Strategyproof profit sharing in partnerships: Improving upon autarky*

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Abstract

Several producers decide to form a partnership, to which they contribute both capital and labor. We propose a *group-strategyproof* mechanism under which no single agent is tempted to secede from the partnership: the *inverse marginal product proportions* (or *IMPP*) *mechanism.*

The IMPP mechanism combines aspects of common ownership with the requirement that private property rights be respected: when an agent decides to stop exploiting her own capital, the latter is shared between the remaining agents in proportion to the productivity of their *own* capital. The IMPP is in fact the only *fixed-path method* (as introduced in Friedman, 2002) to satisfy autarkic individual rationality; its path is uniquely determined by the capital contributions of the agents. Thus, our results provide one of the first economic motivation for the asymmetry of fixed-path methods.

Keywords: Autarky, incentive compatibility, joint production, partnerships, surplus sharing, serial rule.

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1 Introduction

Because partnerships are usually comprised of a small number of partners who can communicate and coordinate their actions easily, it is important that joint profits be shared by a mechanism with very strong incentives properties.

One common rule to sharing profits in law firms, for instance, consists in assigning each partner's share according to her seniority in the firm. While seniority (or "lockstep") methods are not devoid of appeal[1], they are vulnerable to the criticism that senior partners—who have no prospects for advancement—have a tremendous incentive to free-ride on the effort of younger partners.

Popular alternatives to seniority-based methods are productivity-based: profit shares mirror each partner's billings. While these so-called "eat-what-you-kill" methods provide unambiguous incentives once cases are assigned, how to assign them is a non-trivial question. Gilson and Mnookin (1985, p. 370) express the difficulty of designing satisfactory productivity-based methods due to the fact that partners often negotiate their profit share by threatening to leave the partnership with their own client base (a maneuver called "grabbing"):

> Because the ability to leave with the firm' clients is critical to the strength of a lawyer's threat, [...] lawyers have an important incentive to make sure that clients in fact remain *theirs*, and not the firm's [...].

We feel that this concern calls for a mechanism which combines aspects of common ownership with the requirement that private property rights be respected. We propose such a mechanism.

1.1 The case for strategyproofness

The question of incentive compatible profit-sharing is certainly not new and has already been addressed in the Bayesian framework (see, e.g., Groves 1973, d'Aspremont and Gérard-Varet 1979, Holmstrom 1982, McAfee and McMillan 1991, Levin and Tadelis 2005). However, a drawback of the Bayesian approach is that it requires some informational superstructure: the mechanism designer must assume a prior distribution of characteristics (or types), a certain attitude towards risk (typically, risk neutrality) and a particular formula for updating beliefs (typically, using Bayes' rule). And, even if these assumptions are justified,

[1] See Gilson and Mnookin (1985) for a defense of lockstep methods.

the system manager must bear the cost of estimating the prior distribution of types.

On the other hand, the incentives criterion we propose does not rely on such heavy informational assumptions. Under *strategyproofness*, it is a dominant strategy for each agent to reveal her true type *regardless of the belief structure*. I.e., an individual agent does not need to have any information about what other agents know (their beliefs) or want (their preferences) in order to act in her own best interest. In that sense, strategyproofness is a more robust requirement than Bayesian incentive compatibility. Under *group-strategyproofness*, agents also behave truthfully *regardless of their cooperative possibilities*; we contend that the latter is an appropriate requirement in the context of partnerships. We refer the reader to Sprumont (1995) and Barberà (2001) for recent surveys on strategyproofness.

1.2 The partnership problem

Several producers of a common private good decide to pool their private production possibilities. In addition to technological (or capital) contributions, each producer makes input (or labor) contributions to the cooperative. We assume that input is transferable across technologies. Two classical questions are: how to jointly utilize their private capital and how to share the proceeds of their cooperation (see Israelsen 1980, Sen 1966, Weitzman 1974)?

In a stylized version of the problem, each agent makes her privately owned machine (her capital) available to all the group-members and can supply labor to any machine. Practical examples include farmers pooling their land in a cooperative; here, land is capital and input can be labor or seeds to be planted. Examples of such cooperatives can also be found in the fishing sector (Townsend, 1994) and in the plywood industry (Craig and Pencavel, 1995). A similar situation arises whenever a group of experts (e.g. lawyers, physicians, financial advisors, car salesmen, etc.) who can rank their clients in decreasing order of productivity decide to engage in a partnership; each agent's client base then amounts to a decreasing-returns technology. By pooling their clienteles, agents can reallocate their time or resources (the input) across the total pool of clients.

We assume that individual technologies are known to the planner and exhibit diminishing marginal returns. However, information about the leisure/consumption trade-off of the agents is private, potentially leaving room for misrepresentation. The first requirement is that production possibilities and input contributions be pooled efficiently. Under diminishing marginal returns, there is a unique efficient way to reallocate a given amount of input across the various technologies.[2] Thus, the autarkic use of the production possibilities, where agent i only supplies input to her own technology, can be Pareto-improved. The *aggregate* production function (of the individual technologies) summarizes these production opportunities.

While production efficiency is easily obtained, allocative efficiency cannot be achieved, precisely because the preferences of the agents are private information (see Leroux, 2004). In particular, it follows that a standard market mechanism—which computes the competitive wage and the equilibrium amounts of labor and consumption for each agent—is manipulable; it would be in some agents' best interest to artificially restrict her labor-supply schedule in order to command a higher wage.[3]

In what follows we insist on strategyproofness while relaxing full efficiency in favor of the milder requirement of voluntary participation, i.e., every participant should be at least as well off under the pooling method as she would be by using her own capital by herself; we refer to this condition as *autarkic individual rationality* (or *AIR*), a term introduced by Saijo (1991) in the public good context.

Our main contribution is the design of a simple, group-strategyproof and autarkically individually rational mechanism to manage a partnership: the *inverse marginal product proportions mechanism* (or *IMPP*).[4]

1.3 The IMPP mechanism

For the sake of exposition, we describe the IMPP mechanism in the case of three agents (farmers) pooling their private capital (land). In Figure 1 are depicted the marginal product (MP) curves of each farmer's own land (MP_1, MP_2, and MP_3). We assume for clarity that each agent's utility is quasilinear in the amount of profit, y_i, she receives: $u_i(x_i, y_i) = y_i - v_i(x_i)$, where $v_i(x_i)$ is agent i's disutility of supplying x_i units of labor expressed in monetary terms.[5]

[2] For instance, if machine 1 is always more productive than the others, productive efficiency requires that the $(n-1)$ other agents work on machine 1 instead of their own.

[3] A larger-scale manipulation of the sort contributed to the collapse of the California energy market in 2001. Manipulation was successful even when as many as two hundred electricity providers were involved! See Oppel, 2003. We thank Simon Grant for this observation.

[4] The mechanism we propose is actually Nash-implementable with unique equilibrium. In this model the corresponding direct revelation mechanism is then group-strategyproof; see Section 2 for a discussion of other, weaker interpretations of incentive compatibility.

[5] Our results hold on more general preference domains.

IMPP works as follows. Consider a dynamic production process where labor is allocated to land so that marginal product is equalized across fields at all times (but decreases over time). In the first stage, each agent works on her own land and reaps the fruit of her labor, like in autarky, until one agent decides to stop working. That agent then "leaves" the procedure with the output she has secured thus far but her unused capital is now available for agents 2 and 3 to utilize in addition to their own. In our example, agent 1 leaves the procedure after supplying x_1 units of labor: the intersection of her marginal disutility of effort (v_1') with the marginal product of her *own* land (MP_1) occurs "first", i.e. for a higher marginal product level than the other agents ($\lambda_1 > \lambda_2$, λ_3).

With agent 1 gone, her left-over capital is divided between agents 2 and 3 in proportion to the inverse marginal product of lands 2 and 3, respectively. For any marginal product level, $\lambda < \lambda_1$, compare the amounts of labor which can be assigned to fields 2 and 3 before reaching the marginal product λ. Suppose, as in Figure 2, that twice as much labor can be supplied to field 2 as can be supplied to field 3: 2α and α, respectively. In a sense, field 2 is twice as productive as field 3. Accordingly, agent 2 will be allowed to supply twice as much labor as agent 3 to field 1 at the margin—$\frac{2}{3}\beta$ and $\frac{1}{3}\beta$, respectively— and receive the corresponding marginal increment of output.

The procedure is carried out until another agent decides to stop working (in our example, agent 2 departs after having supplied x_2 units of labor, see Figure 3) and the remaining agent (agent 3) can then freely operate all three fields at will.

Clearly IMPP satisfies AIR: agent 1 is indifferent between autarky and taking part in the partnership while agents 2 and 3 effectively see their production possibilities enhanced by the "departure" of agent 1. In addition, the mechanism is strategy-proof: just like in an ascending-price auction, it is a strictly dominant strategy for each agent to "drop out" when the (common) marginal product equals her (private) marginal disutility of effort.

1.4 IMPP and the commons problem

Upon noticing that the production possibilities of the partnership are summarized by the aggregate production function, as stated above, it becomes clear that the IMPP mechanism can be applied to situations where a group of users jointly operate a single facility exhibiting diminishing marginal returns: the familiar commons problem.

If agents have equal rights to the facility, by assigning to each of them virtual property rights to "$1/n^{\text{th}}$ of the facility" and by applying IMPP one obtains the output-sharing version of the well-known serial mechanism discussed in Moulin and Shenker (1992). Thus, much better incentives are achieved than under, say, average- or marginal-product pricing[6].

Similarly, if agents do not have equal access to the common facility for exogenous reasons (e.g. social status), these differences can be taken into account by assigning unequal virtual production functions before applying IMPP. The corresponding sharing rule allocates marginal increments of input, and their corresponding increments of output, along a path in the input space of the agents. The slope of this path describes each individual's relative access to the facility. After an agent's supply of input is met, the procedure carries on along the projection of the path onto the space of the remaining "active" agents. In other words, the active agents split the increments of input which would have been assigned to inactive agents (i.e. their "rights to using the facility") in the relative proportions granted by the original path. We call such procedures *fixed-path methods (FPMs).*

1.5 Organization of the article

After a brief review of some related literature, the remainder of the paper is organized in reverse order relative to the introduction. We first define fixed-path methods in a single technology model (Section 3) and recall their strong incentives properties. We then show the close relationship between FPMs and the IMPP mechanism (Section 4). Finally, we discuss the appeal of the IMPP mechanism as a reasonable compromise between simplicity and responsiveness to capital contributions (Section 5). Most proofs can be found in the Appendix.

2 Relation to the literature

This work contributes to the large literature exploring the trade-off between efficiency and incentive compatibility in the production and distribution of private goods.

Our main result (Theorem 2) can be viewed as a follow-up on work by Friedman (2002, 2004). The *FPM*s described above are the output-sharing version

[6] We refer the reader to Moulin and Shenker (1992) for a discussion of the serial cost-sharing rule in comparison to average- and marginal-cost pricing.

of the mechanisms introduced in Friedman (2002) as non-anonymous generalizations of the Moulin and Shenker serial cost-sharing mechanism. Theorem 2 establishes that to each partnership problem (i.e. any profile of capital contributions) corresponds a unique FPM sharing the aggregate production function while satisfying AIR. Conversely, we show (Theorem 3) that to each FPM corresponds a unique (virtual) decomposition of the common facility into individual capital contributions; the FPM is in fact the IMPP mechanism applied to this very decomposition. We thus establish a bijection between the class of FPM to manage a commonly owned facility and possible distributions of property rights to the facility. This result provides an economic justifications for the non-anonymity of FPMs.

Many rules outside of the class of FPMs meet our high standards of incentive compatibility; their more complex path structure is similar to the "path functions" of Sprumont (1998) (see Section 3). This contrasts with Moulin (1999) where it is shown in the discrete framework that FPMs—and the corresponding IMPP mechanisms—are in fact characterized by an incentive compatibility requirement fairly close to ours (see Remark 1). This discrepancy illustrates a subtle difference between the discrete and continuous versions of the model and is worthy of future research.

Recent related literature on the common production of private goods considers weaker interpretations of incentive compatibility (see, e.g., Corchón and Puy 2002, Shin and Suh 1997). For instance, Corchón and Puy establish that any continuous sharing rule admits a Pareto-efficient allocation which can be Nash-implemented. Yet, any game implementing such an outcome must have several, non-welfare-equivalent Nash equilibria at some profiles. Here we insist on the uniqueness of the Nash equilibrium, a much more demanding requirement than the above kind of Nash-implementability.

3 The single technology model

Let $N = \{1, ..., n\}$ be the set of agents. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a production function which is increasing, strictly concave such that $F(0) = 0$. We denote by $\mathcal{F}$ the class of such functions. Each agent i provides a non-negative amount x_i of input to the common technology, and receive a non-negative quantity y_i of output such that $\sum_i y_i = F\left(\sum_i x_i\right)$. We write $x = (x_1, ..., x_n)$ and for any $i \in N$, (x_i', x_{-i}) is the vector of inputs where the ith entry of x has been replaced by $x_i' \in \mathbb{R}_+$. A *bundle* is an element $z_i = (x_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$; we define an *allocation*, z, to be a list of n bundles, one for each agent. We denote by $Z_F = \left\{ z \in (\mathbb{R}_+ \times \mathbb{R})^N \mid \sum_i y_i \leq F\left(\sum_i x_i\right) \right\}$ the set of feasible allocations under F.

Each agent i can supply up to M_i units of input (with M_i possibly very large). Her *preferences* over bundles are defined on $\mathbb{R}_+ \times \mathbb{R}$; they are continuous, convex, strictly increasing in y_i, strictly decreasing in x_i and representable by a utility function u_i. While all our results are purely ordinal, we will use utility representations rather than the more cumbersome binary relation notation. We adopt the convention $u_i(x_i, y_i) = -\infty$ if $x_i > M_i$. We denote by $\mathcal{U}$ the class of preferences. A *preference profile* (or *utility profile*) is a list of n preferences, $u = (u_1, ..., u_n) \in \mathcal{U}^N$, one per agent. For any $j \in N$, we will sometimes abuse notations and write $u = (u_j, u_{-j})$.

Definition 1 *An F-*sharing method *(or F-*sharing rule*) is a mapping*

$$\begin{aligned} \xi: \quad & \mathbb{R}_+^N \to \mathbb{R}^N \\ & x \mapsto (\xi_1(x), ..., \xi_2(x)) \quad s.t. \quad \textstyle\sum_{i \in N} \xi_i(x) = F\left(\sum_{i \in N} x_i\right) \end{aligned}$$

which satisfies the following monotonicity property: $\frac{\partial \xi_i}{\partial x_i} > 0$ *for all* $i \in N$.

We denote by $\mathcal{S}_F$ the class of F-sharing rules.

Monotonicity is a normatively appealing requirement. It states that an agent should receive strictly more output as her input contribution increases: it gives agents an incentive to supply input. Also, from the point of view of fairness, it implies that every agent will receive a positive fraction of the output resulting from her input contribution.

For any preference profile $u \in \mathcal{U}^N$ and any F-sharing method $\xi \in \mathcal{S}_F$, we denote by $G(\xi; u)$ the *game* in which each agent's strategy space is $\mathbb{R}_+$ and agent i's payoff is $u_i(x_i, \xi_i(x))$ when x_j is the strategy played by each agent $j \in N$.

We now define what we mean by "sharing a technology along a path". A *path* is a mapping

$$\begin{aligned} \phi: \quad & \mathbb{R}_+ \to \mathbb{R}_+^N \\ & t \mapsto (\phi_1(t), ..., \phi_n(t)) \end{aligned}$$

such that for all $i \in N$ the following two properties hold:

(a) ϕ_i is non-decreasing and differentiable almost everywhere on $\mathbb{R}_+$ with respect to the Lebesgue measure,

(b) $\sum_j \phi_j(t) = t$ for any $t \in \left[0, \sum_j M_j\right]$ and $\phi_i(t) = M_i$ for any $t \geq \sum_j M_j$.

We denote by $\mathcal{P}$ the class of paths.

Fix $\phi \in \mathcal{P}$. For any $i \in N$, define the mapping δ_i as follows:

$$\begin{aligned} \delta_i : \quad & [0, M_i] \to \mathbb{R}_+ \\ & x_i \mapsto \min\{t | \phi_i(t) \geq x_i\}. \end{aligned} \tag{1}$$

Note that if ϕ_i is increasing, its inverse exists and $\delta_i \equiv \phi_i^{-1}$.

Given a path $\phi \in \mathcal{P}$, we define the *fixed-path method generated by* ϕ, denoted ξ^ϕ, as follows. Let $x \in \times_i [0, M_i]$, without loss we relabel the agents such that $\delta_1(x_1) \leq \delta_2(x_2) \leq ... \leq \delta_n(x_n)$; i.e. such that the coordinates of x are met along ϕ in the natural order. Let $t \geq 0$ be such that $\phi(t) \leq x$, i.e. such that no agent's supply level has yet been met. ξ^ϕ recommends that the marginal product $F'(t)$ be split between the agents according to the vector of proportions $\left(\phi_1'(t), ..., \phi_n'(t)\right)$ (recall that $\sum_i \phi_i'(t) = 1$). Once the input supply of the first agent is met along the path ($\phi_i(t) \geq x_i$), we freeze her output share and continue the sharing procedure with the remaining "active" agents. The subprocedure shares the remainder of F along the projection of ϕ on the subspace $\left\{s \in \mathbb{R}_+^n | s_1 = x_1\right\}$ until agent 2's supply is met. And so on. We next give a formal definition.

Definition 2 *The* fixed-path method generated by ϕ, *denoted* ξ^ϕ, *is the* F-*sharing rule defined by:*

$$\begin{aligned} &\xi_1^\phi(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_1(t) \\ &\xi_2^\phi(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_2(t) + \int_{\delta_1(x_1)}^{\delta_2(x_2)} F'\left(x_1 + \sum_{i \geq 2} \phi_i(t)\right) d\phi_2(t) \\ &\vdots \\ &\xi_n^\phi(x) = \int_0^{\delta_1(x_1)} F'(t) d\phi_n(t) + ... + \int_{\delta_{n-1}(x_{n-1})}^{\delta_n(x_n)} F'\left(\sum_{i=1}^{n-1} x_i + \phi_n(t)\right) d\phi_n(t) \end{aligned} \tag{2}$$

for any $x \in \times_i [0, M_i]$.

A more compact notation is used by Friedman (2002, 2004): for any $i \in N$,

$$\xi_i^\phi(x) = \int_0^\infty F'\left(|\phi(t) \wedge x|\right) d(\phi_i(t) \wedge x_i)$$

where $|\cdot|$ returns the sum of the coordinates of a vector and $\wedge$ is the componentwise minimum of two vectors.

It follows easily from the monotonicity of F and the δ_j's that ξ^ϕ is monotonic ($\frac{\partial \xi_i^\phi}{\partial x_i} > 0$ for all i); hence, $\xi^\phi \in \mathcal{S}_F$. Moreover, one can check (or see Friedman 2002, Lemma 1) that ξ_i^ϕ is strictly concave in x_i.

We next give two examples of fixed-path methods:

Example 1: Incremental sharing. ($n = 2$) This method gives agent 1 full access to F; once she is served, agent 2 can use $F(x_1 + \cdot)$ at will. The corresponding path is

$$\phi^I : t \mapsto \begin{cases} (t, 0) & \text{if } t \leq M_1 < +\infty \\ (M_1, t - M_1) & \text{if } M_1 \leq t \leq M_1 + M_2 \end{cases}$$

i.e., ϕ^I is a parametrization of the horizontal axis up to $x_1 = M_1$. Output is awarded as follows:

$$\begin{cases} \xi_1^{\phi^I}(x) = F(x_1) \\ \xi_2^{\phi^I}(x) = F(x_1 + x_2) - F(x_1) \end{cases}$$

Example 2: Weighted serial rule. Assume $M_1 = M_2 = +\infty$[7]. Let $\alpha_1, ..., \alpha_n > 0$ and consider the path $\phi^S : t \mapsto (\alpha_1 t, ..., \alpha_n t)$. Let $x \in \mathbb{R}_+^N$ and assume without loss that $\frac{x_1}{\alpha_1} \leq \frac{x_2}{\alpha_2} \leq ... \leq \frac{x_n}{\alpha_n}$. Expression (2) then yields:

$$\xi_i^{\phi^S}(x) = \frac{\alpha_i}{\alpha^i} F(x^i) - \sum_{k=1}^{i-1} \frac{\alpha_i \alpha_k}{\alpha^k \alpha^{k+1}} F(x^k) \quad \text{for all } i = 1, ..., n,$$

where $\alpha^k = \sum_{j=k}^n \alpha_j$, and $x^k = x_1 + ... + x_{k-1} + \frac{\alpha^k}{\alpha_k} x_k$. As a particular case, the usual serial rule assigns identical weight to each agent.

The family of fixed-path methods (FPMs) was introduced in Friedman (2002) as a non-anonymous generalization of the serial rule retaining its strong incentives properties.

Theorem 1 *Let ξ be an FPM, the following statements are true:*
i) $G(\xi; u)$ has a unique Nash equilibrium,
ii) every Nash equilibrium of $G(\xi; u)$ is strong.

Proof. It is shown in Friedman (2002) that for any production function $F \in \mathcal{F}$, any path $\phi \in \mathcal{P}$ and any preference profile $u \in \mathcal{U}^N$, the game induced by ξ^ϕ satisfies a more demanding equilibrium property called O-solvability. ■

[7] Although M_1 and M_2 were defined as real numbers, the definition of the weighted serial rule readily extends to the case where they are infinite.

It follows from a standard result of the implementation literature (see Theorem 7.2.3 in Dasgupta et al. 1979) that the associated direct revelation mechanism is group-strategyproof.

While Moulin and Shenker (1992) established that the serial rule could be characterized by the equilibrium properties of Theorem 1 along with Anonymity ($x_i = x_j \implies \xi_i(x) = \xi_j(x)$), these properties do not characterize the class of FPMs when we relax Anonymity.

For instance, assume $n = 3$, $F \in \mathcal{F}$ and let $\phi \in \mathcal{P}$. Consider an F-sharing rule ξ that coincides with ξ^{ϕ} until one of the agents is served, say agent i, but then shares the remainder of F between the remaining two agents along a strictly increasing subpath, $\psi(i, x_i)$, depending on the identity of the first-served agent and her input supply level. Note that $\psi(i, x_i)$ may differ from the projection of ϕ onto the plane $\left\{s \in \mathbb{R}^n_+ | s_i = x_i\right\}$ for some pair (i, x_i). It is clear that agent i has the same unique dominant strategy under ξ and under ξ^{ϕ}. A straightforward application of Theorem 1 yields that the remaining agents also have a unique dominant strategy regardless of ψ. Yet, ξ is *not* a fixed-path method. These path structures are called *path functions* in Sprumont (1998), though his use of path functions is ultimately quite different from ours.[8] Characterizing the class of strategy-proof mechanisms is still a very large open question and is beyond the scope of this paper. We refer the reader to a companion paper (Leroux, 2005) for a two-agent characterization result.

Remark 1 *In the discrete version of our model, Moulin (1999) establishes that "incremental sharing rules" (the discrete equivalent of fixed-path methods) are characterized by similar strategic properties for any number of agents. Interestingly, the continuous framework allows for a much richer class of incentive compatible rules.*

We show on a straighforward example why some of these more complex rules do not meet our incentive compatibility requirement in the discrete setting. Consider a technology given by the discrete increments $\partial F : 4, 2, 1, 0$ (i.e. $F(1) = 4$, $F(2) = 4 + 2$,...) to be shared between 3 agents, each of whom can supply 0 or 1 unit of input. Suppose that the path structure used to share F yields the following priority orderings: $1 \rightarrow 2 \rightarrow 3$ if $x_1 = 1$ and $1 \rightarrow 3 \rightarrow 2$ if $x_1 = 0$. If preferences are such that agent 1 is indifferent between bundles (1,4) and

[8] Note that when $n = 2$, the type of methods just described cannot be distinguished from fixed-path methods.

(0,0), and if agent 2 prefers (1,2) to (0,0), then agent 1 can help out agent 3 by deciding not to work, thus giving her access to the bundle (1,2) instead of (1,1).

The above rule is immune to coalitional deviations in a weak sense: at least one agent in the deviating coalition does not strictly benefit (agent 1). Yet, not every Nash equilibrium of the supply game is strong due to agent 1's indifference between two bundles. Such indifferences are ruled out by the specifications of the continuous model.

4 Pooling private technologies

Consider now a situation where each agent privately owns a technology, $f_i \in \mathcal{F}$, which she decides to contribute to a partnership along with an input level $x_i \in [0, M_i]$. One can think of the individual technologies as being machines (or capital) and of input as being labor time. Labor is transferable, meaning that agents are able to work on machines other than their own. The manager of the partnership (the planner) allocates the labor time of the workers across the various machines; e.g., if $x_1 = 3$, agent 1 may be asked to spend, say, two units of input on machine 1 and one unit on machine 4. The resulting total output is distributed between the agents according to their labor (the x_i's) and capital (the f_i's) contributions. Technologies are known to the planner, but the preferences of the agents are private information.

Define F^* to be the *aggregated production function* resulting from the efficient usage of the combined individual technologies:

$$\forall t \in \mathbb{R}_+ \quad F^*(t) = \max_{\substack{(x_1, ..., x_n) \in \mathbb{R}_+^N \\ \sum_i x_i = t}} \sum_{i=1}^{n} f_i(x_i). \tag{3}$$

Notice that because the f_i's belong to $\mathcal{F}$, F^* must also belong to $\mathcal{F}$. We call $f = (f_1, ..., f_n) \in \mathcal{F}^N$ the *technology profile* (or *capital profile*).

Thus, the pooling framework is tantamount to the previous context of sharing a single technology. Here, however, autarkic individual rationality is a concern: no agent should be better off by using her private technology on her own. This voluntary participation requirement will end up determining uniquely the fixed-path method to use.

Definition 3 (AIR) *An* f-pooling method *is an* F^**-sharing rule* ξ *such that* *for any preference profile u and any Nash equilibrium x^* of $G(\xi; u)$ the following holds:*

$$u_i\left(x_i^*, \xi_i(x^*)\right) \geq sa_i(u_i) \equiv \max\left\{u_i(x_i, y_i) | y_i \leq f_i(x_i)\right\} \qquad \forall i \in N. \qquad (4)$$

We say ξ pools f *(or satisfies* AIR with respect to f*) and we denote by $\mathcal{S}_f$ the class of f-pooling methods.*

Define by ϕ^* the mapping assigning to each $t \geq 0$ the unique solution vector of (3); notice that ϕ^* is a path. The following theorem motivates the use of FPMs.

Theorem 2 *ξ^{ϕ^*} is the unique FPM which pools f.*

The following comments concerning ϕ^* will prove useful. Because ϕ^* is the unique solution of expression (3), it follows that

$$F^{*\prime}(t) = f_i'(\phi_i^*(t)) \qquad (5)$$

whenever $\phi_i^*(t) > 0$ (technology i is in use). I.e., $\phi_i^*(t)$ is the level of input that can be assigned to technology i before its productivity falls below $F^{*\prime}(t)$. Hence, for a given $t > 0$, the larger $\phi_i^*(t)$, the more productive technology i is.

We now give some intuition as to why ξ^{ϕ^*} not only satisfies AIR but also improves upon autarky. As long as all agents are active ($t \leq \min_j \delta_j^*(x_j)$), ξ^{ϕ^*} shares the marginal product $F^{*\prime}(t)$ according to the vector of ratios $\left(\phi_1^{*\prime}(t), ..., \phi_n^{*\prime}(t)\right)$. Hence, assuming for clarity that $\delta_1^*(x_1)$ is the smallest of the $\delta_j^*(x_j)$'s, then

$$\xi_1^{\phi^*}(x) = \int_0^{\delta_1^*(x_1)} F^{*\prime}(t)\phi^{*\prime}(t)dt = f_1(x_1)$$

and agent 1 receives her stand-alone level of output. Now, for $\delta_1^*(x_1) \leq t \leq \min_{j\neq 1} \delta_j^*(x_j)$, ξ^{ϕ^*} shares the marginal output $F^{*\prime}(t)$ between agents 2,...,n according to the ratios $\left(\phi_2^{*\prime}(t), ..., \phi_n^{*\prime}(t)\right) \times \frac{1}{\sum_{j>1} \phi_j^{*\prime}(t)}$. Clearly, for any $i \neq 1$, $\frac{\phi_i^{*\prime}(t)}{\sum_{j>1} \phi_j^{*\prime}(t)} \geq \phi_i^{*\prime}(t)$ and agent i receives no less (typically more) than her stand-alone share of output. And so on. Improvement upon autarky obtains by integration. In words, when an agent leaves the procedure what is left of her technology is shared between the remaining agents in proportion to their technological contributions to F^*. The formal proof of Theorem 2 can be found in Appendix A.1.

Remark 2 *ξ^{ϕ^*} is the IMPP mechanism described in the introduction.*

Remark 3 *Among the rules generated by path stuctures as in Sprumont (1998), all those (and only those) whose main path is ϕ^* are f-pooling methods, but their subpaths may be arbitrary.*

Theorem 2 has an interesting converse interpretation. Given a production function F^*, to any path ϕ^* corresponds a unique decomposition of F^* into a "virtual" production profile, f, such that ξ^{ϕ^*} is the unique FPM pooling f.

Theorem 3 *For any $F^* \in \mathcal{F}$ and any $\phi^* \in \mathcal{P}$, there exists a unique technology profile f decomposing F^* in the sense of (3) such that ξ^{ϕ^*} pools f. For any $i \in N$, f_i is given by*

$$f_i(x_i) = \int_0^{x_i} F^{*\prime}(\delta_i^*(t))dt$$

for all $0 \leq x_i \leq M_i$, where δ_i^* is defined relative to ϕ_i^* as in expression (1).

Proof. Immediate from Theorem 2. Let $F^* \in \mathcal{F}$, $\phi^* \in \mathcal{P}$ and $f \in \mathcal{F}^N$ decomposing F^* in the sense of (3) such that ξ^{ϕ^*} pools f. For any $i \in N$, expression (5) holds almost everywhere. I.e.,

$$f_i'(t) = F^{*\prime}(\delta_i^*(t)) \quad \text{almost everywhere.}$$

The result follows from integrating between 0 and x_i (recall $f_i(0) = 0$). ■

To illustrate Theorem 3, we provide the virtual production profiles corresponding to examples of Section 3.

Example 1. ξ^{ϕ^I} gives priority to agent 1. It is equivalent to pooling the production profile where agent 2's technology is useless compared to that of agent 1 on $[0, M_1 + M_2]$.

Example 2. Agents contribute to F^* in proportion to the α_i's: $f_i(t) = \alpha_i F^*(\frac{t}{\alpha_i})$.

Theorems 2 and 3 together establish a striking bijection between the family of FPMs and the possible distribution of property rights on F^*.

5 Discussion

As made clear in the previous section, the IMPP mechanism is essentially an FPM and, as such, meets our high standards of incentive compatibility. Yet, sharing rules outside the class of FPMs—like the ones generated by the path functions in Sprumont (1998)—also meets these standards, along with AIR.

However, the specification of path functions can potentially be quite complex, requiring the specification of a subpath $\psi(i, x_i)$ for each agent i at every level of input x_i, whereas the unique FPM satisfying AIR is entirely determined by the capital profile $(f_1, ..., f_n)$.

Admittedly, not all path functions need to be complex. Consider the rule coinciding with ξ^{ϕ^*} until the departure of agent 1 (subject to the usual relabeling of agents) and then giving full priority to agent 2, then to agent 3; or the one sharing the remainder of the technology, $G(\cdot) = F(\cdot - x_1 - \phi_2(\delta_1(x_1)) - \phi_3(\delta_1(x_1)))$, according to the (two-agent) Moulin and Shenker serial rule. From Remark 3, both these simple rules satisfy AIR. Yet, they are not responsive to capital contributions from the agents; in particular, they do not provide any incentives for the agents to supply capital, f_i , to the partnership. On the other hand, the IMPP mechanism rewards agents in proportion to the productivity of the capital contributed and thus encourages the supply of capital.

We contend that the IMPP mechanism is a reasonable compromise between simplicity and responsiveness to capital contributions, two appealing features for any practical profit-sharing mechanism in producer cooperatives or professional partnerships.

A Proofs

A.1 Proof of Theorem 2

Before proving Theorem 2, we present a lemma establishing that under any FPM, ξ^{ϕ}, any positive level of output, x_i, can be guaranteed at equilibrium by some preference u_i^* for agent i. Its proof can be found in Appendix A.2.

Lemma 1 *Let $\phi \in \mathcal{P}$, $i \in N$. For any $x_i > 0$, there exists a preference $u_i^* \in \mathcal{U}$ such that the following holds:*

$$\forall u_{-i} \in \mathcal{U}^{N \setminus i} \quad x_i^* = x_i;$$

where x^* denotes the unique Nash equilibrium of $G(\xi^{\phi}; u_i^*, u_{-i})$.

Now to the proof of Theorem 2. Let $\phi \in \mathcal{P}$ such that ξ^{ϕ} pools f. For the rest of the proof we will write F instead of F^* as no confusion is possible.

Fix $x \in \times_i [0, M_i]$ such that $\delta_i^*(x_i) = \delta_j^*(x_j)$ for all $i, j \in N$; i.e. x is a point on the graph of ϕ^*. From Lemma 1, there exists a preference profile $u \in \mathcal{U}^N$ such that x is the unique Nash equilibrium of $G(\xi^\phi; u)$. It follows that ξ^ϕ pools f only if for any $i \in N$ and any $x_i > 0$ the following holds:

$$\int_0^{\delta_i(x_i)} F'(t) d\phi_i(t) \geq \int_0^{x_i} f_i'(t) dt.$$

By (5) and the definitions of δ_i and δ_i^*, this transforms into

$$\int_0^{x_i} F'(\delta_i(t)) dt \geq \int_0^{x_i} F'(\delta_i^*(t)) dt \tag{6}$$

for all $i \in N$ and all $x_i > 0$. Let $i \in N$ and define $H_i(x_i) = \int_0^{x_i} F'(\delta_i(t)) dt$ for any $x_i \geq 0$; H_i is strictly increasing and strictly concave. Hence,

$$\begin{aligned} H_i(x_i) &\leq H_i(\phi_i \circ \delta_i^*(x_i)) + H_i'(\phi_i \circ \delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \text{i.e.} \quad H_i(x_i) &\leq H_i(\phi_i \circ \delta_i^*(x_i)) + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \end{aligned} \tag{7}$$

with equality if and only if $x_i = \phi_i \circ \delta_i^*(x_i)$. It follows from equations (6) and (7) that

$$\begin{aligned} & \int_0^{x_i} F'(\delta_i^*(t)) dt \leq \int_0^{\phi_i \circ \delta_i^*(x_i)} F'(\delta_i(t)) dt + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \iff \quad & \int_0^{x_i} F'(\delta_i^*(t)) dt \leq \int_0^{\delta_i^*(x_i)} F'(t) d\phi_i(t) + F'(\delta_i^*(x_i)) \cdot (x_i - \phi_i \circ \delta_i^*(x_i)) \\ \iff \quad & \int_0^{x_i} F'(\delta_i^*(t)) dt \leq - \int_0^{\delta_i^*(x_i)} \phi_i(t) F''(t) du + F'(\delta_i^*(x_i)) \cdot x_i \end{aligned}$$

the last expression is obtained by integrating by parts. Rearranging yields:

$$\int_0^{\delta_i^*(x_i)} \phi_i(t) F''(t) dt \leq F'(\delta_i^*(x_i)) \cdot x_i - \int_0^{x_i} F'(\delta_i^*(t)) dt.$$

Recall that $\delta_i^*(x_i) = \delta_j^*(x_j)$ for all $i \in N$; and write $z = \delta_i^*(x_i)$ for any i. Summing up over all $i \in N$ and using the fact that $\sum_i \phi_i(t) = t$ for any $t \geq 0$ and $\sum_i x_i = \sum_i \phi_i^*(z) = z$, we get:

$$\begin{aligned} & \int_0^z t F''(t) dt \leq F'(z) \cdot z - \sum_{i=1}^n \int_0^{\phi_i^*(z)} F'(\delta_i^*(t)) dt \\ \iff \quad & \int_0^z t F''(t) dt \leq F'(z) \cdot z - \sum_{i=1}^n \int_0^z F'(t) d\phi_i^*(t) \end{aligned}$$

From $\sum_i \phi_i^*(t) = t$ and integrating by parts again, this yields an equality. Therefore, equation (6) must be an equality for all $i \in N$. The choice of j and x_j

being arbitrary, it follows that $\delta_i(x_i) = \delta_i^*(x_i)$ for all $x_i \in [0, M_i]$ and for all $i \in N$. That is to say that $\phi_i \equiv \phi_i^*$ for all $i \in N$, proving the theorem.

Remark 4 *In the definition of an f-pooling method, we could replace the voluntary participation requirement with the following weaker one and Theorem 2 would still hold:*

"For any profile $u \in \mathcal{U}^N$ and any Nash equilibrium x^* of $G(\xi; u)$ the following holds:

$$\xi_i(x^*) \geq f_i(x_i^*) \qquad \forall i \in N."$$

A.2 Proof of Lemma 1

Notation: We fix a production function $F \in \mathcal{F}$, a path $\phi \in \mathcal{P}$ and a preference profile $u \in \mathcal{U}^N$. As no confusion may arise, we shall write ξ instead of ξ^ϕ. We denote by F'_- (resp. F'_+) the left (resp. right) derivative of F. Similarly, $\frac{\partial^-}{\partial\lambda}$ (resp. $\frac{\partial^+}{\partial\lambda}$) is the left-derivative (resp. right-derivative) operator. Also, we write:

(i) $\delta(x_1, ..., x_n) = (\delta_1(x_1), \delta_2(x_2), ..., \delta_n(x_n))$ for any $x \in \times_{i \in N}[0, M_i]$,

(ii) $(t_1, t_2, ..., t_{i-1}, t_i \cdot (n-i))$ is the vector of $\mathbb{R}_+^N$ with the last $(n-i)$ coordinates equal to t_i,

(iii) for any $(t_1, ..., t_n) \in \mathbb{R}_+^N$, $\phi(t_1, ..., t_n) = (\phi_1(t_1), \phi_2(t_2), ..., \phi_n(t_n))$ with a slight abuse of notation.

Let $i \in N$ and $x_i > 0$. Consider a preference (utility) u_i^* which is quasi-linear with respect to y_i such that its indifference curves are piecewise linear with a single kink at (x_i, y_i) for any $y_i \in \mathbb{R}$. Set the slope of these indifference curves to be no greater than $F'_-(\delta_i(x_i))$ before x_i and no smaller than $F'_+(x_i)$ after x_i; where "before x_i" (resp. "after x_i") stands for "at any point of $\mathbb{R}_+ \times \mathbb{R}$ with first coordinate smaller (resp. greater) than x_i".

We show below that the former quantity is the smallest variation in output that agent i can obtain via ξ by deviating infinitesimally from x_i: it corresponds to the case where she is the first one served along the path (i.e., the agent with smallest $\delta_j(x_j)$). On the other hand, $F'_+(x_i)$ is the largest variation in output obtainable via ξ at x_i by deviating marginally from x_i; it corresponds to the case where she receives all the output up to $F(x_i)$ ($\delta_j(x_j) = 0$ for all $j \neq i$).

Indeed, let $x_{-i} \in \mathbb{R}_+^{N \setminus i}$; then, from the definition of ξ, and keeping in mind that $|\cdot|$ returns the sum of the coordinates of a vector and $\wedge$ is the componentwise minimum of two vectors,

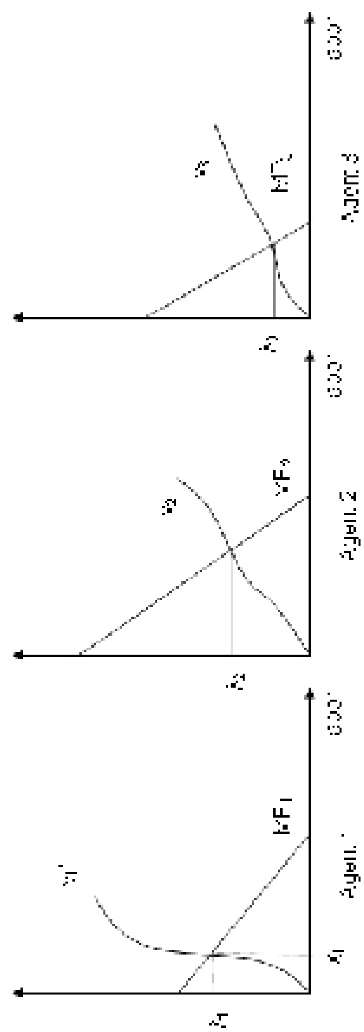
$$\tfrac{\partial^-}{\partial \lambda} \xi_i(\lambda, x_{-i}) = F'_-\left(|(\lambda, x_{-i}) \wedge \phi\left(\delta_i(\lambda) \cdot n\right)|\right) \quad \text{and} \quad \tfrac{\partial^+}{\partial \lambda} \xi_i(\lambda, x_{-i}) = F'_+\left(|(\lambda, x_{-i}) \wedge \phi\left(\delta_i(\lambda) \cdot n\right)|\right) \ .$$

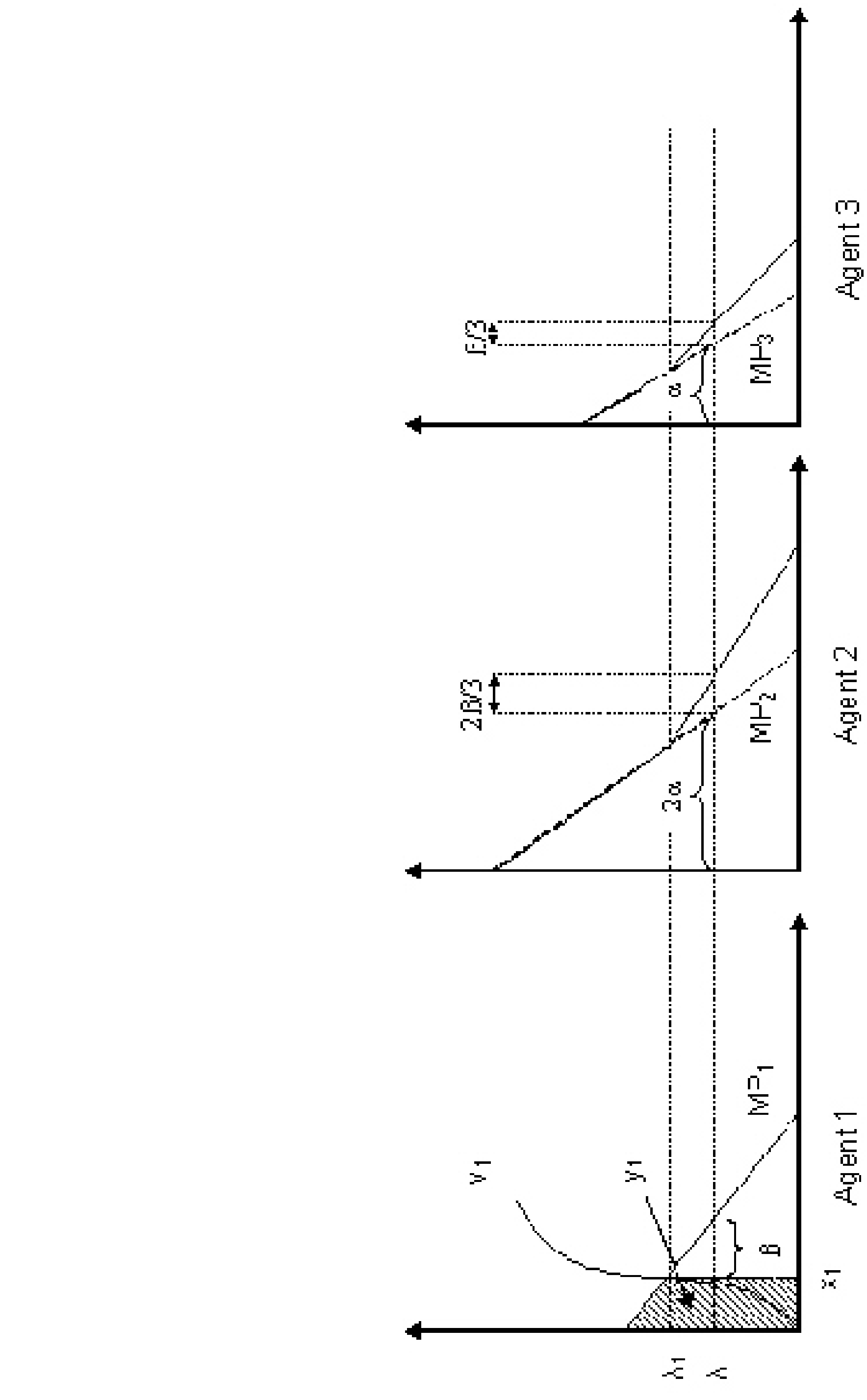
As the ith component of both vectors x and $\phi(\delta_i(x_i) \cdot n)$ is equal to x_i, the concavity of F yields $F'_+\left(|x \wedge \phi\left(\delta_i(x_i) \cdot n\right)|\right) \leq F'_+(x_i)$. Moreover, the concavity of F also yields $F'_-\left(|x \wedge \phi\left(\delta_i(x_i) \cdot n\right)|\right) \geq F'_-\left(|\phi\left(\delta_i(x_i) \cdot n\right)|\right)$; notice that this last term equals $F'_-(\delta_i(x_i))$. It follows from these two facts that:

$$\left.\frac{\partial^-}{\partial \lambda} \xi_i(\lambda, x_{-i})\right|_{\lambda = x_i} \geq F'_-(\delta_i(x_i)) \quad \text{and} \quad \left.\frac{\partial^+}{\partial \lambda} \xi_i(\lambda, x_{-i})\right|_{\lambda = x_i} \leq F'_+(x_i)$$

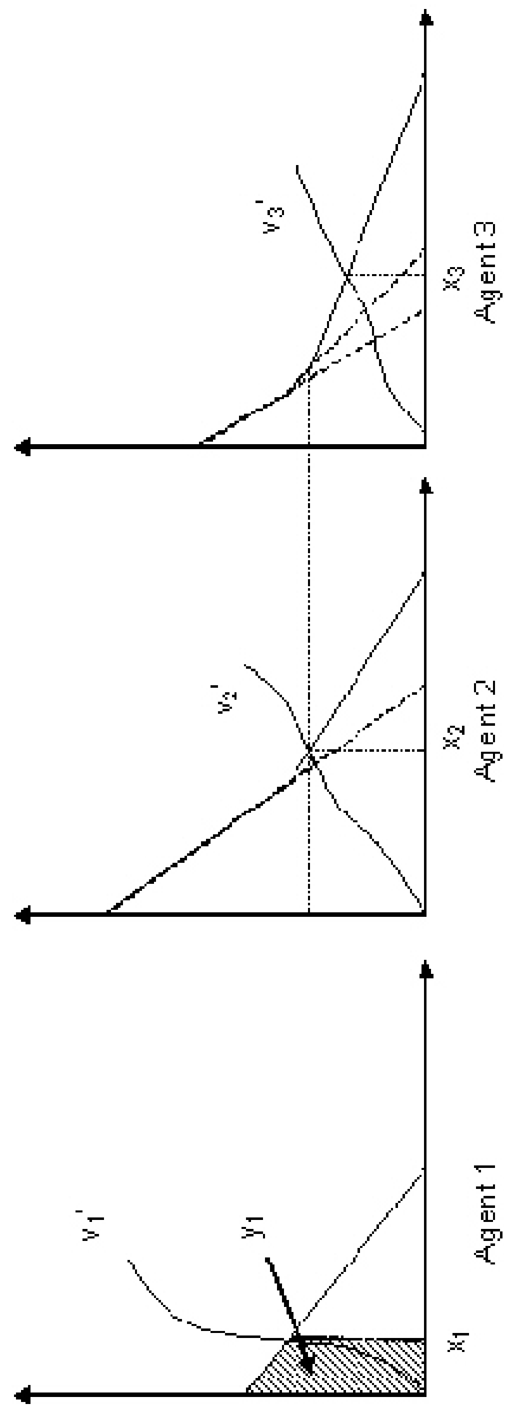
Hence, for any $x_{-i} \in \mathbb{R}_+^{N \setminus i}$, the slope of $\xi_i(\lambda, x_{-i})$ at $\lambda = x_i$ lies between $F'_-(\delta_i(x_i))$ and $F'_+(x_i)$. It follows from the strict concavity of $\xi_i(\cdot, x_{-i})$ that x_i maximizes $u_i^*(\lambda, \xi_i(\lambda, x_{-i}))$ on $\mathbb{R}_+^{N \setminus i}$ for any $x_{-i} \in \mathbb{R}_+^{N \setminus i}$, completing the proof of the lemma.

B Figures





Agent 1
Agent 2
Agent 3
MP1
MP2
MP3
2β/3
β/3
2α
α
β
v1
y1


v1'
y1
x1
Agent 1
v2'
x2
Agent 2
v3'
x3
Agent 3

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