and general formula for $\gamma(n, C, INC)$. Fix a strictly positive demand vector x with $x_1 = \max_i x_i$; the associated utility profile (see at the beginning of section 7) is $u_i = C'(x_N)$ for all i, and the corresponding efficient and equilibrium surplus are

$$es(x) = \Gamma(x_N);\ \sigma(x) = x_N C'(x_N) - \sum_N (C(x_N) - C(x_{N \setminus i}))$$

Fixing $z = x_N$, a consequence of the convexity of C is that the minimum of $\sum_N C(x_{N \setminus i})$ is achieved at $x_i = \frac{z}{n}$ for all i. Therefore

$$\gamma(n, C, INC) = \inf_{z \geq 0} \frac{zC'(z) - n(C(z) - C(\frac{n-1}{n} z))}{\Gamma(z)} \tag{28}$$

7.4.1 Propositions 2 and 3

For the cost function C_p, this expression reduces to

$$\gamma(n, C_p, INC) = 1 - \frac{1}{p} \{ \frac{n^{p+1} - (n-1)^{p+1}}{n^p} - 1 \}$$

7.4.2 Theorem 1 statement i)

From the concavity of C' one deduces easily

$$\begin{aligned} C(z) - C(\frac{n-1}{n} z) \quad &\geq \quad \frac{z}{n} \frac{C'(z) + C'(\frac{n-1}{n} z)}{2} \Rightarrow \\ zC'(z) - n(C(z) - C(\frac{n-1}{n} z)) \quad &\leq \quad \frac{z \left(C'(z) - C'(\frac{n-1}{n} z) \right)}{2} \end{aligned} \tag{29}$$

We give now a lower bound for $\Gamma(z)$. Let α be the slope of a line supporting the graph of C' at $\frac{n-1}{n} z$. For all $t, 0 \leq t \leq z$, we have

$$\begin{aligned} C'(t) \quad &\leq \quad \min \{ C'(\frac{n-1}{n} z) + \alpha (t - \frac{n-1}{n} z), C'(z) \} \\ &\Rightarrow \quad C(z) \leq \int_0^{\frac{n-1}{n} z} \{ C'(\frac{n-1}{n} z) + \alpha (t - \frac{n-1}{n} z) \} dt + \frac{z}{n} C'(z) \\ &= \quad z(\frac{n-1}{n} C'(\frac{n-1}{n} z) + \frac{1}{n} C'(z)) - \frac{\alpha}{2} (\frac{n-1}{n} z)^2 \end{aligned}$$

From this we get

$$\Gamma(z) \geq \frac{n-1}{n} z \left(C'(z) - C'(\frac{n-1}{n} z) \right) + \frac{\alpha}{2} (\frac{n-1}{n} z)^2$$

On the other hand, concavity of C' implies that α is bounded below by the slope of the line from $(\frac{n-1}{n} z, C'(\frac{n-1}{n} z))$ to $(z, C'(z))$, namely

$$\alpha \geq \frac{n}{z} \left(C'(z) - C'(\frac{n-1}{n} z) \right)$$

Combining the above two inequalities

$$\Gamma(z) \geq \frac{n^2-1}{2n} z \left(C'(z) - C'(\frac{n-1}{n} z) \right)$$

which, together with (29), yields the bound

$$\gamma(n, C, INC) \leq \frac{n}{n^2-1} \leq \frac{4}{3n}$$

7.4.3 Theorem 1, statement ii)

We fix z and compute an upper bound of the ratio in the right hand term of (28). Convexity of C' and the upper bound on its elasticity imply inequality (23), that we integrate between $\frac{n-1}{n} z$ and z :

$$C(z) - C(\frac{n-1}{n} z) \geq \frac{1}{n} (1 - \frac{p}{2n}) z C'(z) \Rightarrow z C'(z) - n(C(z) - C(\frac{n-1}{n} z)) \leq \frac{p}{2n} z C'(z)$$

On the other hand, as in subsection 7.3.2, $\Gamma(z) \geq \frac{1}{2} z C'(z)$. Combining these two inequalities yields the desired upper bound.

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