Choice under Uncertainty with the Best and Worst in Mind: Neo-additive Capacities*.

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Abstract

The concept of a non-extreme-outcome-additive capacity (*neo-additive capacity*) is introduced. Neo-additive capacities model optimistic and pessimistic attitudes towards uncertainty as observed in many experimental studies. Moreover, neo-additive capacities can be applied easily in economic problems, as we demonstrate by examples. This paper provides an axiomatisation of Choquet expected utility with neo-capacities in a framework of purely subjective uncertainty.

"That the chance of gain is naturally over-valued we may learn from the universal success of lotteries. [...] The vain hope of gaining some of the great prizes is the sole cause of this demand. The soberest people scarce look upon it as a folly to pay a small sum for the chance of gaining ten or twenty thousand pounds."

ADAM SMITH (1776)
"The Wealth of Nations" (p.210).

"Overconfidence, however generated, appears to be a fundamental factor promoting the high volume of trade we observe in speculative markets. Without such confidence, one would think that there would be little trading in financial markets.

ROBERT SHILLER (2001)
"Irrational Exuberance" (p.144/5).

1. Introduction

Optimism and pessimism are important features of a person's attitude towards uncertainty. On an aggregate level, business cycles and stock market fluctuations have been attributed to "irrational" optimism and pessimism. Economic theory, however, finds it difficult to see in such moods a major factor determining economic behavior. With large amounts of money and wealth at stake, as in the investment behavior of traders in financial markets, one hesitates to attribute major influence on decisions to vague notions of belief.

Faced with uncertainty economists like to think of investors as cool analysts, carefully weighing likelihoods of events relevant for their decisions. Yet, many observers of investment behavior in financial markets, from KEYNES (1921) to ROBERT SHILLER (2001), could not escape the conclusion that psychological effects seem to interact with probabilistic information in shaping investors' behavior.

Embracing RAMSEY's (1926) and DE FINETTI's (1937) personalistic view of probability, SAVAGE (1954) provided a set of behavioral postulates showing that it is possible to view decision makers' behavior in the face of uncertainty as guided by a consistent system of probabilistic beliefs. His axioms gave researchers an opportunity to put these postulates to direct tests. ALLAIS (1953) and ELLSBERG (1962) were the most promi-

nent articles reporting choice behavior of people which contradicts Savage's postulates. In particular, the Sure-Thing-Principle which allows one to decompose a decision problem, omitting "equivalent parts" and focussing choice on the remaining parts, was quickly identified as especially problematic.

There are behavioral regularities which influence individuals' betting behavior. People distinguish categorically between situations which they consider as certain, just possible, or strictly impossible. These consistently observed *certainty* and *impossibility effects* cannot be modeled by a transition from zero probability of an event to a positive probability, or from a positive probability to the probability of one.

A typical lottery with a high prize on a very unlikely event can turn the certainty of low wealth for a poor person into the possibility of great riches, providing a reason for accepting an unfair gamble. Conversely, rich people may find the possibility of loosing substantial amounts of wealth so dangerous that high expected returns are necessary to induce them to an investment.

BELL (1985) interprets these psychological biases as *disappointment aversion* or *elation-seeking* behavior. He studies situations where these biases determine the behavior, such as the process of releasing information, behavior in auctions and the Ellsberg paradox. Based on these observations he argues for an inverse-S shaped pattern of decision weights as an adequate representation of individual attitudes towards uncertainty.

Optimistic behavior overestimates the likelihood of good outcomes while pessimistic attitudes exaggerate the likelihood of bad outcomes. Based upon mounting experimental evidence for certainty and impossibility effects, WAKKER (2001) extends these notions to arbitrary events with rank-ordered outcomes and characterizes *optimistic* and *pessimistic* attitudes. In the context of the Choquet expected utility (CEU) model, concave capacities reflect optimistic attitudes towards uncertainty, while convex capacities model pessimism.

1.1   Experimental evidence

CAMERER (1995) reviews numerous studies refuting the validity of the expected utility

approach as a description of individual behavior. More recently, however, experiments find evidence for typical patterns of deviation from the expected utility model. In particular, one often observes subjects willing to bet on high outcomes with low probability while refusing to accept even small risks. For subjects choosing between lotteries, one can explain such behavior by a function $w(p)$ weighting the probability p of events. Experimental studies by GONZALEZ & WU (1999), ABDELLAOUI (2000), BLEICHRODT & PINTO (2000) and others show a pattern of probability weights as in Figure 1.
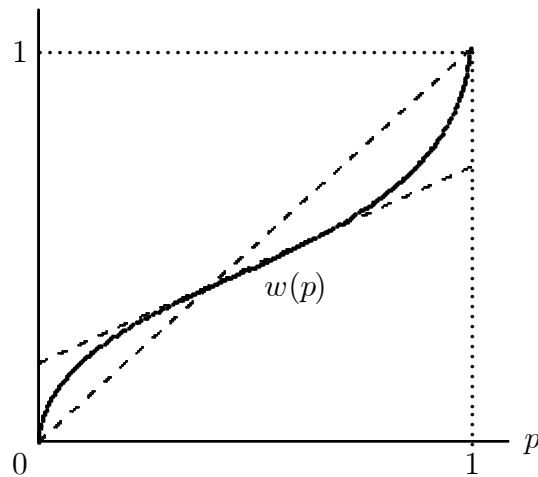

Figure 1: Probability weighting function

The decision weight of an event E, $w(p(E))$, measured by the willingness to bet on this event, differs usually from the probability of the event $p(E)$. Figure 1 shows an inverse-S shaped weighting function, overweighting probabilities close to zero and underweighting probabilities close to one as. TVERSKY & WAKKER (1995) study the relationship between decision weights and attitudes towards risk and characterize the *possibility* and *certainty effects*. WAKKER (2001) defines *optimism* and *pessimism* in terms of decision weights. This article contains also a brief survey of the relevant experimental literature. KILKA & WEBER (2001) demonstrate how decision weights and subjective probabilistic beliefs can be distinguished in experiments.

1.2 Axiomatic treatments

A simple version of an inverse-S shaped weighting function $w(p)$,

$$w(p) := \begin{cases} 1 & \text{for} \quad p = 1 \\ \lambda + (1 - \lambda - \gamma) \cdot p & \text{for} \quad 0 < p < 1 \\ 0 & \text{for} \quad p = 0 \end{cases} ,$$
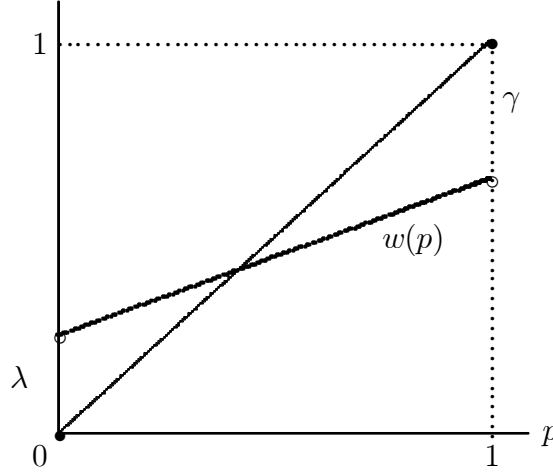
is depicted in Figure 2.

Figure 2: Neo-additive capacity

This weighting scheme, which we call *neo-additive capacity* for reasons explained below, has a simple Choquet integral. Integrating over a lottery yields a weighted average of the expected utility of the lottery and its maximal and minimal outcomes.

In the context of choice of objective lotteries there are a few studies providing behavioral axioms for functionals which combine expected utility with minimum utility and maximum utility approach. GILBOA (1988) and JAFFRAY (1988) independently axiomatize a functional which aggregates the expected utility functional and the minimum utility functional in a monotonic function. Their representations capture only pessimistic attitudes towards uncertainty. COHEN (1992), in contrast, provides axioms for a representation of preferences over lotteries as a weighted average of expected utility, minimum utility and maximum utility. For the context of choice over lotteries, her paper provides the behavioral foundations for the representation studied in this paper. To our knowledge

there is no axiomatization for choice over acts in either the Anscombe-Aumann or the Savage framework.

The *neo-additive* weighting scheme provides an easy way to model the certainty and the impossibility effects. Combined with a probability function $\Pr(.)$ over ranges of monetary outcomes, this weighting scheme models an individual who overweights the likelihood of a monetary outcome x exceeding $\overline{x}$,

$$
\begin{aligned}
w(\Pr(x \ > \ \overline{x})) &= \lambda + (1 - \lambda - \gamma) \cdot \Pr(x > \overline{x}) \\
&> \ \Pr(x > \overline{x}),
\end{aligned}
$$

whenever $\Pr(x > \overline{x}) < \lambda / (\lambda + \gamma)$. In contrast, outcomes below $\underline{x}$ occurring with low probability $\Pr(x \leq \underline{x})$ obtain a weight,

$$
\begin{aligned}
\overline{w}(\Pr(x \ \leq \ \underline{x})) &\equiv 1 - w(1 - \Pr(x > \underline{x})) \\
&= (1 - \lambda) - (1 - \lambda - \gamma)(\Pr(x \leq \underline{x})) \\
&> \ \Pr(x \leq \underline{x}),
\end{aligned}
$$

whenever $\Pr(x \leq \underline{x}) < \gamma / (\lambda + \gamma)$.

In the next section, we introduce some notation and concepts necessary for our analysis. Section 3 studies the *neo-additive* weighting scheme in the context of the CEU model. This parameterized CEU model can be easily applied to economic models in order to analyse the implications of the certainty and impossibility effect. Section 4 illustrates the potential of the *neo-additive* CEU representation for economic applications in the context of a portfolio choice model. Section 5 provides an axiomatic treatment of neo-additive capacities in a framework of purely subjective uncertainty. Proofs are collected in an appendix.

2. Capacities and the Choquet integral

We assume that the uncertainty a decision maker faces can be described by a non-empty set of *states*, denoted by S. This set may be finite or infinite. Associated with the set of states is the set of events, taken to be a sigma-algebra of subsets of S, denoted by $\mathcal{E}$.

6

We assume that for each s in S, $\{s\}$ is in $\mathcal{E}$. *Capacities* are real-valued functions defined on $\mathcal{E}$, that generalize the notion of probability distributions. Formally, a capacity is a normalized monotone set function.

Definition 2.1 *A* capacity *is a function* $\nu : \mathcal{E} \to \mathbb{R}$ *which assigns real numbers to events, such that*

      (i) $E, F \in \mathcal{E}, E \subseteq F$ *implies* $\nu(E) \leqslant \nu(F),$ monotonicity

      (ii) $\nu(\emptyset) = 0$ *and* $\nu(S) = 1.$ normalization

A capacity ν is called *convex* if $\nu(E \cup F) \geq \nu(E) + \nu(F) - \nu(E \cap F)$ holds for arbitrary events $E, F \in \mathcal{E}$. If the reverse inequality holds then the capacity is called *concave*. Probability distributions are special cases of capacities which are both concave and convex. For each capacity ν there is a *dual* or *conjugate capacity* $\overline{\nu}$ defined by $\overline{\nu}(E) = 1 - \nu(S - E)$ for all $E \in \mathcal{E}$. If the dual capacity $\overline{\nu}$ is *convex*, then the capacity ν is *concave*. The most common way to integrate functions with respect to a capacity is the *Choquet integral*. Let $f : S \to \mathbb{R}$ be a $\mathcal{E}$-measurable real-valued function. We consider finite outcome acts and suppose that f has finite range, that is, the set $f(S)$ is finite. We call a function f with these properties a *simple function*. The Choquet integral can therefore be written in the following intuitive form.

Definition 2.2 *For any simple function f the* Choquet integral *with respect to the capacity ν is defined as*

$$V(f|\nu) = \sum_{u \in f(S)} u \cdot \left[\nu\left(\{s|\, f(s) \geq u\}\right) - \nu\left(\{s|\, f(s) > u\}\right) \right].$$

Each element in the range has a decision weight equal to the difference between the capacity of the states yielding an element better or equal than the one under consideration and the capacity of the states yielding a strictly better outcome. The Choquet integral is interpreted as the expected value of the function f with respect to the capacity ν. The decision weights used in the computation of the Choquet integral will overweight high outcomes if the capacity is concave and will overweight low outcomes if the capacity is convex. It is therefore well-suited to model such responses to ambiguity as *optimism* or *pessimism*.

Sᴀʀɪɴ & Wᴀᴋᴋᴇʀ (1998) provide a detailed discussion of decision weights.

3.   Neo-additive capacities

In this section we introduce a special kind of capacity which we call a *neo-additive capacity* because it is *additive* on *non-extreme outcomes*. We begin by considering a partition of the set of events $\mathcal{E}$ into the following three subsets; the set of 'null' events, the set of 'universal' events and the set of 'essential' events, denoted $\mathcal{N}, \mathcal{U}$ and $\mathcal{E}^*$, respectively. As its name suggests, a set is 'null' if 'loosely speaking' it is impossible for it to occur. Formally, we assume that this set satisfies the following properties: (i) $\varnothing \in \mathcal{N}$, (ii) if $A \in \mathcal{N}$, then $B \in \mathcal{N}$, for all $B \subset A$. A 'universal' set is one that is viewed as being certain to occur. Formally, it is the set of events obtained by taking the complements of each member of the set of null events, that is, $\mathcal{U} = \{E \in \mathcal{E} : S - E \in \mathcal{N}\}$. Notice that since $\varnothing \in \mathcal{N}$, it follows from the definition of the set of universal events that $S \in \mathcal{U}$. Finally, every other set is 'essential' in the sense that is neither impossible nor certain, that is, $\mathcal{E}^* = \mathcal{E} - (\mathcal{N} \cup \mathcal{U})$. A capacity $\nu : \mathcal{E} \rightarrow [0, 1]$ will be deemed congruent with the set of null events $\mathcal{N}$ if $\nu(E) = 0$, for all $E \in \mathcal{N}$ and $\nu(E) = 1$, for all $E \in \mathcal{U}$.

Neo-additive capacities can be viewed as a convex combination of an additive capacity and two capacities, one of which reflects *complete ignorance* or *complete ambiguity* in everything bar a universal event *occurring*, denoted $\mu^0(.)$, and the second which reflects *complete confidence* in everything bar a null set, denoted $\mu^1(.)$. Fix $\mathcal{N}, \mathcal{U}$ and $\mathcal{E}^*$

Definition 3.1   *For all $E \in \mathcal{E}$,*

$$\mu^0(E) := \begin{cases} 1 & for \quad E \in \mathcal{U} \\ 0 & otherwise \end{cases} , \mu^1(E) := \begin{cases} 0 & for \quad E \in \mathcal{N} \\ 1 & otherwise \end{cases} .$$

The capacity $\mu^0(.)$ is *convex*. The capacity $\mu^1(.)$ is the conjugate of $\mu^0(.)$, that is $\mu^1(E) = 1 - \mu^0(S - E)$ for all E in $\mathcal{E}$, and therefore is *concave*.

Let $\pi(.)$ be a finitely additive probability distribution defined over $\mathcal{E}$, such that $\pi(A) = 0$, for all $A \in \mathcal{N}$ (and hence $\pi(B) = 1$ for all $B \in \mathcal{U}$) and denote by $\Delta := \{(\alpha, \beta) | \alpha \geq 0, \beta \geq 0, \alpha + \beta \leq 1\}$ the simplex in $\mathbb{R}^2$.

Definition 3.2 *For a given finitely additive probability distribution π on $(S, \mathcal{E})$, and a pair of numbers $(\gamma, \lambda) \in \Delta$, a* neo-additive capacity $\nu(\cdot|\pi, \gamma, \lambda)$ *is defined as*

$$\nu(E|\pi, \gamma, \lambda) := \gamma \cdot \mu^0(E) + \lambda \cdot \mu^1(E) + (1 - \gamma - \lambda) \cdot \pi(E)$$

for all E in $\mathcal{E}$.

It is straightforward to derive the Choquet integral of a simple function f with respect to a neo-additive capacity in terms a weighted sum of the infimum, the supremum and the expectation with respect to π of the act. We say $z = \inf(f)$ if $f^{-1}(x : x \geq z) \in \mathcal{U}$ and for every $y > z$, $f^{-1}(x : x > y) \notin \mathcal{U}$. Similarly, we say $z = \sup(f)$ if $f^{-1}(x : x > z) \in \mathcal{N}$ and for every $y < z$, $f^{-1}(x : x \geq y) \notin \mathcal{N}$.

Lemma 3.1 *The Choquet expected value of a simple function $f : S \to \mathbb{R}$ with respect to the neo-additive capacity $\nu(E|\pi, \gamma, \lambda)$ is given by:*

$$V(f|\nu(\cdot|\pi, \gamma, \lambda)) := \gamma \cdot \inf(f) + \lambda \cdot \sup(f) + (1 - \gamma - \lambda) \cdot \mathrm{E}_\pi[f]. \tag{1}$$

Proof. To see this note that $V(f|\mu^0(\cdot)) = \inf(f)$, $V(f|\mu^1(\cdot)) = \sup(f)$ and $V(f|\pi) = \mathrm{E}_\pi[f]$. The result then follows from the linearity of the Choquet integral with respect to the capacity (DENNEBERG (2000), Properties (ix) and (x) on page 49). ∎

Notice that we have do *not* require that $\pi(E) = 0$ implies $E \in \mathcal{N}$. Indeed nothing in the discussion and definitions above, prevent $\mathcal{N}$ consisting only of the empty set, $\varnothing$ (and hence $\mathcal{U}$ contains only the element S). In this case, even if for some event E, $\pi(E) = 0$, the capacities $\nu(E) = \lambda$ and $1 - \nu(S - E) = \gamma$ are still both positive. That is, an event E may receive a zero decision weight in the evaluation of the Choquet expected value for any act with neither its infimum nor its supremum on E, but for an act, for which the infimum (respectively, supremum) results in a state of E the decision weight on E is γ (respectively, λ). For further illustration of this point, consider the following example.

Example 3.1 *Suppose S is the unit interval, $[0, 1]$, $\mathcal{E}$ is power set of $[0, 1]$ and $\mathcal{N}$ is the set of singleton states and pairs of states. That is, $E \in \mathcal{N}$ if E is an event consisting of no more than two states. Suppose further that π is a finitely additive probability measure*

9

for which $\pi(s) = 0$ *for all* s *in* $[0, 1]$. *From the finite additivity of* π, *it follows that* $\pi(\{0, 1\}) = \pi(\{0, 0.5, 1\}) = 0$ *but only* $\{0, 1\}$ *is in* $\mathcal{N}$. *In particular,* $\nu(\{0, 1\}) = 0$ *and* $\nu(\{0, 0.5, 1\}) = \lambda$. *Furthermore, although* $\pi(S - \{0, 1\}) = \pi(S - \{0, 0.5, 1\}) = 1$, *only* $S - \{0, 1\}$ *is in* $\mathcal{U}$, *and hence* $\nu(S - \{0, 1\}) = 1 > \nu(S - \{0, 0.5, 1\}) = 1 - \gamma$.

∎

Several well-known decision criteria can be viewed as special cases of the Choquet integral of a neo-additive capacity:

(i)	$\gamma = \lambda = 0$	*expected utility,*
(ii)	$1 \geq \gamma > 0, \ \lambda = 0$	*pure pessimism,*
(iii)	$\gamma = 0, \ 1 \geq \lambda > 0$	*pure optimism,*
(iv)	$\gamma + \lambda = 1$	*Hurwitz criterion.*

Neo-additive capacities satisfy three conditions:

- They are additive for pairs of events which are not null and do not form a partition of a universal event.
- They exhibit uncertainty aversion for some events.
- They exhibit uncertainty preference for some other events.

Indeed, as the following proposition shows, these conditions characterize neo-additive capacities completely.

Proposition 3.1 *Let* ν *be a capacity on* $(S, \mathcal{E})$, *where* $\mathcal{E}^*$ *contains at least three elements* E_1, E_2 *and* E_3 *which are pairwise disjoint (that is,* $E_i \cap E_j = \emptyset$ *for all* $i \neq j$). *Then the following statements are equivalent:*

(i) ν *is a neo-additive capacity,*

(ii) the capacity ν *satisfies the following properties:*

(a) for any three events $(E, F, G) \in \mathcal{E}^* \times \mathcal{E}^* \times \mathcal{E}^*$ *such that* $E \cap F = \emptyset = E \cap G$, $E \cup F \notin \mathcal{U}$, *and* $E \cup G \notin \mathcal{U}$,

$$\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G),$$

(b) for some $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ *such that* $E \cap F = \emptyset$ *and* $E \cup F \notin \mathcal{U}$,

$$\nu(E \cup F) \leq \nu(E) + \nu(F),$$

10

(c) for some $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$,

$$\overline{\nu}(E \cup F) \leq \overline{\nu}(E) + \overline{\nu}(F).$$

Proof. Appendix. ∎

Note that for a neo-additive capacity ν on $(S, \mathcal{E})$, where $\mathcal{E}^*$ contains at least three elements, as assumed throughout the paper, uniqueness of the pessimism and optimism coefficients and of the underlying probability measure π is guaranteed. This is proved in a lemma preceding the proof of Proposition 3.1 in the appendix.

Property (iia) establishes additivity of the neo-additive capacity for events that yield non-extreme outcomes. According to property (iib), the capacity overweights the event in which the most preferred prize is obtained, hence $\lambda \geq 0$. Property (iic) says the capacity overweights also the event with the least preferred prize. It implies $\gamma \geq 0$.

3.1 Optimism and pessimism

In this section we will provide two arguments why one may be justified to interpret the overweighting of the extreme outcomes with the notions of *optimism* and *pessimism*. The first argument shows that neo-additive capacities are a special case of the behavioural concept of optimism and pessimism advanced in WAKKER (2001). The second argument appeals to the intuitive notion of optimism and pessimism suggested by the context of the multiple prior approach.

3.1.1 The behavioural approach of Wakker (2001)

Inspired by the Allais and Ellsberg paradox ,WAKKER (2001) suggests a notion of optimism and pessimism based on choice behaviour over acts. This approach derives its appeal from its immediate testability in experiments and its natural representation by properties of capacities.

Properties (iib) and (iic) of Proposition 3.1 imply the neo-additive capacity to be concave on some events, which corresponds to the notion of *optimism* suggested in WAKKER (2001), and convex on some others, hence *pessimistic* in the sense of WAKKER (2001). To see this, consider the following four acts f_1, f_2, f_3, f_4 defined on a partition of the state space (B, A, I, L) with lottery outcomes $M \succ m \succ 0$.

	B	A	I	L
f_1	M	m	m	0
f_2	M	M	0	0
f_3	m	m	m	m
f_4	m	M	0	m

$$(*)$$

Assume that m is chosen such that $V(f_1) = V(f_2)$. WAKKER (2001) calls a decision maker *pessimistic* if $V(f_3) > V(f_4)$ and *optimistic* if $V(f_3) < V(f_4)$. Of course, an expected utility maximiser must be indifferent between f_3 and f_4.

If the decision maker is indifferent between acts f_1 and f_2, then m measures the willingness to pay for the gamble M on A and 0 on I conditional on the gamble M on B and 0 on L. If f_3 is preferred to f_4, then the gamble M on A and 0 on I is worth less than m because there is no chance of losing in event L. This special attention given to bad outcomes is associated with pessimism. In contrast, an optimist, will be willing to pay more for the gamble M on A and 0 on I if there is no chance of winning in event B.

Consider a neo-additive capacity $\nu(\cdot|\pi, \gamma, \lambda)$. Since neo-additive capacities exhibit pessimism for some acts and optimism for others, we have to distinguish two cases.

Case (i): *pessimism*

Assume $B \in \mathcal{N}$. From $V(f_1|\nu(\cdot|\pi, \gamma, \lambda)) = V(f_2|\nu(\cdot|\pi, \gamma, \lambda))$, we conclude that

$$[\lambda + (1 - \lambda - \gamma) \cdot \pi(A \cup I)] \cdot m = [\lambda + (1 - \lambda - \gamma) \cdot \pi(A)] \cdot M.$$

Hence,

$$
\begin{aligned}
& V(f_4|\nu(\cdot|\pi, \gamma, \lambda)) \\
= \ & [\lambda + (1 - \lambda - \gamma) \cdot \pi(A)] \cdot M + (1 - \lambda - \gamma) \cdot \pi(L) \cdot m \\
= \ & [\lambda + (1 - \lambda - \gamma) \cdot \pi(A \cup I \cup L)] \cdot m \\
= \ & (1 - \gamma) \cdot m < m = V(f_3|\nu(\cdot|\pi, \gamma, \lambda)).
\end{aligned}
$$

Case (ii): *optimism*

Assume $L \in \mathcal{N}$. From $V(f_1|\nu(\cdot|\pi, \gamma, \lambda)) = V(f_2|\nu(\cdot|\pi, \gamma, \lambda))$, we conclude now

$$[\gamma + (1 - \lambda - \gamma) \cdot \pi(A \cup I)] \cdot m = (1 - \lambda - \gamma) \cdot \pi(A) \cdot M.$$

Therefore,

$$V(f_4|\nu(\cdot|\pi, \gamma, \lambda))$$

$$= \lambda \cdot M + (1 - \lambda - \gamma) \cdot \pi(A) \cdot M + (1 - \lambda - \gamma) \cdot \pi(B) \cdot m$$

$$= \lambda \cdot M + [\gamma + (1 - \lambda - \gamma) \cdot \pi(A \cup I \cup B)] \cdot m$$

$$= \lambda \cdot M + (1 - \lambda) \cdot m > m = V(f_3|\nu(\cdot|\pi, \gamma, \lambda)).$$

It is easy to check that condition $B \in \mathcal{N}$. is necessary for pure pessimism, and $L \in \mathcal{N}$ for pure optimism. If none of the four elements of the partition are elements of $\mathcal{N}$, then $V(f_1|\nu(\cdot|S, \pi, \gamma, \lambda)) = V(f_2|\nu(\cdot|S, \pi, \gamma, \lambda))$ does not imply an unambiguous ranking of $V(f_3|\nu(\cdot|S, \pi, \gamma, \lambda))$ and $V(f_4|\nu(\cdot|S, \pi, \gamma, \lambda))$. Neo-additive capacities show both optimism and pessimism as they relate to the certainty and the impossibility effect[1].

3.1.2 The multiple-prior approach

The multiple-prior approach assumes that uncertainty of a decision maker can be represented by a set of probability distributions $\mathcal{D}$. The evaluation of an act is the expected utility with respect to the probability distributions in the set $\mathcal{D}$. Attitudes towards the ambiguity about the correct probability distribution can then be captured by weight which the decision maker puts on the best or worst expected utility over the set $\mathcal{D}$.

Without loss of generality, but for the ease of exposition, assume there is a finite set of states $S = \{1, ..., n\}$. The set of probability distributions over S is the simplex Δ^n in this case. Let the set of probability distributions $\mathcal{D} \subseteq \Delta^n$, which the decision maker considers possible, be compact and convex. In the multiple-prior approach preferences over acts $(x_1, ..., x_n)$ are represented by the preference functional

$$\mathcal{V}(x_1, ..., x_n|\alpha, \mathcal{D}) := \alpha \cdot \min_{\widetilde{\pi} \in \mathcal{D}} \left[\sum_{s \in S} u(x_s) \cdot \widetilde{\pi}_s \right] + (1 - \alpha) \cdot \max_{\widetilde{\pi} \in \mathcal{D}} \left[\sum_{s \in S} u(x_s) \cdot \widetilde{\pi}_s \right].$$

The parameter $\alpha \in [0, 1]$ represents the weight given to the worst expected utility and $(1 - \alpha)$ is the weight given to the best expected utility. Optimism corresponds to the weight given the best expected utility and pessimism to the weight given the worst expected utility. A nice feature of this approach is the natural separation between ambiguity,

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[1]   They are *cavex* in the sense of WAKKER (2001, P. 1049), that is concave for events which are revealed unlikely and convex for events which are revealed as likely.

reflected by the set $\mathcal{D}$, and attitudes towards ambiguity, reflected by the degrees of pessimism α and optimism $(1 - \alpha)$.

The Choquet expected utility of a neo-additive capacity defined in Equation 1 can be viewed as a multiple-prior expected utility. Applied to the case of a finite state space, it is not difficult to verify the following equality:

$$
V(x_1, ..., x_n | \nu(\cdot | \pi, \gamma, \lambda)) :=
$$

$$
\gamma \cdot \min_{s \in S}\ u(x_s) + \lambda \cdot \max_{s \in S}\ u(x_s) + (1 - \gamma - \lambda) \cdot \left[\sum_{s \in S} u(x_s) \cdot \pi_s \right]
$$

$$
= \quad \frac{\gamma}{\gamma + \lambda} \cdot \min_{\widetilde{\pi} \in \mathcal{D}} \left[\sum_{s \in S} u(x_s) \cdot \widetilde{\pi}_s \right] + \frac{\lambda}{\gamma + \lambda} \cdot \max_{\widetilde{\pi} \in \mathcal{D}} \left[\sum_{s \in S} u(x_s) \cdot \widetilde{\pi}_s \right]
$$

$$
= \quad : \mathcal{V}(x_1, ..., x_n | \frac{\gamma}{\gamma + \lambda}, \mathcal{D}),
$$

with

$$
\mathcal{D} := \left\{ \widetilde{\pi} \in \Delta^n |\ \widetilde{\pi}_s \geq (1 - \gamma - \lambda) \cdot \pi_s,\ s \in S \right\}.
$$

Note that the set of probabilities $\mathcal{D}$ is convex and compact. Moreover, it has a nice geometric structure as Figure 3 illustrates for the case of $n = 3$. For a neo-additive capacity
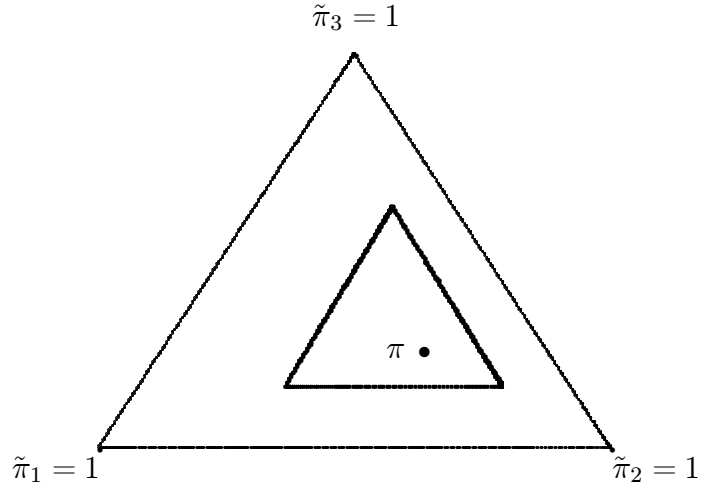

Figure 3: Multiple priors

14

$\nu(\cdot|\pi, \gamma, \lambda)$ the set of possible probability distributions $\mathcal{D}$ is centered around the probability distribution π and has a size determined by $\gamma + \lambda$. Hence, $\gamma + \lambda$ can be viewed as the degree of ambiguity about the additive probability distribution π. The degree of pessimism $\frac{\gamma}{\gamma+\lambda}$ and the degree of optimism $\frac{\lambda}{\gamma+\lambda}$ measure the decision maker's attitude towards this ambiguity. Thus, neo-additive capacities have also a natural interpretation in the context of the multiple-prior approach.

It is worth noting that it is well-known that the Choquet expected utility approach is equivalent to the multiple prior approach if capacities are convex. It is worth noting that, in general, neo-additive capacities are neither convex nor concave.

4. Economic Applications

The ups and downs of economic activity during the business cycle which are usually accompanied by swings in investors' sentiments, ranging from bull to bear spirits in financial markets, provide numerous examples of the impact of uncertainty on economic behavior.

Neo-additive capacities provide a natural way for modelling optimism and pessimism influencing economic activities. The parameters of a neo-additive capacity can be interpreted as measuring confidence in beliefs and degrees of optimism and pessimism. A neo-additive capacity $\nu(E|\pi, \gamma, \lambda)$ is based on an additive probability distribution π reflecting the subjective beliefs of the decision maker. It represents an assessment of the likelihood of events consistent with the individual's belief. The weight $(1 - \gamma - \lambda)$ given to π is a measure of the *degree of confidence* which the individual holds in this belief. The core belief of a neo-additive capacity represented by the additive probability distribution π can be determined endogenously in equilibrium[2]. Thus, standard equilibrium analysis is always the special case of full confidence, $\gamma = \lambda = 0$.

Positive parameters γ and λ represent the impact of pessimism and optimism respectively. Neo-additive capacities can therefore model psychological phenomena such as

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[2]   EICHBERGER & KELSEY (2000) provide a thorough analysis of strategic games when beliefs are modelled as non-additive capacities.

excessive optimism and pessimism which have been put forward as explanations for economic behavior in depressions or bubbles and which have been confirmed in laboratory experiments.

In this section we show by example that optimism and pessimism can explain behavior inconsistent with expected utility maximization. In theses cases, optimism and pessimism can help to explain well-known economic puzzles. We will reconsider the paradox of people buying insurance and gambling, and we will review portfolio choice behavior where one observes unreasonably high risk premia (the *equity premium puzzle*) and a willingness to invest in high-risk stock of unknown start-up companies (the *small stock puzzle*).

4.1 Insurance and gambling

The same individual is often observed to buy both insurance against risk and lottery tickets. As our introductory quotation of Adam Smith illustrates, such behavior is ubiquitous but hard to reconcile with rational decision making based on probabilistic calculus. For expected utility maximizers with a von Neumann-Morgenstern utility function such behavior is hard to explain.[3] Buying insurance suggests a preference for reduced risk, while paying for a lottery implies preference for a risky gamble, often at very unfair odds.

To see how both types of behavior can be accommodated by a neo-additive capacity, consider an individual endowed with wealth x, whose preferences over lotteries can be represented by the Choquet expected utility of a neo-additive capacity, with parameters $\gamma > \lambda \geq 0$ for the neo-additive capacity and utility index u (taken to be concave). This individual faces a (small) probability π_L of incurring a loss of size L. Insurance coverage is available at a premium q. Also available at a price p is a lottery ticket that 'wins' with (a very small) probability π_W and pays out the single prize of size W and otherwise pays out nothing. Suppose that the individual views the event in which he incurs the loss and the event in which he wins the lottery (should he purchase a ticket) are independent.

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[3]   FRIEDMAN & SAVAGE (1948) suggest an S-shaped von Neumann-Morgenstern utility function. This approach to reconcile such behaviour has been critisised by MARKOWITZ (1952). See HIRSHLEIFER & RILEY (1992) for a discussion of the Friedman-Savage approach (pp. 26-28).

Further suppose that $[\gamma + (1 - \gamma - \lambda) \cdot \pi_L] \cdot L > q \geq \pi_L \cdot L$. The weak inequality is a feasibility condition for the insurance premium to cover at least the expected loss (and if strict it means that the insurance coverage is actuarially unfair). The strict inequality is satisfied if the individual has a positive degree of pessimism γ and if the potential loss L is sufficiently large.

The difference in the Choquet expected utilities between buying and not buying the insurance is

$$u\left(x - q\right) - \left(\left[\lambda + (1 - \gamma - \lambda) \cdot (1 - \pi_L)\right] \cdot u\left(x\right) + \left[\gamma + (1 - \gamma - \lambda) \cdot \pi_L\right] \cdot u\left(x - L\right)\right)$$

$$\geq \; u\left(x - q\right) - u\left(\left[\lambda + (1 - \gamma - \lambda) \cdot (1 - \pi_L)\right] \cdot x + \left[\gamma + (1 - \gamma - \lambda) \cdot \pi_L\right] \cdot (x - L)\right)$$

$$= \; u\left(x - q\right) - u\left(x - \left[\gamma + (1 - \gamma - \lambda)\pi_L\right] L\right) > 0$$

The first inequality follows from Jensen's inequality applied to the convex function $-u$, and the second inequality follows from monotonicity of u. Figure 4 illustrates the desirability of purchase of full coverage at the unfair premium for the case where u is affine.
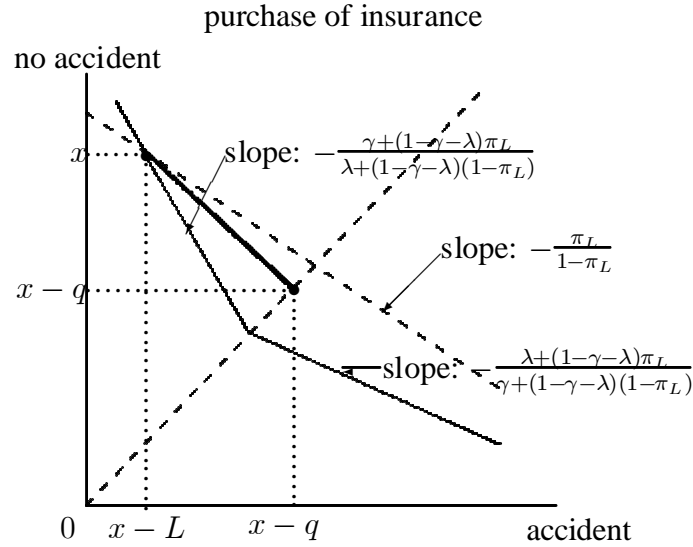

Figure 4: Insurance

Having purchased the insurance, the difference in Choquet expected utilities between

buying the lottery ticket and not may now be expressed as

$$[\lambda + (1 - \gamma - \lambda) \cdot \pi_W] \cdot u(x + W - p - q)$$

$$+ [\gamma + (1 - \gamma - \lambda) \cdot (1 - \pi_W)] \cdot u(x - p - q) - u(x - q)$$

$$> \quad \lambda \cdot u(x + W - p - q) + (1 - \lambda) \cdot u(x - p - q) - u(x - q)$$

$$\geq \quad \lambda \cdot [u(x + W - p - q) - u(x - q)] - p \cdot u'(x - q).$$

The last inequality follows from the concavity of u. For $\lambda > 0$ and u strictly increasing, there is a lottery (W, π_W, p) with W high enough and π_W small enough such that $\pi_W \cdot W < p$ and

$$\frac{\lambda \cdot [u(x + W - p - q) - u(x - q)]}{p} > u'(x - q).$$

Notice that this is true for any degree of concavity of u. Optimism makes lotteries with high prizes and low probabilities of winning attractive even for individuals who are averse to accepting actuarially fair fifty-fifty gambles. Figure 5 illustrates the desirability of the purchase of an unfair lottery ticket for the case where u is affine.

4.2   Portfolio choice

There are numerous puzzles in portfolio choice theory. THALER (2000) provides a stimulating exposition of some well-known irregularities. These puzzles highlight inconsistencies between standard economic theories and empirical regularities. Naturally, not all can be related to optimism or pessimism. The following two puzzles however can be explained easily by a small degree of optimism and pessimism.

The *equity premium puzzle* refers to the large difference between the average return on a stock portfolio and the return of a fixed interest bearing bond which was first noted by MEHRA & PRESCOTT (1985). The implied risk premium appears to be too big to be explained by risk aversion as modelled by a concave von Neumann-Morgenstern utility function. The conservative behavior in the face of uncertainty suggested by such a high risk premium stands in stark contrast to the observation that small firms with high-risk stocks seem to attract investors' interest more than is warranted by their average returns.
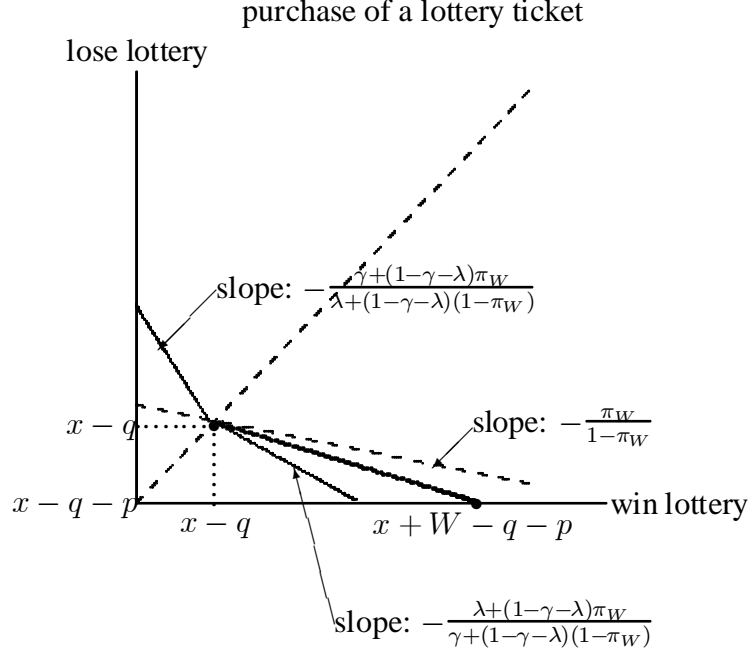
purchase of a lottery ticket

lose lottery

slope: $-\dfrac{\gamma+(1-\gamma-\lambda)\pi_W}{\lambda+(1-\gamma-\lambda)(1-\pi_W)}$

slope: $-\dfrac{\pi_W}{1-\pi_W}$

$x-q$

$x-q-p$

$x-q$ $x+W-q-p$ win lottery

slope: $-\dfrac{\lambda+(1-\gamma-\lambda)\pi_W}{\gamma+(1-\gamma-\lambda)(1-\pi_W)}$

Figure 5: Gambling

To invest in stock of young "promising" companies appears to be extraordinarily risky. Yet such uncertainty did not deter investors who otherwise requested a surprisingly high risk premium.

We study a simple financial market system with a representative investor, one risky and one riskless asset and an exogenous supply of assets. This framework suffices to illustrate the impact of optimism and pessimism on portfolio choice. With well-known modifications these results carry over to more general models of financial markets.

Consider an investor with initial wealth W_0 who can invest in two assets, a stock with uncertain returns and a bond with a certain payoff. The following table summarizes the notation of the assets.

asset	quantity	price	payoff in state $s \in S$
stock	a	q	r_s
bond	b	1	r

Preferences of the investor are represented by a Choquet expected utility $V(W_1, ..., W_S)$

of end-of-period wealth, $W_s = r_s \cdot a + r \cdot b$, with respect to a neo–additive capacity

$$V(W_1, ..., W_S)$$
$$:= \gamma \cdot \min\{u(W_1), ..., u(W_S)\} + \lambda \cdot \max\{u(W_1), ..., u(W_S)\}$$
$$+(1 - \gamma - \lambda) \cdot \sum_{s \in S} \pi_s \cdot u(W_s).$$

Using the budget constraint, $W_0 = q \cdot a + b$, to substitute for the bond, one gets wealth as a function of stock transactions a,

$$W_s = r \cdot W_0 + [r_s - q \cdot r] \cdot a.$$

Denoting by $\overline{r} = \max\{r_1, ..., r_S\}$ and $\underline{r} = \min\{r_1, ..., r_S\}$ the maximal and minimal returns of the risky stock, one can write the Choquet expected utility from a stock investment $a > 0$ as

$$V(a) := \ \gamma \cdot u(r \cdot W_0 + [\underline{r} - q \cdot r] \cdot a)$$
$$+\lambda \cdot u(r \cdot W_0 + [\overline{r} - q \cdot r] \cdot a)$$
$$+(1 - \gamma - \lambda) \cdot \sum_{s \in S} \pi_s \cdot u(r \cdot W_0 + [r_s - q \cdot r] \cdot a).$$

For a stock market equilibrium price q^* with an aggregate endowment of equity $A > 0$ and bonds $B = 0$ where the single investor maximizes Choquet expected utility $V(a)$,

$$V'(A) = \ \gamma \cdot u'(r \cdot W_0 + [\underline{r} - q^* \cdot r] \cdot A) \cdot [\underline{r} - q^* \cdot r]$$
$$+\lambda \cdot u'(r \cdot W_0 + [\overline{r} - q^* \cdot r] \cdot A) \cdot [\overline{r} - q^* \cdot r]$$
$$+(1 - \gamma - \lambda) \cdot \sum_{s \in S} \pi_s \cdot u'(r \cdot W_0 + [r_s - q^* \cdot r] \cdot A) \cdot [r_s - q^* \cdot r] \ = 0$$

must hold in equilibrium. Substituting for the initial wealth $W_0 = q^* \cdot A$, this equilibrium condition can be solved explicitly for the equilibrium stock price q^*,

$$q^* = \frac{\gamma \cdot u'(\underline{r} \cdot A) \cdot \underline{r} + \lambda \cdot u'(\overline{r} \cdot A) \cdot \overline{r} + (1 - \gamma - \lambda) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \left[\gamma \cdot u'(\underline{r} \cdot A) + \lambda \cdot u'(\overline{r} \cdot A) + (1 - \gamma - \lambda) \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}. \quad (*)$$

The case of subjective expected utility, $\gamma = \lambda = 0$, is the reference situation against which we can assess the impact of optimism or pessimism. Denote by q_0^* the equity price in this case,

$$q_0^* = \frac{\sum_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \sum_{s \in S} \pi_s \cdot u'(r_s \cdot A)}.$$

The *equity premium* is defined as the ratio

$$\alpha(q^*) := \frac{\sum_{s \in S} \pi_s \cdot r_s}{q^* \cdot r}.$$

20

The smaller q^* the greater the equity premium.

Consider first the case of a risk-neutral investor, $u'(\cdot) = k$. In this case, the equilibrium stock price equals the discounted expected return of the stock plus an optimism and pessimism related premium

$$
\begin{aligned}
q^* &= \frac{\sum\limits_{s \in S} \pi_s \cdot r_s}{r} + \frac{\gamma \cdot \left(\underline{r} - \sum\limits_{s \in S} \pi_s \cdot r_s\right) + \lambda \cdot \left(\overline{r} - \sum\limits_{s \in S} \pi_s \cdot r_s\right)}{r} \\
&= q_0^* + \frac{1}{r} \cdot \left[\gamma \cdot (\underline{r} - \mathcal{E}_\pi r_s) + \lambda \cdot (\overline{r} - \mathcal{E}_\pi r_s)\right],
\end{aligned}
$$

where we denote the expected return of the stock by $\mathcal{E}_\pi r_s := \sum\limits_{s \in S} \pi_s \cdot r_s$.

Since $\overline{r} > \mathcal{E}_\pi r_s > \underline{r}$, optimism will add a positive premium, $\lambda \cdot (\overline{r} - \mathcal{E}_\pi r_s)$, to the reference price q_0^*, while pessimism will make the premium negative, $\gamma \cdot (\underline{r} - \mathcal{E}_\pi r_s)$. For a risk-neutral investor, we can note that

- the equity premium will be the higher the more pessimistic the investor is, i.e. the smaller γ;

- if there is no optimism, $\lambda = 0$, then the equity premium required by a pessimistic investor will be strictly higher than the one based on the subjective probability distribution π alone, $\alpha(q^*) > \alpha(q_0^*)$;

- if both optimism and pessimism prevail, but $\gamma > \lambda$ as in most experimental studies, then a sufficient condition for an equity premium $\alpha(q^*)$ exceeding $\alpha(q_0^*)$ is an average return exceeding the average of the minimum and the maximum return,

$$
\frac{\overline{r} + \underline{r}}{2} < \mathcal{E}_\pi r_s.
$$

For a risk-averse investor with a strictly decreasing marginal utility function $u'(\cdot)$ similar results can be deduced which we summarize in a lemma.

Lemma 4.1 *A* risk-averse *and* pessimistic *investor, i.e., with $\lambda = 0$, $\gamma > 0$, $u'(\cdot)$ strictly decreasing, requires an equity premium exceeding the equity premium with risk but no pessimism,*

$$
\alpha(q^*) > \alpha(q_0^*).
$$

21

Proof. Appendix. ■

In recent years "new stock markets" have emerged in many developed countries where stock of start-up firms is traded. These markets were opened in order to provide venture capital for new high-risk enterprises with great potential. In the light of the rather conservative behavior reflected in the equity premium puzzle it is even more surprising that investors were willing to bet substantial amounts of wealth on firms with no record of earnings.

Optimism and pessimism as modelled with a neo-additive capacity enables us to explain such behavior. In fact, we can show that for an arbitrary small degree of optimism there are maximal returns of a firm high enough to induce a positive stock price for high-risk firms with potentially high returns. Reconsider the stock market equilibrium price of Equation (*) and assume, without loss of generality, that the firm's stock pays off a return R only in state 1. Hence, $\overline{r} = r_1 = R$ and $\underline{r} = r_s = 0$ for all $s \neq 1$. Suppose the expected return of the firm is bounded away from zero, $\pi_1 \cdot R \geq \kappa > 0$. Then the equilibrium price satisfies

$$
\begin{aligned}
q^* &= \frac{1}{r} \cdot \frac{\lambda \cdot u'(R \cdot A) \cdot R + (1 - \gamma - \lambda) \cdot \pi_1 \cdot u'(R \cdot A) \cdot R}{\gamma \cdot u'(0) + \lambda \cdot u'(R \cdot A) + (1 - \gamma - \lambda) \cdot \left[\pi_1 \cdot u'(R \cdot A) + \sum_{s \neq 1} \pi_s \cdot u'(0) \right]} \\
&= \frac{R}{r} \cdot \frac{u'(R \cdot A) \cdot [\lambda + (1 - \gamma - \lambda) \cdot \pi_1]}{u'(R \cdot A) \cdot [\lambda + (1 - \gamma - \lambda) \cdot \pi_1] + u'(0) \cdot [\gamma \cdot + (1 - \gamma - \lambda) \cdot (1 - \pi_1)]} \\
&\geq \frac{R}{r} \cdot [\lambda + (1 - \gamma - \lambda) \cdot \pi_1] \\
&> \frac{R}{r} \cdot \lambda,
\end{aligned}
$$

where the first inequality follows from $u'(0) \geq u'(R \cdot A)$ and the second strict inequality from the positive expected return.

It is clear that with some optimism, $\lambda > 0$, even a vanishing probability of success π_1 will not deter investors provided the return rises sufficiently, $R \geq \kappa/\pi_1$. The stock market price will not collapse. There is no contradiction if investors buy high-risk stock because of optimism, $\lambda > 0$, and require an "excessive" equity premium. Adam Smith's observa-

tion that even "sober people" do play lotteries and Robert Shiller's observed "exuberance" in the stock market can be reconciled with rational decision making under uncertainty, if one allows for optimism and pessimism as modelled by neo-additive capacities.

5.   Behavioral axioms

We present our theory in the context of a variant of SAVAGE'S (1954) purely subjective uncertainty framework employed by GHIRARDATO & MARINACCI (2001) and GHI-RARDATO, MACCHERIONI, MARINACCI & SINISCALCHI (2002) (hereafter, GMMS). The state space S is taken to be the same as was defined in section 2 above. Let X, the set of outcomes, be a connected and separable topological space. An *act* is a function (measurable with respect to $\mathcal{E}$) $f : S \to X$ with *finite* range, $\mathcal{F}$ denotes the set of such acts and is endowed with the product topology induced by the topology on X. We shall identify each $x \in X$ with the constant act, $f(s) = x$ for all $s \in S$. For any pair of acts f, g in $\mathcal{F}$ and any event $E \in \mathcal{E}$, $f_E g$ will denote the act $h \in \mathcal{F}$, formed from the concatenation of the two acts f and g, in which $h(s)$ equals $f(s)$ if $s \in E$, and equals $g(s)$ if $s \notin E$.

Let $\succsim$ denote the individual's preference relation on $\mathcal{F}$. For any $f \in \mathcal{F}$, the *certainty equivalent* of f, denoted by $m(f)$, is the set of constant acts that are indifferent to f. That is, $x \in m(f)$, if $x \sim f$. Although many constant acts may be equivalent, when there is no risk of confusion, we shall write $m(f)$ to indicate an arbitrary member of the set.

We say f and g are comonotonic if for every pair of states s and s' in S, $f(s) \succ f(s')$ implies $g(s) \succsim g(s')$. We say an event $E \in \mathcal{E}$ is null if $f_E g \sim g$ for all $f, g \in \mathcal{F}$. Let $\mathcal{N}$ denote the set of null events. An event E is *universal*, if its complement is null, that is, $S - E \in \mathcal{N}$. We shall denote by $\mathcal{E}^*$, the set of events that are neither impossible nor certain, that is the set $\mathcal{E} - (\mathcal{N} \cup \mathcal{U})$.

For ease of exposition and without any essential loss of generality we assume there exist outcomes 0 and M in X, that are, respectively, the "*best*" and "*worst*" outcomes in X, in the sense that $M \succ 0$ and $M \succsim x \succsim 0$ for all $x \in X$.

Neo-additive capacities are a special case of the Choquet expected utility theory. In order

23

to obtain a behavioral characterization, we seek to modify the axioms of GMMS appropriately. Their key innovation is to define a behavioral definition of 'subjective mixtures' of acts which allows them to define in a Savage framework of purely subjective uncertainty, analogs to axioms based on probability mixtures that play such a key role in the Anscombe-Aumann framework.

The first is the standard ordering axiom.

Axiom 1 (*Ordering*)

The preference relation $\succsim$ on $\mathcal{F}$ is complete, reflexive and transitive.

The neo-additive expected utility representation allows for the 'discontinuous over-weighting' of events on which extreme, i.e. either best or the worst, outcomes obtain. Hence, standard continuity with respect to the product topology cannot be expected to hold for the whole preference relation. Following GHIRARDATO & MARINACCI (2001) we only require a weaker notion of pointwise convergence, where in this product topology, we say a net $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$ converges pointwise to $f \in \mathcal{F}$, if and only if $f_\alpha(s) \to f(s)$ for all $s \in S$.

Axiom 2 (*Continuity*).

Let $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$ be a net that converges pointwise to f and such that all f_αs and f are measurable with respect to the *same* finite partition.. If $f_\alpha \succsim g$ (respectively, $g \succsim f_\alpha$) for all $\alpha \in D$, then $f \succsim g$ (respectively, $g \succsim f$).

We also adopt the monotonicity axiom of CHEW & KARNI (1994) which combines statewise dominance with a weakening of Savage's axiom P3.

Axiom 3 (*Eventwise Monotonicity*).

For any pair of acts, $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$. In addition, for any triple of outcomes $x, y, z \in X$, and any event $E \notin \mathcal{N}$

(a) if $x \succsim z$, $y \succsim z$ then $x \succ y \Rightarrow x_E z \succ y_E z$;

(b) if $z \succsim x$, $z \succsim y$ then $x \succ y \Rightarrow x_E z \succ y_E z$.

The next axiom due to GHIRARDATO & MARINACCI (2001) builds on the idea of

NAKAMURA (1990) and GUL (1992) of a 'subjective mixture' of two acts f and g. Fix some event E, and then construct state by state an act which yields at each state s, the certainty equivalent of the bet $f(s)_E g(s)$. Formally, the statewise (event) E-mixture of f and g, denoted as $f^E g$, is taken to be the act

$$f^E g(s) = m(f(s)_E g(s)).$$

Adopting the shorthand $\{x, y\} \succsim z$ for $x \succsim z$ and $y \succsim z$, and $z \succsim \{x, y\}$ for $z \succsim x$ and $z \succsim x$, the next axiom may be stated as follows.

Axiom 4 (*Binary Comonotonic Act Independence*)

For any event $A \in \mathcal{E}^*$ (that is, neither A nor $S - A$ is null), any event $B \in \mathcal{E}$, and for all $f, g, h \in \mathcal{F}$, such that $f = x_A y$, $g = x'_A y'$, $h = x''_A y''$. If f, g, h are pairwise comonotonic, and $\{x, x'\} \succsim x''$ and $\{y, y'\} \succsim y''$ (or $x'' \succsim \{x, x'\}$ and $y'' \succsim \{y, y'\}$), then

$$f \succsim g \Rightarrow f^B h \succsim g^B h.$$

As its names suggests, Binary Comontonic Act Independence, means that the preference relation restricted to acts that are measurable with respect to two-element partitions, conforms to the theory of Choquet Expected Utility. With these four axioms, GHIRARDATO & MARINACCI (2001) were able to prove that the preference relation admits what they dubbed a (canonical) biseparable representation, namely, a Choquet Expected Utility representation defined on this restricted set of acts.

Proposition 5.1 (Ghirardato and Marinacci [2001], Theorem 11) *Let X be a connected and separable topological space and let $\succsim$ be a binary relation on $\mathcal{F}$ for which there exist outcomes 0 and M in X, such that $M \succ 0$ and $M \succsim x \succsim 0$ for all $x \in X$. Then the following are equivalent:*

(i) $\succsim$ satisfies Axioms 1-4 and there exists an event A such that A and $S - A$ are both non-null.

(ii) There exist a unique continuous utility index $u : X \rightarrow [0, 1]$, with $u(0) = 0$ and $u(M) = 1$, and a unique capacity $\nu : \mathcal{E} \rightarrow [0, 1]$ such that for all x, y, x', y', such that

25

$x \succsim y$ and $x' \succsim y'$ and all $E, E' \in \mathcal{E}$

$$x_E y \quad \succsim \quad x'_{E'} y' \tag{2}$$

$$\Leftrightarrow \quad \nu(E) u(x) + (1 - \nu(E)) u(y) \geq \nu(E') u(x') + (1 - \nu(E')) u(y')$$

It remains to impose an appropriate version of an independence-type axiom that extends the biseparable CEU representation obtained in Proposition 5.1 to the whole domain $\mathcal{F}$ and moreover entails that the capacity in that representation is neo-additive. To do this, we first need to define GMMS's notion of a 'subjective mixture' of two acts. We begin with their definition of a 'preference average' of two consequences..

Definition 5.1 *Fix* $x, y \in X$, *such that* $x \succ y$. *We say that a consequence* $z \in X$ *is a* preference average *of* x *and* y *(given E) if* $x \succsim z \succsim y$ *and*

$$x_E y \sim m (x_E z)_E m (z_E y)$$

The reason for their nomenclature becomes apparent if we consider for a preference relation that satisfies Axioms 1-4, the preference average of x and y given an event E that is not null and whose complement is also not null. From Proposition (5.1) we obtain the equality

$$\nu(E) u(x) + (1 - \nu(E)) u(y)$$
$$= \quad \nu(E) u(m(x_E z)) + (1 - \nu(E)) u(m(z_E y))$$
$$= \quad [\nu(E)]^2 u(x) + 2\nu(E)(1 - \nu(E)) u(z) + (1 - \nu(E))^2 u(y).$$

Notice that if neither the event E nor its complement is null then $0 < \nu(E) < 1$, and so solving for $u(z)$ yields

$$u(z) = \frac{1}{2} u(x) + \frac{1}{2} u(y),$$

which is independent of E. We shall therefore denote by $(1/2) x \oplus (1/2) y$ the preference average of the outcomes x and y. To deliver weighted averages of x and y, we follow the line of argument detailed in GMMS. That is, by using iterated averages (for example, $(1/2) x \oplus ((1/2) x \oplus (1/2) y)$ corresponds to a $(3/4, 1/4)$−weighted average of x and y) and appealing to standard continuity arguments, it is possible to identify, for any α in

26

$[0, 1]$ and every x and y in X, the weighted preference averages characterized by

$$u(z) = \alpha u(x) + (1 - \alpha) u(y). \tag{3}$$

With slight abuse of notation, we shall let $\alpha x \oplus (1 - \alpha) y$ (or, equivalently, $(1 - \alpha) y \oplus \alpha x$) denote an arbitrary element of the indifferent set of outcomes for such preference averages. We are now in a position to define subjective mixtures of acts.

Definition 5.2 *Fix $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$. A* subjective mixture *of f and g with weight α is any act $h \in \mathcal{F}$ such that $h(s) \sim \alpha f(s) \oplus (1 - \alpha) g(s)$ for each $s \in S$.*

As GMMS note, all subjective mixtures of f and g with weight α are state-wise indifferent, and hence by Axiom 3 (i), indifferent. So we follow them and denote by $\alpha f \oplus (1 - \alpha) g$ any one of them.

Our final axiom is key to characterizing the decision maker's attitudes towards events that yield extreme outcomes. We first need, however, to define for each act which events the decision maker views as yielding the extreme outcomes. We begin with preference-based definitions for the infimum and the supremum of an act.

Definition 5.3 *Fix $f \in \mathcal{F}$. An outcome $z \in X$ is said to be in the indifference set of the infimum of f, $z \in \inf_{\succeq}(f)$, if for $A := f^{-1}(x : z \succ x)$, $z_A f \sim f$ and if for every $y \succ z$ and $B := f^{-1}(x : y \succeq x)$, $y_B f \succ f$. Similarly, an outcome $z \in X$ is said to be in the indifference set of the supremum of f, $z \in \sup_{\succeq}(f)$ if for $A := f^{-1}(x : x \succ z)$, $z_A f \sim f$ and if for every y such that $z \succ y$ and $B := f^{-1}(x : x \succeq y)$, $f \succ y_B f$.*

Although $\inf_{\succeq}(f)$ and $\sup_{\succeq}(f)$ are defined to be indifference sets of outcomes, when there is no risk of confusion, we shall write $\inf_{\succeq}(f)$ and $\sup_{\succeq}(f)$ to indicate arbitrary members of these respective sets.

From the definition of a subjective mixture and equation (3) it follows that for every $f, g \in \mathcal{F}$, $\alpha \in [0, 1]$ and $s \in S$

$$u(\alpha f(s) \oplus (1 - \alpha) g(s)) = \alpha u(f(s)) + (1 - \alpha) u(g(s)).$$

Hence, if there is a non-null event on which both acts f and g attain their supremum,

that is, the set $E = \left\{ s \in S : f\left(s\right) \in \sup_{\succeq}\left(f\right) \right\} \cap \left\{ s \in S : g\left(s\right) \in \sup_{\succeq}\left(g\right) \right\}$ is not null, then a subjective mixture of these two acts will also attain its supremum on this set. That is, for any α in $\left(0, 1\right)$, $\alpha f \oplus \left(1 - \alpha\right) g\left(s\right) \in \sup_{\succeq}\left(\alpha f \oplus \left(1 - \alpha\right) g\right)$, for all $s \in E$. On the other hand, if the set E is null, then a subjective mixture of these acts need not necessarily attain its supremum on any event in $\left\{ s \in S : f\left(s\right) \in \sup_{\succeq}\left(f\right) \right\}$ nor on any event in $\left\{ s \in S : g\left(s\right) \in \sup_{\succeq}\left(g\right) \right\}$. In this case, the supremum attained will be *less* than that which would have been attained if there had been a common non-null event on which both acts attained their supremum. Similarly, if the set $F = \left\{ s \in S : f\left(s\right) \in \inf_{\succeq}\left(f\right) \right\}$ $\cap \left\{ s \in S : g\left(s\right) \in \inf_{\succeq}\left(g\right) \right\}$ is non-null, then for any $\alpha \in \left(0, 1\right)$, $\alpha f \oplus \left(1 - \alpha\right) g\left(s\right) \in$ $\inf_{\succeq}\left(\alpha f \oplus \left(1 - \alpha\right) g\right)$, for all $s \in F$. While the infimum attained for that subjective mixture would be no smaller if the set F were null.

The idea behind the next axiom is that if the decision maker is sensitive about extreme events, both for good and bad outcomes, then subjective mixtures of acts for which supremum outcomes are obtained on common non-null events will be viewed relatively favorably while subjective mixtures of acts for which infimum outcomes are obtained on common non-null events will be viewed relatively unfavorably.

To facilitate the formulation of this final axiom, denote by $\underline{\mathcal{E}}(f)$ the set of acts with some infimum outcomes on a common non-null event as the act f

$$\underline{\mathcal{E}}(f) := \left\{ h \in \mathcal{F} \mid \left\{ s \in S : f\left(s\right) \in \inf_{\succeq}\left(f\right) \right\} \cap \left\{ s \in S : g\left(s\right) \in \inf_{\succeq}\left(g\right) \right\} \notin \mathcal{N} \right\}.$$

Similarly, denote by $\overline{\mathcal{F}}(f)$ the set of acts with some maximum outcomes on a common non-null event as the act f,

$$\overline{\mathcal{F}}(f) := \left\{ h \in \mathcal{F} \mid \left\{ s \in S : f\left(s\right) \in \sup_{\succeq}\left(f\right) \right\} \cap \left\{ s \in S : g\left(s\right) \in \sup_{\succeq}\left(g\right) \right\} \notin \mathcal{N} \right\}.$$

In GMMS's axiomatization of Choquet Expected Utility, their key axiom is the restriction of an independence type axiom to subjective mixtures of co-monotonic acts. The Choquet integral of a neo-additive capacity satisfies the independence axiom for all acts with the best and worst outcomes on the same events, respectively. Hence, we effectively strengthen their co-monotonic independence axiom by requiring it to hold for all acts which obtain their infimum (and, respectively, their supremum) on some common

28

non-null event in common.

Axiom 5 (*Extreme Events Sensitivity*)

For any $f, g, h \in \mathcal{F}$ such that $f \sim g$ and $h \in \underline{\mathcal{E}}(g) \cap \overline{\mathcal{F}}(g)$, and any $\alpha \in (0, 1]$,

1. If $h \in \underline{\mathcal{E}}(f)$ then $\alpha g \oplus (1 - \alpha)h \succsim \alpha f \oplus (1 - \alpha)h$,
2. If $h \in \overline{\mathcal{F}}(f)$ then $\alpha f \oplus (1 - \alpha)h \succsim \alpha g \oplus (1 - \alpha)h$.

Acts that are comonotonic with f are elements of $\underline{\mathcal{E}}(f) \cap \overline{\mathcal{F}}(f)$. Hence Axiom 5 implies comonotonic independence for the indifference relation. In the proof of our representation result, we show that in conjunction with the other axioms it characterizes a subclass of the family of Choquet expected utility functionals.

Theorem 5.1 *Let X be a connected and separable topological space, let $\mathcal{E}^*$ contain at least four elements E_1, E_2, E_3 and E_4 that together form a partition of S, and let $\succsim$ be a binary relation on $\mathcal{F}$ for which there exist outcomes 0 and M in X, such that $M \succ 0$ and $M \succsim x \succsim 0$ for all $x \in X$. The following two statements are equivalent:*

1. *The preference relation $\succsim$ on $\mathcal{F}$ satisfies Ordering, Continuity, Eventwise Monotonicity, Binary Comonotonic Act Independence and Extreme Events Sensitivity.*

2. *There exists a unique neo-additive capacity ν on $\mathcal{E}$ and a unique continuous real-valued function u on X, with $u(0) = 0$ and $u(1) = 1$, such that for all $f, g \in \mathcal{F}$*

$$f \succsim g \iff \int u \circ f \, d\nu \geq \int u \circ g \, d\nu.$$

Proof. Appendix. ∎

Remark: Comparing the conditions of Proposition 3.1 and Theorem 5.1, one may wonder why the set of possible events $\mathcal{E}^*$ was required to have *at least three non-intersecting* elements but had to have *at least four such events* in the latter. In Proposition 3.1 three such events are required to make statement (ii a) meaningful but this statement had not to be derived. In Theorem 5.1 four non-null and non-universal events are necessary in order to prove that the capacity ν which we deduce satisfies statement (ii a) of Proposition 3.1. In order to see that four states are necessary for ν to satisfy statement (ii a), consider the

following *counterexample.* The capacity ν on $S = \{s_1, s_2, s_3\}$ defined by

$$\nu(s_1) = \tfrac{1}{2}, \quad \nu(s_2) = \tfrac{1}{3}, \quad \nu(s_3) = \tfrac{1}{4},$$

$$\nu(\{s_1, s_2\}) = \nu(\{s_1, s_3\}) = \nu(\{s_2, s_3\}) = \tfrac{1}{2}$$

satisfies statements (ii b) and (ii c) but not (ii a) of Proposition 3.1 Hence, it is not neo-additive.

6. Concluding remarks

Optimism and pessimism have long been recognized as important determinants of economic behavior. Subjective expected utility theory assumes that the impact of uncertainty can be reduced to the statistical properties of a probability distribution. This reduction extends economic analysis to situations under uncertainty where one could rightfully neglect psychological aspects relating to the focal attraction of the best and worst outcome of economic choices.

In this paper we have introduced a special case of capacity and its Choquet integral which captures aspects of optimism and pessimism without abandoning the subjectively probabilistic approach all together. In particular, subjective expected utility is always contained as a special parametric case in this approach. Moreover, as in EICHBERGER & KELSEY (2000), the additive part of a neo-additive capacity can be determined endogenously in equilibrium.

Most importantly neo-additive capacities open new avenues of research. It appears natural to view the degree of confidence which a decision maker holds in a probabilistic assessment of an uncertain situation as dependent on past experience and subject to influence from other people's beliefs. Optimism and pessimism may spread in a population. Attitudes towards uncertain outcomes may be contagious leading to general swings in optimism and pessimism. So "irrational exuberance" as observed by SHILLER (2001) may become amenable to formal economic analysis after all.

Appendix A

The following preliminary lemma establishes uniqueness of γ, λ, and π.

Lemma: Let ν be a capacity on $(S, \mathcal{E})$, where $\mathcal{E}$ contains at least three elements E_1, E_2, and E_3, such that $E_i \cap E_j = \varnothing$, for all $i \neq j$, and $\nu(E_i) > 0$, for all i. Then (γ, λ, π) is unique.

Proof. Let (γ, λ, π) be a given vector of parameters of ν. Denote by $\delta := 1 - \gamma - \lambda$ and $\widetilde{\pi}(E) := \delta \cdot \pi(E)$ for all $E \in \mathcal{E}$.

For any $E \in \mathcal{E}^*$ and $F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$, $\widetilde{\pi}(E)$ is uniquely defined by $\widetilde{\pi}(E) := \nu(E \cup F) - \nu(F)$.

Assume now that $E \in \mathcal{E}^*$, then there exists $(E_1, E_2) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. Hence, $\widetilde{\pi}(E)$ is uniquely defined by $\widetilde{\pi}(E) = \widetilde{\pi}(E_1) + \widetilde{\pi}(E_2)$.

This implies that $\widetilde{\pi}(E)$ is uniquely defined for all $E \in \mathcal{E}^*$ and that λ is uniquely defined by $\lambda := \nu(E) - \widetilde{\pi}(E)$ for all $E \in \mathcal{E}^*$.

Let $\{E_i \in \mathcal{E}^* |\ i = 1, .., 3\}$ be a partition of S. Then α is uniquely defined by $\alpha := \sum\limits_{i=1}^{3} \widetilde{\pi}(E_i)$. Hence, γ is unique. For, either $\alpha = 0$ and there is no π in the expression of ν, or $\alpha > 0$ and π is uniquely defined for any $E \in \mathcal{E}^*$ by $\pi(E) := \frac{\widetilde{\pi}(E)}{\alpha}$. $\blacksquare$

A.1   Proof of Proposition 3.1

(i) $\Longrightarrow$ (ii). This follows from the definition of a neo-additive capacity.

(ii) $\Longrightarrow$ (i).

(a) First we define a non-negative simple additive measure $\widetilde{\pi}$ on $\mathcal{E}^*$.

Case 1: $E \in \widetilde{\mathcal{E}} := \{E \in \mathcal{E}^* |\ \exists\, F \in \mathcal{E}^*,\ E \cap F = \emptyset,\ E \cup F \notin \mathcal{U}\}$.

Define $\widetilde{\pi}(E) := \nu(E \cup F) - \nu(F)$ for all $F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$, $E \cup F \notin \mathcal{U}$. Property (a) implies that $\widetilde{\pi}(E)$ is well-defined.

Case 2: $E \in \mathcal{E}^* - \widetilde{\mathcal{E}}$. Hence $F = S - E \notin \mathcal{E}^*$ is a singleton and there exists a finite partition

of E, $\{E_i \in \widetilde{\mathcal{E}} | \ i = 1, ..., n\}$. Define $\widetilde{\pi}(E) := \sum\limits_{i=1}^{n} \widetilde{\pi}(E_i)$. It is well defined because

$$
\begin{aligned}
\sum_{i=1}^{n} \widetilde{\pi}(E_i) \ &= \ [\nu(E_1 \cup ... \cup E_n) - \nu(E_2 \cup ... \cup E_n)] \\
&\quad + ... + [\nu(E_{n-1} \cup E_n) - \nu(E_n)] + \widetilde{\pi}(E_n) \\
&= \ \nu(E) - \nu(E_n) + \widetilde{\pi}(E_n) \\
&= \ \nu(E) - \nu(F) + \widetilde{\pi}(F).
\end{aligned}
$$

Monotonicity of ν implies that $\widetilde{\pi}(E) \geq 0$ for all $E \in \mathcal{E}^*$.

Let us check now:

$$
\widetilde{\pi}(E \cup F) = \widetilde{\pi}(E) + \widetilde{\pi}(F)
$$

for all $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$, $E \cup F \notin \mathcal{U}$.

If $E \cup F \in \mathcal{E}^* - \widetilde{\mathcal{E}}$ then this follows directly from Case 2.

If $E \cup F \in \widetilde{\mathcal{E}}$ then, for $G \in \mathcal{E}^*$ such that $G \cap (E \cup F) = \emptyset$ and $G \cup (E \cup F) \notin \mathcal{U}$

one obtains

$$
\begin{aligned}
\widetilde{\pi}(E \cup F) \ &= \ \nu(E \cup F \cup G) - \nu(G) \\
&= \ [\nu(E \cup F \cup G) - \nu(F \cup G)] + [\nu(F \cup G) - \nu(G)] \\
&= \ \widetilde{\pi}(E) + \widetilde{\pi}(F).
\end{aligned}
$$

(b) Next we extend $\widetilde{\pi}$ on all of $\mathcal{E}$ to a non-negative simple additive measure.

Clearly, defining $\widetilde{\pi}(E) = 0$ for all $E \in \mathcal{N}$ is consistent with the restricted additivity of $\widetilde{\pi}$, i.e. $\widetilde{\pi}(E \cup F) = \widetilde{\pi}(E) + \widetilde{\pi}(F)$ for all $E, F \in \mathcal{E}$ such that $E \cap F = \emptyset$, $E \cup F \notin \mathcal{U}$. Consider a finite partition of S, $\{E_i \in \mathcal{E}^* | \ i = 1, ..., n\}$, and define $\widetilde{\pi}(S) := \sum\limits_{i=1}^{n} \widetilde{\pi}(E_i)$. To check that $\widetilde{\pi}(S)$ is well-defined, let $\{F_j \in \mathcal{E}^* | \ j = 1, ..., m\}$ be another finite partition of S, then one obtains

$$
\begin{aligned}
\widetilde{\pi}(S) \ : \ &= \sum_{i=1}^{n} \widetilde{\pi}(E_i) = \sum_{i=1}^{n} \left[\sum_{j=1}^{m} \widetilde{\pi}(E_i \cap F_j) \right] \\
&= \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \widetilde{\pi}(E_i \cap F_j) \right] = \sum_{j=1}^{m} \widetilde{\pi}(F_j).
\end{aligned}
$$

32

Let $\alpha := \widetilde{\pi}(S)$. Clearly, $\alpha \geq 0$.

It remains to prove that $\widetilde{\pi}(E \cup F) = \widetilde{\pi}(E) + \widetilde{\pi}(F)$ for all $E, F \in \mathcal{E}^*$ such that $E \cap F = \emptyset$, $E \cup F \notin \mathcal{U}$. Since $\mathcal{E}^*$ contains at least three elements, one can assume, without loss of generality, that $E = E_1 \cup E_2$ with $E_1, E_2 \in \mathcal{E}^*$, $E_1 \cap E_2 = \emptyset$. Hence, $\widetilde{\pi}(E) = \widetilde{\pi}(E_1) + \widetilde{\pi}(E_2)$ and, from the definition of $\widetilde{\pi}(S)$, the desired result follows.

(c) We prove that there exists $\lambda \in \mathbb{R}_+$ such that $\nu(E) = \lambda + \widetilde{\pi}(E)$ for all $E \in \mathcal{E}^*$.

If $E \in \mathcal{E}^* - \widetilde{\mathcal{E}}$ then, from Case 2, there exists $A \in \mathcal{E}^*$ such that

$$\nu(E) = \widetilde{\pi}(E) + \nu(A) - \widetilde{\pi}(A).$$

If $E \in \widetilde{\mathcal{E}}$ then there exists $F \in \mathcal{E}^*$ and $B \in \mathcal{E}^*$ such that $E \cup F \cup G = S$. Hence, one has

$$\widetilde{\pi}(E \cup F) = \nu(E \cup F) + \nu(B) - \widetilde{\pi}(B),$$

and, therefore,

$$\widetilde{\pi}(E) + \widetilde{\pi}(F) = \nu(E \cup F) - \nu(E) + \nu(E) + \widetilde{\pi}(B) - \nu(B),$$

which gives

$$\nu(E) = \widetilde{\pi}(E) + \nu(B) - \widetilde{\pi}(B).$$

Hence,

$$\nu(A) - \widetilde{\pi}(A\}) = \nu(B) - \widetilde{\pi}(B)$$

for all $A, B \in \mathcal{E}^*$.

It remains to check that the common value $\lambda := \nu(A) - \widetilde{\pi}(A)$ is non-negative. Let $E, F \in \mathcal{E}^*$ satisfy Property (b) and consider $A \subset E$ and $B \subset F$. Applying Property (a) twice gives

$$\begin{aligned} 0 &\leq \nu(E) + \nu(F) - \nu(E \cup F) = \nu(A) + \nu(B) - \nu(A \cup B) \\ &= \nu(A) - \widetilde{\pi}(A) = \lambda. \end{aligned}$$

(d) We prove that $\alpha + \lambda \leq 1$.

By Property (c), there is $E, F \in \mathcal{E}^*$ such that

$$
\begin{aligned}
0 &\leq \overline{\nu}(E) + \overline{\nu}(F) - \overline{\nu}(E \cup F) \\
&= 1 - \nu(S - E) - \nu(S - F) + \nu(S - (E \cup F)) \\
&= 1 - (\lambda + \widetilde{\pi}(S - E)) - (\lambda + \widetilde{\pi}(S - F)) + (\lambda + \widetilde{\pi}(S - (E \cup F))) \\
&= 1 - \lambda - \alpha.
\end{aligned}
$$

(e) Setting $\gamma := 1 - \lambda - \alpha$, we finally obtain: $\nu(E) = \gamma \cdot \mu^0(E) + \lambda \cdot \mu^1(E) + (1 - \gamma - \lambda) \cdot \pi(E)$ for all $E \in \mathcal{E}$ where, for $1 - \gamma - \lambda > 0$, the probability measure $\pi(E)$ is defined by $\pi(E) := \frac{1}{1-\gamma-\lambda} \cdot \widetilde{\pi}(E)$. $\blacksquare$

A.2 Proof of Lemma 4.1

Proof. From Equation (*), we get

$$
\begin{aligned}
q^* &= \frac{\gamma \cdot u'(\underline{r} \cdot A) \cdot \underline{r} + (1 - \gamma) \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{r \cdot \left[\gamma \cdot u'(\underline{r} \cdot A) + (1 - \gamma) \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} \\
&= \frac{\gamma \cdot u'(\underline{r} \cdot A) \cdot \underline{r} + (1 - \gamma) \cdot \left[r \cdot q_0^* \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}{r \cdot \left[\gamma \cdot u'(\underline{r} \cdot A) + (1 - \gamma) \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} \\
&< \frac{\gamma \cdot u'(\underline{r} \cdot A) \cdot r \cdot q_0^* + (1 - \gamma) \cdot \left[r \cdot q_0^* \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]}{r \cdot \left[\gamma \cdot u'(\underline{r} \cdot A) + (1 - \gamma) \cdot \sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)\right]} \\
&= q_0^*.
\end{aligned}
$$

The inequality follows because

$$
r \cdot q_0^* = \frac{\sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A) \cdot r_s}{\sum\limits_{s \in S} \pi_s \cdot u'(r_s \cdot A)} < \underline{r}.
$$

$\blacksquare$

A.3 Proof of Theorem 5.1

We begin with an observation and a couple of preliminary results. The observation is that

34

any act $f \in \mathcal{F}$ may be expressed as $[x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$, where $\{E_1, \ldots, E_n\}$ is the coarsest finite $\succsim$-ordered partition of S with respect to which f measurable. By that we mean for any pair of states $s, t \in S$, if both s and t are in some $E \in \{E_1, \ldots, E_n\}$ then $f(s) = f(t)$, otherwise $f(s) \neq f(t)$. Furthermore for any $s \in E_i$ and $t \in E_j$, $i < j$ implies $f(s) \succsim f(t)$. Throughout this proof, if an act is expressed in the form $[x_1 \text{ on } E_1; \ldots; x_n \text{ on } E_n]$ then it should be taken as given that $x_i \succsim x_{i+1}$, for $i = 1, \ldots, i-1$. We also note that Axiom 5 (Extreme Events Sensitivity) and Axiom 1 (Ordering) imply that if the preference relation expresses indifference between two co-monotonic acts then indifference is preserved when those two acts are each mixed with a third act that is pairwise co-monotonic with both.

Lemma: *(Comonotonic Independence of Indifference)* Axiom 5 implies that $\succsim$ satisfies the following *independence* property for pairwise comonotonic acts. For any $\alpha \in [0,1]$ and any three acts $f, g, h \in \mathcal{F}$, that are pairwise comonotonic, if $f \sim g$ then $\alpha f \oplus (1-\alpha) h \sim \alpha g \oplus (1-\alpha) h$.

Proof: From the pairwise co-monotonicity of h with both f and g, it follows that $h \in \underline{\mathcal{E}}(g) \cap \overline{\mathcal{F}}(g)$ and $h \in \underline{\mathcal{E}}(f) \cap \overline{\mathcal{F}}(f)$. Hence Axiom 5 implies that $\alpha f \oplus (1-\alpha) h \succsim \alpha g \oplus (1-\alpha) h$ and $\alpha g \oplus (1-\alpha) h \succsim \alpha f \oplus (1-\alpha) h$, as required. $\square$

Finally we report GMMS's result (2002, Proposition 6) that the triple $(X, \sim, \oplus)$ constitutes a *mixture set*. That is, for all $x, y \in X$ and all α, β in $[0,1]$, (M0) $\alpha x \oplus (1-\alpha) y \subset X$, (M1) $x \in (1x \oplus 0y)$, (M2) $\alpha x \oplus (1-\alpha) y = (1-\alpha) y \oplus \alpha x$ (commutative law), and (M3) $\beta(\alpha x \oplus (1-\alpha) y) \oplus (1-\beta) y = \alpha \beta x \oplus (1-\alpha\beta) y$ (distributive law). Applying this result state by state, to Definition 5.2 (the definition of a subjective mixture of f and g with weight α in $[0,1]$) it readily follows that the triple $(\mathcal{F}, \sim, \oplus)$ is also a *mixture set* and hence exhibits the analogous properties.

Proof of Theorem 5.1-

1. Sufficiency

We first show that $\succsim$ has a CEU representation, Part (i), and then that the capacity is

35

neo-additive, Part (ii).

Part (i). $\succsim$ admits a CEU-representation. Let $u(.)$ and $\nu(.)$ be the continuous utility index and capacity of the canonical biseperable representation that from Proposition (5.1) we know $\succsim$ admits. Recall that $u(.)$ represents $\succsim$ restricted to the constant acts, and that $V\left([x_1 \text{ on } E; \; x_2 \text{ on } S-E]\right) = \nu(E)u(x_1) + (1-\nu(E))u(x_2)$ represents $\succsim$ restricted to the set of acts that are measurable with respect to a two-element partition of S.

Fix, $f = [x_1 \text{ on } E_1; \ldots; \; x_n \text{ on } E_n]$. For each $i = 1, \ldots, n$, it follows from the definition of $\oplus$ and the connectedness of X, that there exists a unique $\lambda_i \in [0,1]$ for which $x_i \in \lambda_i M \oplus (1-\lambda_i) 0$ and a unique ν_i for which

$$[M \text{ on } E_1 \cup \ldots \cup E_i; \; 0 \text{ on } E_{i+1} \cup \ldots \cup E_n] \sim \nu_i M \oplus (1-\nu_i) 0.$$

Equation (3) implies that $1 \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $0 \leq \nu_1 \leq \ldots \leq \nu_{n-1} \leq 1$.

Hence we have, by construction and the mixture set properties of $(\mathcal{F}, \sim, \oplus)$ that

$$f = \begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix} = \begin{bmatrix} \lambda_1 M \oplus (1-\lambda_1) 0 & \text{on } E_1 \\ \vdots & \vdots \\ \lambda_n M \oplus (1-\lambda_n) 0 & \text{on } E_n \end{bmatrix}$$

$$= (1-\lambda_1) \begin{bmatrix} 0 & \text{on } E_1 \\ 0 & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix} \oplus (\lambda_1 - \lambda_2) \begin{bmatrix} M & \text{on } E_1 \\ 0 & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix} \oplus (\lambda_2 - \lambda_3) \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ 0 & \text{on } E_3 \\ \vdots & \vdots \\ 0 & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix}$$

$$\oplus \cdots \oplus (\lambda_{n-1} - \lambda_n) \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ M & \text{on } E_3 \\ \vdots & \vdots \\ M & \text{on } E_{n-1} \\ 0 & \text{on } E_n \end{bmatrix} \oplus \lambda_n \begin{bmatrix} M & \text{on } E_1 \\ M & \text{on } E_2 \\ M & \text{on } E_3 \\ \vdots & \vdots \\ M & \text{on } E_{n-1} \\ M & \text{on } E_n \end{bmatrix}$$

By applying the *comonotonic independence of indifference* property of Lemma 6.3 $n-1$ times and utilizing the distributive law of $(\mathcal{F}, \sim, \oplus)$, we obtain

$$
\begin{aligned}
f \;\sim\; & (1-\lambda_1)\, 0 \oplus (\lambda_1 - \lambda_2)\left[\nu_1 M \oplus (1-\nu_1)\, 0\right] \oplus (\lambda_2 - \lambda_3)\left[\nu_2 M \oplus (1-\nu_2)\, 0\right] \\
& \oplus \ldots \oplus (\lambda_{n-1} - \lambda_n)\left[\nu_{n-1} M \oplus (1-\nu_{n-1})\, 0\right] \oplus \lambda_n M \\
=\; & \left[\sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\,\nu_i + \lambda_n\right] M \oplus \left[1 - \lambda_n - \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\,\nu_i\right] \delta_0.
\end{aligned}
$$

36

Hence it follows from equation (3) that for any pair of acts

$$f = \begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix} \text{ and } f' = \begin{bmatrix} x_1' & \text{on } E_1' \\ \vdots & \vdots \\ x_{n'}' & \text{on } E_{n'}' \end{bmatrix}$$

applying the above methods we have

$$f \succsim f' \text{ if and only if}$$

$$\left[\sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) \nu_i + \lambda_n\right] \geq \left[\sum_{j=1}^{n'-1} (\lambda_j' - \lambda_{j+1}') \nu_i' + \lambda_n'\right]$$

By construction, $u(0) = 0$, $u(M) = 1$, $u(x_i) = \lambda_i$, $\nu(\emptyset) := 0$, $\nu(S) := 1$ and $\nu\left(\cup_{j=1}^{i} E_i\right) = \nu_i$. Thus we have established that $\succsim$ can be represented by the Choquet expected utility functional

$$\text{CEU}\left(\begin{bmatrix} x_1 & \text{on } E_1 \\ \vdots & \vdots \\ x_n & \text{on } E_n \end{bmatrix}\right) = \sum_{i=1}^{n-1} (u(x_i) - u(x_{i+1})) \nu\left(\cup_{j=1}^{i} E_j\right) + u(x_n)$$

as required.

2. Necessity.

ν satisfies conditions (ii) of Proposition 3.1.

(a) We prove that for any three events $(E, F, G) \in \mathcal{E}^* \times \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \notin \mathcal{U}$, $E \cup G \notin \mathcal{U}$.

$$\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G),$$

Since there are at least four pairwise disjoint events in $\mathcal{E}^*$, we can assume that there are $E, F, G \in \mathcal{E}^* \times \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = E \cap G = F \cap G = \emptyset$ and $E \cup F \cup G \notin \mathcal{U}$. The following lemma contains the key argument.

Lemma: If there are $\mathcal{E}^* \times \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = E \cap G = F \cap G = \emptyset$ and $E \cup F \cup G \notin \mathcal{U}$. Then

$$\nu(E \cup F \cup G) - \nu(F \cup G) = \nu(E \cup F) - \nu(F). \tag{*}$$

Proof: Assume, without loss of generality, $\nu(E \cup F) \leq \nu(F \cup G)$ and let $\beta \in [0, 1]$ be

37

such that $\nu(E \cup F) = \beta \cdot \nu(F \cup G)$. Consider

$$f \quad := \quad \left[\begin{array}{lll} M & \text{on} & E \cup F \\ 0 & \text{on} & S - (E \cup F) \end{array} \right],$$

$$g \quad := \quad \left[\begin{array}{lll} \beta \cdot M \oplus (1 - \beta) \cdot 0 & \text{on} & F \cup G \\ 0 & \text{on} & S - (F \cup G) \end{array} \right],$$

$$h \quad := \quad \left[\begin{array}{lll} M & \text{on} & F \cup G \\ 0 & \text{on} & S - (F \cup G) \end{array} \right].$$

Clearly, $f \sim g$, $h \in \mathcal{E}(f) \cap \overline{\mathcal{F}}(f)$ and $h \in \mathcal{E}(g) \cap \overline{\mathcal{F}}(g)$. By Axiom 5, Extreme Event Sensitivity, we have

$$\frac{1}{2} \cdot g \oplus \frac{1}{2} \cdot h \sim \frac{1}{2} \cdot f \oplus \frac{1}{2} \cdot h.$$

Hence,

$$\begin{aligned} \frac{1}{2} \cdot [\nu(E \cup F \cup G) + \nu(F)] &= \frac{1}{2} \cdot (1 + \beta) \cdot \nu(F \cup G) \\ &= \frac{1}{2} \cdot (\nu(F \cup G) + \nu(E \cup F)) . \end{aligned}$$

Thus, we conclude $\nu(E \cup F \cup G) - \nu(F \cup G) = \nu(E \cup F) - \nu(F)$. $\qquad\square$

Let us now show that $(E, F, G) \in \mathcal{E}^* \times \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset = E \cap G$, $E \cup F \notin \mathcal{U}$, $E \cup G \notin \mathcal{U}$ implies

$$\nu(E \cup F) - \nu(F) = \nu(E \cup G) - \nu(G).$$

Several cases have to be considered when $F \neq G$.

Case 1.1: $F \subset G$. Using Equation (*), we get

$$\begin{aligned} \nu(E \cup G) - \nu(G) &= \nu(E \cup F \cup (G - F)) - \nu(F \cup (G - F)) \\ &= \nu(E \cup F) - \nu(F). \end{aligned}$$

Case 1.2: $G \subset F$. Similar to Case 1.1.

Case 2.1: $F - G \neq \emptyset \neq G - F$ and $F \cap G \neq \emptyset$ Using Equation (*), we get

$$\begin{aligned} \nu(E \cup F) - \nu(F) &= \nu(E \cup (F \cap G) \cup (F - G)) - \nu((F \cap G) \cup (F - G)) \\ &= \nu(E \cup (F \cap G)) - \nu(F \cap G) \\ &= \nu(E \cup (F \cap G) \cup (G - F)) - \nu((F \cap G) \cup (G - F)) \\ &= \nu(E \cup G) - \nu(G). \end{aligned}$$

38

Case 2.2: $F - G \neq \emptyset \neq G - F$ and $F \cap G = \emptyset$ If $E \cup F \cup G \notin \mathcal{U}$, then the result follows immediately from Equation (*).

Suppose $E \cup F \cup G \in \mathcal{U}$. Since $\mathcal{E}^*$ contains at least four elements we may assume that one of the events E, F or G can be partitioned into two events. Without loss of generality, suppose E can be partitioned into E_1 and E_2. Then Equation (*) implies

$$
\begin{aligned}
\nu(E \cup F) - \nu(F) &= \nu(E_1 \cup E_2 \cup F) - \nu(F) \\
&= [\nu(E_1 \cup E_2 \cup F) - \nu(E_1 \cup F)] + [\nu(E_1 \cup F) - \nu(F)] \\
&= [\nu(E_2 \cup F) - \nu(F)] + [\nu(E_1 \cup F) - \nu(F)] \\
&= [\nu(E_2 \cup G) - \nu(G)] + [\nu(E_1 \cup G) - \nu(G)] \\
&= [\nu(E_1 \cup E_2 \cup G) - \nu(E_1 \cup G)] + [\nu(E_1 \cup G) - \nu(G)] \\
&= \nu(E \cup G) - \nu(G).
\end{aligned}
$$

(b) We prove that for some $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$,

$$
\nu(E \cup F) \leq \nu(E) + \nu(F).
$$

Consider $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$. Assume, without loss of generality, that $\nu(E) = \beta \cdot \nu(F)$ for some $\beta \in [0, 1]$. Let

$$
f \quad := \quad \left[\begin{array}{ccc} M & \text{on} & E \\ 0 & \text{on} & S - E \end{array} \right],
$$

$$
g \quad := \quad \left[\begin{array}{ccc} \beta \cdot M \oplus (1 - \beta) \cdot 0 & \text{on} & F \\ 0 & \text{on} & S - F \end{array} \right],
$$

$$
h \quad := \quad \left[\begin{array}{ccc} M & \text{on} & F \\ 0 & \text{on} & S - F \end{array} \right].
$$

Clearly, $f \sim g$, $h \in \underline{\mathcal{E}}(f)$ and $h \in \underline{\mathcal{E}}(g) \cap \overline{\mathcal{F}}(g)$. By Axiom 5(1), Extreme Event Sensitivity, we have

$$
\frac{1}{2} \cdot g + \frac{1}{2} \cdot h \succeq \frac{1}{2} \cdot f + \frac{1}{2} \cdot h.
$$

Hence,

$$
\begin{aligned}
\frac{1}{2} \cdot \nu(E \cup F) &\leq \frac{1}{2} \cdot (1 + \beta) \cdot \nu(F) \\
&= \frac{1}{2} \cdot (\nu(F) + \nu(E)).
\end{aligned}
$$

39

Thus, we conclude $\nu(E \cup F) \le \nu(E) + \nu(F)$.

(c) We prove that for some $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$,

$$\overline{\nu}(E \cup F) \le \overline{\nu}(E) + \overline{\nu}(F).$$

Consider $(E, F) \in \mathcal{E}^* \times \mathcal{E}^*$ such that $E \cap F = \emptyset$ and $E \cup F \notin \mathcal{U}$. Assume, without loss of generality, that $\nu(S - E) = \beta \cdot \nu(S - F)$ for some $\beta \in [0, 1]$. Let

$$f \; : \; = \left[\begin{array}{ll} M & \text{on} \quad S - E \\ 0 & \text{on} \quad E \end{array} \right],$$

$$g \; : \; = \left[\begin{array}{ll} \beta \cdot M \oplus (1 - \beta) \cdot 0 & \text{on} \quad S - F \\ 0 & \text{on} \quad F \end{array} \right],$$

$$h \; : \; = \left[\begin{array}{ll} M & \text{on} \quad S - F \\ 0 & \text{on} \quad F \end{array} \right].$$

Clearly, $f \sim g$, $h \in \overline{\mathcal{F}}(f)$ and $h \in \underline{\mathcal{E}}(g) \cap \overline{\mathcal{F}}(g)$. By Axiom 5(2), Extreme Event Sensitivity, we have

$$\frac{1}{2} \cdot f + \frac{1}{2} \cdot h \succeq \frac{1}{2} \cdot g + \frac{1}{2} \cdot h.$$

Hence,

$$\frac{1}{2} + \frac{1}{2} \cdot \nu((S - E) \cup (S - F)) \;\; \ge \;\; \frac{1}{2} \cdot (1 + \beta) \cdot \nu(S - F)$$
$$= \;\; \frac{1}{2} \cdot (\nu(S - F) + \nu(S - E))$$

or

$$\frac{1}{2} \cdot [(1 - \nu(S - F)) + (1 - \nu(S - E))] \ge \frac{1}{2} \cdot [1 - \nu((S - E) \cup (S - F))].$$

Thus, we conclude $\overline{\nu}(E \cup F) \le \overline{\nu}(E) + \overline{\nu}(F)$.

The necessity of the representation follows straightforwardly from the definition of the neo-additive representation and so the proof is omitted. ∎

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