Population Principles with Number-Dependent Critical Levels*

by

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Abstract

This paper introduces and characterizes the number-sensitive critical-level generalized-utilitarian family of population principles which is a generalization of the critical-level generalized-utilitarian family. number-sensitive critical-level utilitarian principles rank alternatives by using a value function that is equal to total utility minus a sum of critical levels that may depend on population size but not on individual utilities, and number-sensitive critical-level generalized-utilitarian principles use transformed utilities and critical levels. Ethical properties of the principles are investigated and the new family is compared to number-dampened generalized utilitarianism whose value functions are equal to transformed representative utility (average utility in the utilitarian case) multiplied by a function of population size.

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1. Introduction

This paper introduces a new family of population principles and compares it to one suggested by Ng [1986]. Both families have same-number principles that are generalized utilitarian, using transformed utilities to rank alternatives. If the transform is concave (strictly concave), the principle exhibits weak (strong) aversion to inequality in utilities, and utilitarianism is a special case.

Because attempts to apply population principles to single periods lead to great difficulties, the utility levels that we employ are indexes of lifetime well-being.[1] In addition, because the principles we investigate are 'welfarist' (Sen [1974, 1977]), using only information about individual well-being to rank alternatives, it is important that utility levels reflect a comprehensive account of well-being such as that of Griffin [1986] or Sumner [1996]. Following standard practice, we normalize utilities so that a lifetime utility level of zero represents neutrality: above neutrality, a life, as a whole, is worth living; below neutrality, it is not. We reject the view that a person can gain or lose by being brought into existence.[2]

The number-sensitive critical-level generalized-utilitarian family has critical levels which may depend on population size but not on utility levels. An important subfamily is comprised of the critical-level generalized-utilitarian principles (Blackorby and Donaldson [1984]) with critical levels that are the same for all population sizes.[3] To understand the idea of a critical level of utility, compare two alternatives in which the second has one additional person and the utility levels of the common population are the same in both. If the utility level of the added person is the critical level for the first alternative, the two are ranked as equally good. Critical levels for the classical generalized-utilitarian principles (and for classical utilitarianism) are zero and critical levels for average utilitarianism are average utility.

The value functions for principles in the number-dampened utilitarian family (Ng [1986]) are equal to average utility multiplied by a function of population size. If the function is any positive multiple of population size, the principle is classical utilitarianism and, if the function is a positive constant, it is average utilitarianism. In addition, critical levels for the number-dampened utilitarian principles are equal to average utility multiplied by a function of population size. A subfamily results from making that function independent of population size; such a principle might have critical levels that are equal to one-half average utility. The number-dampened utilitarian family can be extended so that it is consistent with same-number generalized utilitarianism.

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[1] See Blackorby, Bossert and Donaldson [1997a,b].

[2] For discussions, see Broome [1993], Blackorby, Bossert and Donaldson [1997c], Heyd [1992] and Parfit [1984].

[3] See also Blackorby, Bossert and Donaldson [1995].

If a population principle has some negative critical levels, then some additions of persons whose lives are not worth living to utility-unaffected populations are ranked as social improvements. It is desirable, therefore, to use principles that do not have negative critical levels. In addition, it is sensible to employ principles that do not imply the *repugnant conclusion* (Parfit [1976, 1982, 1984]). Such principles rank any alternative in which each person experiences a positive lifetime utility level, no matter how great, as worse than some large-population alternative in which individual lives are barely worth living.

In Section 2, we define the principles that we investigate and explore a few of their properties. Then, in Section 3, we turn to independence axioms that restrict the information needed to rank alternatives. Each requires rankings of pairs of alternatives to be independent of the utility levels and/or existence of individuals whose levels of well-being are the same in both. This would occur, for example, in ranking alternatives that are feasible in the present: the utility levels of Cleopatra and Socrates are the same in them all. Using the different axioms, we present theorems that characterize both number-sensitive critical-level generalized utilitarianism and number-dampened generalized utilitarianism.

Sections 4 and 5 investigate ethical properties of the two families. In addition to the requirements that principles have nonnegative critical levels and avoid the repugnant conclusion, several other requirements on critical levels and the behaviour of best choices from feasible sets of alternatives are considered.

Section 6 concludes the main part of the paper and it compares the two families. We argue that the number-sensitive critical-level principles with nonnegative, nondecreasing critical levels, at least one of which is positive, are superior to all of the number-dampened principles. The critical-level principles with positive critical levels are part of the ethically acceptable family, and we argue that there are good reasons for choosing one of them. Section 7 is an Appendix which contains all formal definitions and proofs of theorems.

Throughout the paper, we restrict attention to principles that order alternatives completely and we do not consider derived rankings of acts that have uncertain consequences. For principles that provide incomplete (but transitive) rankings, see Blackorby, Bossert and Donaldson [1996] and, for extensions to uncertainty, see Broome [1991] and Blackorby, Bossert and Donaldson [1998, 2000]. Although we do not consider the possibility of discounting the utilities of members of future generations in this paper, it is investigated in Blackorby, Bossert and Donaldson [1997b,c].

2. Welfarist Population Principles

Suppose that X is a set of social alternatives that are complete descriptions of the world from distant past to remote future. In each alternative, there is a set of people who are alive at some time. X may or may not contain the null alternative, the one in which no one is alive. Welfarist population principles use information about the levels of lifetime well-being (utility) of those alive to rank alternatives according to their goodness. Principles that rank alternatives in this 'timeless' way avoid the difficulties that arise when they are applied to a single period or to the present and future only (see Blackorby, Bossert and Donaldson [1997a,b]). We employ the standard convention that a utility level of zero represents neutrality: a life, as a whole, is worth living if and only if lifetime well-being is positive.

Welfarist population principles that order alternatives completely and transitively and satisfy same-number anonymity (alternatives with the same number of people alive and the same utility levels are equally good) can be represented by a single ordering[4] of utility vectors of varying dimension: alternative x is ranked as no worse than alternative y if and only if the vector of utility levels in x is ranked as no worse than the vector of utilities in y by the single ordering.[5] Consequently, investigations of welfarist population principles can focus on the ordering of vectors of utility levels.

We work, in this paper, with principles whose same-number subprinciples are anonymous and order vectors of individual utility levels continuously, which ensures that big changes in rankings do not result from trivially small changes in utilities. In addition, we require principles to satisfy the strong Pareto principle.

For any alternative with a positive population, these axioms guarantee the existence of a utility level that we call *representative utility.* It is that level which, if assigned to each person, is associated with an alternative that is as good as the original one. Given this, it can be shown that any population principle has a value function for alternatives other than the null that can be written in terms of population size and representative utility (Blackorby and Donaldson [1984], Blackorby, Bossert and Donaldson [2000]) and is increasing in representative utility. See the end of this section for a way to extend the value function to cover the null alternative.[6]

For any alternative x, we write n_x as population size and, for alternatives other than the null, ξ_x as representative utility. x is better than y if and only if the value function yields a greater number for x than for y. In addition, x and y are equally good if and only if the two values are the same. This can be summarized by the single statement that, for all x and y in X, x is at least as good as y if and only if

$$W(n_x, \xi_x) \geq W(n_y, \xi_y) \tag{1}$$

[4] An ordering is a no-worse-than relation that is reflexive, complete and transitive.

[5] See Blackorby and Donaldson [1984] or Blackorby, Bossert and Donaldson [2000].

[6] This provides a test for any principle that provides complete rankings of alternatives: if it cannot be represented by a value function that can be written in terms of population size and representative utility, it does not rank alternatives transitively.

where W is the value function. If either x or y is the null alternative, the appropriate term in (1) is replaced with its value. In the rest of the paper, value functions are explicitly written out for positive populations only.

If a principle's same-number subprinciples are utilitarian, representative utility is average utility, which we write as

$$\mu = \frac{1}{n}\sum_{i=1}^{n} u_i \tag{2}$$

and, in this case, value functions can be written as $W(n, \mu)$.

The value function for a principle is not unique: any increasing transform, such as the one that results from multiplying by two, represents the same principle. In addition, value functions can be written in other ways. If a principle is consistent with same-number utilitarianism, for example, the value function may be written as a function of population size and total utility because average utility is equal to total utility divided by population size. A value function of this type is given by

$$\tilde{W}(n, \tau) = W(n, \tau/n) = W(n, \mu) \tag{3}$$

where $\tau = \sum_{i=1}^{n} u_i$ is total utility. Alternatively, because population size is equal to total utility divided by average utility, the function W may be replaced by the value function $\hat{W}$, with

$$\hat{W}(\tau, \mu) = W(\tau/\mu, \mu). \tag{4}$$

Although such a value function, which depends on total utility and average utility, is (implicitly) used by Parfit [1984], it is much easier to work with functions that depend on population size and average (or total) utility. The reason is that population size is an integer and average utility is a real number. If the average-total formulation of (4) is employed, the function is defined only at total-average combinations in which total utility is an integer multiple of average utility. For that reason, we have chosen to work with the population-size – representative-utility representations.

The value function for average utilitarianism (AU) is average utility, so

$$W_{AU}(n, \mu) = \mu = \frac{1}{n}\sum_{i=1}^{n} u_i. \tag{5}$$

'Isovalue' lines for average utilitarian principles are displayed in Figure 1. Each bulleted point represents an average-utility – population-size pair, and the dotted isovalue lines join points of equal value which are ranked as equally good by average utilitarianism. The lines have a population of one as a base. In each, a single person has a particular utility level: sixty, thirty, zero and minus thirty for lines I, II, III and IV respectively. Consequently, points on line I are better than points on line II, with points on lines III and IV successively worse. The lines are flat because the value function is independent of population size; any two points with the same average utility are equally good. This means that AU ranks alternatives in a peculiar way: an alternative with any population size in which each person has the same utility level is ranked as worse than an alternative in which *one* person has an arbitrarily higher utility level.

In a similar way, the value function for classical utilitarianism (CU) can be written in terms of population size and average utility. It is

$$W_{CU}(n, \mu) = n\mu = \sum_{i=1}^{n} u_i, \tag{6}$$

the product of population size and average utility which is equal to the sum of utilities. It is illustrated in Figure 2. The four lines again join points ranked as equally good and are based on average utilities of sixty, thirty, zero and minus thirty for a population size of one. When average utility is positive (above neutrality), increases in population size with average utility constant are good and, when average utility is negative (below neutrality), increases in population size that do not change average utility are bad.

A principle leads to the repugnant conclusion (Parfit [1976, 1982, 1984]) if and only if every alternative in which each person experiences a positive utility level, no matter how great, is declared to be worse than an alternative in which each member of a sufficiently large population has a utility level that is above neutrality but arbitrarily close to it. Such principles may recommend the creation of a large population in which each person is poverty stricken. As Heyd [1992, p. 57] remarks, 'What is the good of a world swarming with people having lives barely worth living, even if *overall* the aggregation of the "utility" of its members supersedes that of any alternative, smaller world?' Classical utilitarianism is a principle that implies the repugnant conclusion and many people reject it for that reason.

Although average utilitarianism does not imply the repugnant conclusion, it recommends the addition of individuals to a utility-unaffected population whenever the utilities of the added individuals are above the average utility of the existing population. Thus, when average utility is negative, some additions of people whose lives are not worth living are regarded as good.

This observation can be made more explicit by using critical levels of utility. For any population principle, compare an alternative to another with one added person in which utility levels for the original population are the same. If the two alternatives are equally good, the utility level of the added person is the critical level of utility for the alternative. Given same-number utilitarianism, critical levels (if they exist) depend on population size and the average-utility level of the existing population (but they may be independent of either).[7]

For average utilitarianism, critical levels are equal to average utility. Consequently, critical levels for alternatives in which average utility is negative are themselves negative. For such an alternative, therefore, the addition of a person with a negative but above-average utility is ranked as an improvement. It seems reasonable to reject average utilitarianism because of this, and to require all critical levels to be non-negative.

For classical utilitarianism, all critical levels are equal to zero, the utility level that represents neutrality. It can be shown that, if all critical levels are zero or negative, any principle that is weakly averse to inequality in utilities must satisfy the repugnant conclusion.[8] Consequently, if the repugnant conclusion is regarded as ethically undesirable, some critical levels must be positive.

Critical-level utilitarianism (CLU) is a family of principles, one for each value of a *fixed* utility level which is the critical level for every alternative. CLU's value functions are given by

$$W_{CLU}(n,\mu) = n\big(\mu - \alpha\big) = \sum_{i=1}^{n}\big(u_i - \alpha\big) \tag{7}$$

where α is the critical level for the particular principle represented by the function. The value function can be calculated by subtracting the critical level from average utility and multiplying the resulting number by population size, or by subtracting the critical level from the utility of each person alive and summing. If the critical level is zero, classical utilitarianism results and, therefore, CU is a member of the the CLU family.

Isovalue lines for CLU with a critical level of thirty are illustrated in Figure 3. If average utility is held constant, population expansions are good if it is above thirty, neither good nor bad if it is equal to thirty, and bad if average utility is below thirty. In the general case, for any alternative in which average utility is above the critical level, there is no alternative with a larger population and an average utility below the critical level that is better. Consequently, the repugnant conclusion is not implied as long as the critical level is positive. The critical level can be interpreted as providing a floor on the trade-off between average utility and numbers.

Population principles with positive critical levels give weight to individual lives. Consider alternative x in which population size is $n \geq 2$ and let y and z be alternatives with one additional person. Lifetime utility levels for all but one person (the first $n-1$ people)

[7] If n is population size, μ is average utility and c is the critical level, then the average utility of the larger population is $(n\mu + c)/(n+1)$. Consequently, $W(n,\mu) = W(n+1, (n\mu+c)/(n+1))$ and c depends on n and μ. In general, if the same-number subprinciples are generalized utilitarian and critical levels exist, they can be written in terms of population size and representative utility.

[8] See Arrhenius [1997], Blackorby, Bossert, Donaldson and Fleurbaey [1998], Blackorby and Donaldson [1991], Carlson [1998], McMahan [1981], Ng [1989] and Parfit [1976, 1982, 1984].

are the same in all three alternatives and utility levels for the the remaining two people are given in Table 1. c is the critical level for alternative x.

person	alternative x	alternative y	alternative z
$1 \ldots n-1$	same in all three		
n	100	100	50
$n+1$	–	c	50

Table 1

In z, the utility levels in the table might arise because person the n^{th} person and the additional person live half as long in z as person n did in x. Because the critical level for x is c, x and y are equally good. Therefore, by transitivity, x is better than z if and only if y is better than z. Same-number utilitarianism ranks y and z as equally good if and only if $100 + c = 50 + 50$ or $c = 0$, and x as better than y if and only if $100 + c > 100$ or $c > 0$. Consequently, alternative x, with the smaller number number of people and the same total utility, is ranked as better than z if and only if its critical level is positive.

It follows that members of the critical-level utilitarian family with positive critical levels give weight to individual lives in addition to avoiding the repugnant conclusion. Although average utilitarianism does the latter, it does not always give weight to individual lives. Suppose that, in Table 1, average utility in x is negative (if n were 2, this would result from a utility level of -200 for person one). Then, because the critical level c is negative, the above argument shows that AU ranks z as better than x: two people with utility levels of 50 each are better, other things equal, to one person with a utility level of 100.

This paper introduces a new family of principles which includes the CLU family. It is the number-sensitive critical-level utilitarian (NCLU) family and its critical levels are independent of average utility but not necessarily independent of population size. We write the critical level for population size $n > 0$ as c_n. For $n = 0$, c_0 is the critical level for the null alternative if it is in X, and an arbitrary real number if it is not (the number chosen makes no difference to rankings in this case). The value functions for the NCLU principles can be written, for positive population sizes, as

$$W_{NCLU}(n, \mu) = n\big(\mu - \bar{A}(n)\big) = \sum_{i=1}^{n} \big(u_i - c_{i-1}\big) \tag{8}$$

where

$$\bar{A}(n) = \frac{1}{n}\sum_{i=0}^{n-1} c_i. \tag{9}$$

$\bar{A}(n)$ is the average of the critical levels for population sizes 0 to $n-1$. These value functions can be computed by subtracting the average of the critical levels for population sizes zero to $n-1$ from average utility and multiplying by population size or, equivalently, by subtracting, in turn, the critical levels for population sizes zero to $n-1$ from the utilities of those alive and adding. Alternatively CLU results from making all the critical levels equal to the same real number, so that $\bar{A}(n)$ is equal to α, the fixed critical level. The value functions for members of the NCLU family can also be written as

$$W_{NCLU}(n, \mu) = n\mu - A(n) \tag{10}$$

where

$$A(n) = \sum_{i=1}^{n} c_{i-1}. \tag{11}$$

Number-sensitive critical-level utilitarianism is illustrated in Figure 4. In that example, critical levels are zero for population size one and thirty for all population sizes greater than one. The positive critical levels for higher population sizes ensure that the repugnant conclusion is avoided. If average utility is constant, population expansion is good when it is above thirty, bad when it is negative. And, if average utility is nonnegative and no greater than thirty, expansion may be good, bad or indifferent depending on how big the existing population is. In our example, NCLU coincides with CU for population sizes one and two and, as population size becomes very large, it approximates CLU with a critical level of thirty.

Reasonable conditions for ethical acceptability of members of the NCLU family are that all critical levels be nonnegative and that the repugnant conclusion not be implied. These requirements, together with several others, are discussed in Section 4.

Another family of principles is a generalization of average utilitarianism and of classical utilitarianism. It is the number-dampened utilitarian (NDU) family (Ng [1986]). Its value functions can be written as

$$W_{NDU}(n, \mu) = f(n)\mu = \frac{f(n)}{n}\sum_{i=1}^{n} u_i, \tag{12}$$

where f is a positive-valued function of population size. If $f(n) = n$ or any multiple, CU results and, if $f(n)$ is independent of n, AU results. Both Ng and Hurka [1983] suggested that principles ought to approximate CU for 'small' population sizes and AU for 'large' sizes. Figure 5 illustrates such a case with $f(1) = 1$, $f(2) = 2$, $f(3) = 2.6$, and $f(n) = 3$ for all $n \geq 3$. For population sizes one and two, the principle coincides with CU and, for population sizes of four or more, it coincides with AU. It can be shown (see Section 5) that critical levels for NDU are equal to a multiple of average utility. The multiple can depend on population size. Consequently the critical level for population size n can be written as $h(n)\mu$ where h is a function of n. In our example, $h(n)$ is equal to zero for $n = 1$, .31 for $n = 2$, .47 for $n = 3$, and 1 for $n = 4$ and greater.

An important special case, specializes NDU in a way that is parallel to the way that constant critical levels specialize NCLU to CLU. It requires $h(n)$, the ratio of critical levels to average utilities, to be a constant between zero and one. Conditions on the functions f and h that ensure that the repugnant conclusion is avoided and that critical levels are nonnegative are explored in Section 5.

Utilitarian principles are indifferent to utility inequality: for a fixed population, all distributions of the same total utility are ranked as equally good. Many fixed-population welfarist principles exhibit inequality aversion, however, and can be employed as part of population principles.

Generalized-utilitarian (GU) principles are same-number principles that use the sum of transformed utilities, instead of the sum of utility levels themselves, to rank alternatives. The transforms g must be continuous and increasing and, if concave, endow the principle with inequality aversion. Same-number utilitarianism is same-number generalized utilitarianism with $g(t) = t$ for all t. In this case, g is concave but not strictly so and utilitarianism's inequality aversion is weak. If g is strictly concave, same-number GU exhibits strict inequality aversion: any unambiguous reduction in utility inequality according to the Lorenz criterion (same total with Lorenz curves that do not cross) is ranked as a social improvement. Without loss of generality, we normalize the transform so that the transformed value of a utility level of zero is zero. One simple example is a transform that gives a higher weight to negative than to positive utilities.

The value functions for all of the generalized-utilitarian principles are the same as those for the utilitarian principles with transformed utilities, representative utilities and critical levels replacing their untransformed counterparts. Representative utilities for principles that are consistent with same-number generalized utilitarianism satisfy

$$g(\xi) = \frac{1}{n}\sum_{i=1}^{n} g(u_i). \tag{13}$$

Transformed representative utility is the average of individual transformed utilities.

Average generalized-utilitarian (AGU) principles have value functions that can be written as

$$W_{AGU}(n, \xi) = g(\xi) = \frac{1}{n}\sum_{i=1}^{n} g(u_i). \tag{14}$$

The value function is transformed representative utility, which is equal to average transformed utility.

The value functions for classical generalized utilitarianism (CGU) can similarly be written as

$$W_{CGU}(n,\xi) = ng(\xi) = \sum_{i=1}^{n} g(u_i), \tag{15}$$

which is the sum of transformed utilities.

Members of the critical-level generalized-utilitarian (CLGU) family have value functions that are equal to the difference between transformed representative utility and the transformed critical level multiplied by population size or, equivalently, to the sum of the differences between individual transformed utilities and the critical level. The value functions can be written as

$$W_{CLGU}(n,\xi) = n\big(g(\xi) - g(\alpha)\big) = \sum_{i=1}^{n} \big(g(u_i) - g(\alpha)\big). \tag{16}$$

Value functions for members of the number-sensitive critical-level generalized-utilitarian (NCLGU) family can be written as

$$W_{NCLGU}(n,\xi) = n\Big(g(\xi) - g\big(\bar{A}_g(n)\big)\Big) = \sum_{i=1}^{n} \big(g(u_i) - g(c_{i-1})\big) \tag{17}$$

where

$$g\big(\bar{A}_g(n)\big) = \frac{1}{n} \sum_{i=0}^{n-1} g(c_i). \tag{18}$$

As in the definition of NCLU, $c_0, \ldots, c_{n-1}$ are critical levels for population sizes zero to $n-1$ with c_0 set arbitrarily if the null alternative is not in X. The term $\bar{A}_g(n)$ is the representative utility for the vector of critical levels $(c_0, \ldots, c_{n-1})$ and $g\big(\bar{A}_g(n)\big)$ is the average of the transformed critical levels for population sizes zero to $n-1$. The value function for NCLGU can also be written as

$$W_{CLGU}(n,\xi) = ng(\xi) - A_g(n), \tag{19}$$

where

$$A_g(n) = \sum_{i=1}^{n} g(c_{i-1}). \tag{20}$$

The value functions for members of the number-dampened generalized-utilitarian (NDGU) family can be written as

$$W_{NDGU}(n,\xi) = f(n)g(\xi) = \frac{f(n)}{n} \sum_{i=1}^{n} g(u_i) \tag{21}$$

where $f(n)$ is positive for all $n > 0$. If $f(n)$ is equal to any positive constant, the value function represents average generalized utilitarianism and, if $f(n)$ is equal to n or any positive multiple, the value function represents classical generalized utilitarianism.

Critical levels for NDGU are such that their transformed values are equal to a function h of population size multiplied by transformed representative utility. A special case results when $h(n)$ is a constant: see Section 5 for this result and other explorations.

If the null alternative is in X, it is possible to extend the value function for any principle to cover it as long as it has a critical level. Writing $c_\emptyset$ as the null critical level, the null alternative and one in which a single person experiences a utility level of $c_\emptyset$ are equally good. Consequently, we may define the value for the null alternative to be $w_\emptyset = W(1, c_\emptyset)$. The extensions of the value functions presented above are zero for CGU, CLGU (assuming that $c_\emptyset = \alpha$) and NCLGU (with $c_\emptyset = c_0$). For AGU, $w_\emptyset = g(c_\emptyset)$ and, for NDGU, $w_\emptyset = f(1)g(c_\emptyset)$. It follows that $w_\emptyset$ is zero for CU, CLU and NCLU, $c_\emptyset$ for AU and $f(1)c_\emptyset$ for NDU.

3. Independence Axioms

In general, population principles require a great deal of information. The population in question consists of all those who ever have and ever will live, and their utility levels must be known in addition to population size. Because principles rank alternatives that correspond to complete histories of the world from the remote past to distant future, it is possible that the ranking of changes that affect only those presently alive will be different depending of the existence and/or utility levels of people who are long dead or who will not be born for many centuries. The axioms we introduce in this section have the effect of reducing the information required to make social judgements and, in addition, guarantee the existence of population principles that can be applied to the individuals affected by social changes. In each case, attention is restricted to positive population sizes.

The weakest of the three axioms that we investigate is called *independence of the utilities of unconcerned individuals* (IUUI) and it applies to comparisons in which the same people are alive. Suppose that the utility levels of the members of a population subgroup are the same in two alternatives. Members of the subgroup are called ‘unconcerned’, and this axiom requires that the social ordering of the alternatives must be independent of their utility levels. In conjunction with our basic axioms (welfarism, anonymity, strong Pareto and continuity), this axiom implies that all same-number rankings must be generalized utilitarian (Theorem 1). The transforms used may be different for different population sizes, however.

IUUI ensures that, in fixed-number comparisons, information about the utility levels of the unconcerned individuals need not be known but, if the transforms are different for different population sizes, their number is needed. A consequence of this is that there are well-defined principles for groups such as the population of Canada that can be applied to fixed-number changes that affect subgroup members only. The axiom does *not* imply that the fixed-number principles for Canada are independent of total population size, which is the number of people who ever live. For that, stronger axioms such as the *population substitution principle* (Blackorby and Donaldson [1984]) are needed. It requires that any utility vector and a second vector in which each member of a subgroup receives the subgroup's representative utility are ranked as equally good.

A stronger axiom requires the ranking of utility vectors to be independent of both the utility levels and existence of unconcerned individuals. Suppose that, in any two utility vectors, there is a subgroup of unconcerned individuals. *Extended independence of the utilities of unconcerned individuals* (EIUUI) requires the ranking to be the same as the one that would be made if they did not exist. Thus, in this case, well-defined population principles exist for all groups such as nations. If a change affects the number and utility levels of present-day Canadians only, the overall population principle can safely be applied to them alone. Information about the utilities and numbers of others is not needed.

Extended independence of the utilities of unconcerned individuals implies the weaker axiom IUUI. If we add the requirement that there is at least one alternative for which a critical level of utility exists (weak expansion equivalence), then it, in conjunction with our basic axioms and EIUUI, implies that the population principles must be critical-level generalized utilitarian (Theorem 2).

It is possible to find an axiom that is weaker than EIUUI but stronger than IUUI. We call it intermediate independence of the utilities of unconcerned individuals (IIUUI). Suppose that, for any two alternatives with possibly different population sizes, there is a population subgroup, common to both, whose members have the same utility levels in both. IIUUI requires the ranking of the two alternatives to be independent of the utilities of the members of the subgroup, who are the unconcerned individuals.

IIUUI is implied by EIUUI and it implies IUUI. It has interesting consequences for population principles. If, for each population size, a critical level exists for at least one alternative (intermediate expansion equivalence), then IIUUI in conjunction with our basic axioms implies that the population principles must be number-sensitive critical-level generalized utilitarian (Theorem 3).

All of these axioms imply that the same-number subprinciples of any population principles are generalized utilitarian, but none of them implies that they must be utilitarian. For that, other axioms are needed. One possibility is an axiom that we have called *incremental equity* (Blackorby, Bossert and Donaldson [2000]). It is a fixed-population axiom that requires the effect of an addition to one individual's utility to be independent of the identity and the initial utility of this individual.

4. Number-Sensitive Critical-Level Generalized Utilitarianism

In this section we investigate the properties of the number-sensitive critical-level generalized-utilitarian (NCLGU) principles. Because the number-sensitive critical-level utilitarian principles are members of the larger family, all of our theorems apply to them.

Value functions for the NCLGU principles can be written as in (17) and (18) or, equivalently, as

$$W_{NCLGU}(n,\xi) = ng(\xi) - A_g(n) = \sum_{i=1}^{n} g(u_i) - \sum_{i=1}^{n} g(c_{i-1}) \tag{22}$$

where

$$A_g(n) = ng\big(\bar{A}_g(n)\big) = \sum_{i=0}^{n-1} g(c_i). \tag{23}$$

The first requirement we investigate is that critical levels be nonnegative. This is the case if and only if $g(c_n) \geq 0$ for all n because g is increasing and $g(0) = 0$. For all n, we can write

$$g(c_n) = A_g(n+1) - A_g(n) \geq 0, \tag{24}$$

and this means that $A_g(n)$ is nondecreasing in n (Theorem 4).

In addition to nonnegative critical levels, it is reasonable to require them not to decrease as population size increases. Theorem 5 shows that this is the case if and only if the function A_g is convex, with

$$A_g(n+1) \leq \frac{1}{2}\big[A_g(n) + A_g(n+2)\big] \tag{25}$$

for all positive n.

Next, we turn to the repugnant conclusion. Theorem 6 shows that an NCLGU principle implies the repugnant conclusion if and only if there is a sequence of population sizes and corresponding critical levels with a fairly complex property. It must be the case that the critical levels must fall below any positive number when population size is large enough. This result can be combined with the result of Theorem 4 to show that, if critical levels are nonnegative, the repugnant conclusion is implied if and only if critical levels in the sequence approach zero as population size becomes large (Theorem 7). If critical levels are nondecreasing, the repugnant conclusion is implied if and only if all critical levels are nonpositive (Theorem 8). Consequently, a principle with nondecreasing critical levels avoids the repugnant conclusion if and only if it has one positive critical level. Nondecreasingness of critical levels ensures that, in this case, all critical levels for higher population sizes are also positive.

Suppose now that a member of the NCLGU family is used to choose the best alternative or alternatives in some feasible set. We investigate the consequences of increasing any one of the critical levels (which means that the new best alternative(s) are chosen with a different principle). Theorem 9 shows that, if any of the best alternatives are different in the two cases, a lower optimal population size corresponds to the higher critical level.

The investigations discussed here suggest that ethically acceptable members of the number-sensitive critical-level generalized utilitarian family should have critical levels that are nonnegative and nondecreasing, with at least one that is positive. This means that there is some population size beyond which critical levels are all positive. This can be ensured by choosing a function A_g which is nonnegative, positive beyond some population size, nondecreasing and convex. The same desiderata apply to members of the NCLU family and the function A. Members of the critical-level generalized-utilitarian family with positive critical levels satisfy all of the above requirements.

5. Number-Dampened Generalized Utilitarianism

Members of the number-dampened generalized-utilitarian (NDGU) family have value functions that are given by (21) and, when the same-number principles are utilitarian, number-dampened utilitarianism (NDU) results and the value functions can be written as in (12). In both, $f(n)$ must be positive so that the value function is increasing in representative utility (which is average utility in the utilitarian case). Note that the function f is defined on the set of positive integers.

Suppose that $(u_1, \ldots, u_n)$ is the utility vector for an alternative with population size n and that c is its critical level. Then, using NDU and the definition of critical levels,

$$f(n)\mu = f(n+1)\frac{n\mu + c}{n+1}. \tag{26}$$

Rearranging and solving for c,

$$c = \frac{(n+1)f(n) - nf(n+1)}{f(n+1)}\mu = h(n)\mu, \tag{27}$$

where the function h is defined by

$$h(n) = \frac{(n+1)f(n) - nf(n+1)}{f(n+1)}. \tag{28}$$

Critical levels for NDU are equal to a function of population size multiplied by average utility. In the general case (NDGU),

$$g(c) = \frac{(n+1)f(n) - nf(n+1)}{f(n+1)}g(\xi) = h(n)g(\xi), \tag{29}$$

and transformed critical levels are equal to a function h of population size multiplied by the transformed representative utility (see the Appendix).

Theorem 10 shows that a principle belongs to the NDGU family if and only if its transformed critical levels are equal to a function of population size multiplied by transformed representative utilities, where $h(n) > -n$ for all n. Note that it is possible for $h(n)$ to be negative, in which case critical levels for alternatives with positive representative utilities are negative.

For NDU, critical levels for alternatives with positive population sizes are equal to $h(n)\mu$. Because μ can be positive or negative, some critical levels are negative unless $h(n)$ is equal to zero for all n. Theorem 11 shows that this applies to NDGU as well and that all critical levels are nonnegative if and only if the principle is classical generalized utilitarian. Consequently, the NDGU (NDU) principles that are not classical generalized utilitarian (classical utilitarian) share the defects of AU and AGU: some critical levels are negative and, in some cases, the fragmentation of individual lives is regarded as good.

Theorem 12 examines the conditions under which $f(n)$ is increasing in population size. It shows that $f(n)$ is increasing if and only if $h(n)$ is less than one for all n.

Theorem 13 investigates the conditions under which $h(n)$ is nonnegative for all n. This means that, for NDU, critical levels for alternatives with positive average utilities sizes are nonnegative and critical levels for alternatives with negative /average utilities are nonpositive. The theorem shows that $h(n)$ is nonnegative if and only if $f(n)/n$ is nonincreasing. This condition implies that f is concave but the reverse implication is not true.

It might be thought undesirable to have $h(n)$ decrease as n rises. It could, for example, be zero for small n so that the principle coincides with CGU (or CU) for those population sizes, and approach one for large n, so that the principle approximates AGU (AU) for large population sizes. Theorem 14 shows that this requirement on h is satisfied if and only if

$$(n+2)f(n+1)f(n+1) - f(n+1)f(n+2) - (n+1)f(n)f(n+2) \geq 0 \tag{30}$$

for all values of n. This requirement on f has no transparent interpretation.

Theorem 15 examines the relationship between NDGU and the repugnant conclusion. It shows that an NDGU principle satisfies the repugnant conclusion if and only if $f(n)$ is unbounded. That is, for any number, no matter how large, there must be a population size such that $f(n)$ is greater. This is easily illustrated for NDU. Consider any alternative in which each person's utility level is $\xi > 0$. For any utility level $\varepsilon > 0$, no matter how small, there is a better alternative in which a larger population of m individuals experience each experience utility level ε if and only if

$$f(m)\varepsilon > f(n)\xi \tag{31}$$

or, equivalently,

$$f(m) > f(n)\frac{\xi}{\varepsilon}. \tag{32}$$

By choosing ξ and/or ε, the right side of (31) can take on any positive value. Consequently it must be possible to find a population size m such that $f(m)$ can exceed any value. It follows that, in order to avoid the repugnant conclusion, f must be bounded: that is, $f(n)$ must be less than some finite number for all population sizes.

To summarize, a number-dampened generalized-utilitarian principle cannot avoid having negative critical levels unless they are all zero and it is classical, in which case it implies the repugnant conclusion. If it avoids the repugnant conclusion, the function f is bounded and, in that case, some critical levels are negative. $h(n)$, the ratio of critical levels to transformed representative utility, is less than or equal to one if and only if $f(n)$ is nondecreasing, nonnegative if and only if $f(n)/n$ is nonincreasing, and nondecreasing if and only if (30) is satisfied for all n.

A special case of NDGU arises when $h(n)$ is a positive fraction γ between zero and one. In this case, we write $f(n) = f_\gamma(n)$ and it follows that $f_\gamma(1) = 1$ and, for all $n \geq 2$,

$$f_\gamma(n) = \left(\frac{2}{1+\gamma}\right)\cdots\left(\frac{n}{n-1+\gamma}\right). \tag{33}$$

This subfamily of principles satisfies all of our requirements except, perhaps, the most important one. As proved in Theorem 16, each member of the family implies the repugnant conclusion. Theorem 17 considers choice problems using these principles and investigates the response of best population sizes to increases in γ. It shows that, if a higher value of γ changes the set of best population sizes, any population size must be smaller for the higher value of γ.

6. Concluding Remarks

The families of population principles that we have investigated are very large. Both the number-sensitive critical-level generalized-utilitarian and number-dampened generalized-utilitarian families include the classical generalized-utilitarian subfamily and, thus, classical utilitarianism itself. But most members of the two families have very different ethical properties.

We believe that ethically attractive principles should have at least two fundamental properties. First, they should avoid the repugnant conclusion and, second, their critical levels should all be nonnegative. In addition, it is reasonable to select NCLGU principles whose critical levels do not decrease. Given this, the repugnant conclusion is avoided if at least one critical level is positive (nondecreasingness ensures that all critical levels for larger population sizes are positive as well). These properties are satisfied by any NCLGU member such that the function A_g is nondecreasing and convex and the difference $A_g(n+1) - A_g(n)$ is positive for at least one n. All members of the critical-level generalized-utilitarian family with positive critical levels have these properties but they are not the only NCLGU principles that do.

Consider, however, an NCLGU or NCLU principle whose critical levels are not constant. One level might be chosen for 'small' population sizes with a higher one for 'large' sizes. The foremost difficulty with such a principle is the definition of 'small' and 'large'. Principles must be capable of ranking states of affairs that correspond to complete histories of the universe without incorporating resource constraints such as carrying capacity. Reasonable definitions of 'small' and 'large' are almost certain to refer to resource availability and, if they do that, build the constraints of nature and human history into the axiology represented by the principle. For that reason, we think that it is reasonable to require all critical levels to be the same by choosing a member of the critical-level generalized-utilitarian family with a positive critical level.

NCLGU principles satisfy intermediate independence of the utilities of unconcerned individuals but not extended independence of the utilities of unconcerned individuals. In ranking alternatives, therefore, absolute population sizes must be known in order to use a member of the NCLGU family with more than one critical level. This requires knowledge of the number of people who will ever live, a fact that is almost impossible to ascertain. If a CLGU principle is employed, however, neither the utilities nor the number of unaffected individuals must be known in order to rank pairs of alternatives. For this reason, CLGU principles have a significant advantage over other NCLGU principles.

The number-dampened generalized-utilitarian family has both the classical generalized-utilitarian and average generalized-utilitarian principles as subfamilies. When the same-number principles are utilitarian, critical levels are proportional to average utility, and the proportion may depend on population size. When the same-number principles are generalized utilitarian, transformed critical levels are proportional to transformed representative utility. We have shown, however, that there are *no* members of the NDGU family with non-negative critical levels that avoid the repugnant conclusion. If some negative critical levels are accepted, however, there are additional problems. If our argument about the difficulty of defining 'small' and 'large' is persuasive, it is reasonable to require $h(n)$ (which is the ratio of transformed critical levels to transformed representative utility) to be a positive fraction. We have shown, however, that no such principle avoids the repugnant conclusion. We conclude, therefore, that all members of the number-dampened generalized-utilitarian family of population principles are unsatisfactory.

Some number-sensitive critical-level principles with more than one critical level pass our formal tests. However, informational considerations suggest that extended independence of the utilities of unconcerned individuals should be satisfied, and this restricts the set of acceptable principles to the critical-level generalized-utilitarian principles with positive critical levels.

7. Appendix: Axioms, Theorems, and Proofs

7.1. Notation and Basic Axioms

The set of all (nonnegative, positive) real numbers is denoted by $\mathcal{R}$ ($\mathcal{R}_+$, $\mathcal{R}_{++}$), and $\mathcal{Z}_+$ ($\mathcal{Z}_{++}$) is the set of all nonnegative (positive) integers. $\mathbf{1}_n$ is the vector consisting of $n \in \mathcal{Z}_{++}$ ones. For any nonempty set S and any $n \in \mathcal{Z}_{++}$, S^n is the n-fold Cartesian product of S. A function $F\colon \mathcal{Z}_{++} \longrightarrow \mathcal{R}$ is convex if and only if, for all $n \in \mathcal{Z}_{++}$,

$$F(n+1) \leq \frac{1}{2}[F(n) + F(n+2)]. \tag{34}$$

F is concave if and only if the reverse inequality is satisfied in (34).[9]

Given welfarism and same-number anonymity, which ensures that utility vectors matter but not the identities of the individuals whose levels of well-being they represent, we use a social-evaluation ordering R defined on the set of utility vectors $\Omega = \cup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n$. That is, $R \subseteq \Omega \times \Omega$ is a reflexive, transitive, and complete binary relation. The better-than relation and the equally-good relation corresponding to R are denoted by P and I. For the formulation of some of our axioms, it is convenient to use the null utility vector $u_\emptyset$ which corresponds to the null alternative, and we let $\Omega_\emptyset = \Omega \cup \{u_\emptyset\}$. A utility level of zero represents neutrality.

All the theorems in this Appendix refer to the ordering R on the set Ω which can, with information about individual utility levels, be used to order the non-null alternatives in X. If the null alternative is present, a principle's value function can be extended to cover it by using its critical level (see Section 2).

The following axioms impose restrictions on same-number comparisons.

Continuity: For all $n \in \mathcal{Z}_{++}$ and for all $v \in \mathcal{R}^n$, the sets $\{u \in \mathcal{R}^n \mid uRv\}$ and $\{u \in \mathcal{R}^n \mid vRu\}$ are closed.

Strong Pareto: For all $n \in \mathcal{Z}_{++}$ and for all $u, v \in \mathcal{R}^n$, if $u_i \geq v_i$ for all $i \in \{1, \ldots, n\}$ with at least one strict inequality, then uPv.

The standard formulation of strong Pareto encompasses Pareto indifference, which requires any two alternatives with the same population to be ranked as equally good if each person is equally well off in both. Pareto indifference does not appear in the above definition because it is implied by welfarism; any two identical utility vectors must be equally good because R is reflexive. Same-number anonymity implies

Anonymity: For all $n \in \mathcal{Z}_{++}$, for all $u, v \in \mathcal{R}^n$, for all one-to-one mappings $\rho\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, if $v_i = u_{\rho(i)}$ for all $i \in \{1, \ldots, n\}$, then uIv.

[9] (34) is equivalent to the requirement that the 'step-sizes' $F(n+1) - F(n)$ are nondecreasing in n; that is, $F(n+2) - F(n+1) \geq F(n+1) - F(n)$ for all $n \in \mathcal{Z}_{++}$.

If R satisfies continuity, strong Pareto, and anonymity, there exist continuous, increasing, and symmetric representative-utility functions $\Xi^n\colon \mathcal{R} \longrightarrow \mathcal{R}$ for all $n \in \mathcal{Z}_{++}$, with $\Xi^n(\gamma \mathbf{1}_n) = \gamma$ for all $\gamma \in \mathcal{R}$, such that, for all $n \in \mathcal{Z}_{++}$ and for all $u, v \in \mathcal{R}^n$,

$$uRv \iff \Xi^n(u) \geq \Xi^n(v). \tag{35}$$

See, for example, Blackorby, Bossert, and Donaldson [2000] for a proof.

Although knowledge of the representative utility functions is not sufficient to rank utility vectors of different dimensions, the only additional information needed is population size. If R satisfies the above axioms, there exists a function $W\colon \mathcal{Z}_{++} \times \mathcal{R} \longrightarrow \mathcal{R}$, continuous and increasing in its second argument, that represents it (Blackorby and Donaldson [1984]). That is, for all $n, m \in \mathcal{Z}_{++}$, for all $u \in \mathcal{R}^n$, and for all $v \in \mathcal{R}^m$,

$$uRv \iff W(n, \Xi^n(u)) \geq W(m, \Xi^m(v)). \tag{36}$$

W is unique up to increasing transformations.

7.2. Population Principles

A population principle is same-number utilitarian if, for any population size $n \in \mathcal{Z}_{++}$, utility vectors of dimension n are compared on the basis of their total or average utility. The utilitarian representative-utility functions can be found as follows. For any $n \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$, let representative utility be $\xi = \Xi^n(u)$. Then, for any $u \in \mathcal{R}^n$, $\xi \mathbf{1_n} I u$ and, by same-number utilitarianism,

$$n\xi = \sum_{i=1}^{n} u_i, \tag{37}$$

which implies

$$\xi = \Xi^n(u) = \frac{1}{n}\sum_{i=1}^{n} u_i. \tag{38}$$

Using a similar argument, it is easy to show that representative-utility functions for same-number generalized utilitarianism are given by

$$\Xi^n(u) = (g^n)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g^n(u_i)\right) \tag{39}$$

for all $n \in \mathcal{Z}_{++}$ and $u \in \mathcal{R}^n$, where $g^n\colon \mathcal{R} \longrightarrow \mathcal{R}$ is continuous and increasing. Without loss of generality, g^n can be can be chosen so that $g^n(0) = 0$. Note that the function g^n can be different for different population sizes. Most population principles whose same-number subprinciples are generalized utilitarian have value functions that allow the same transform g to be used for all $n \in \mathcal{Z}_{++}$.

Average generalized utilitarianism compares average transformed utilities to rank utility vectors. The representative-utility functions are same-number generalized utilitarian and the corresponding value function is defined by

$$W_{AGU}(n,\xi) = g(\xi) = \frac{1}{n}\sum_{i=1}^{n} g(u_i) \tag{40}$$

for all $(n,\xi) \in \mathcal{Z}_{++} \times \mathcal{R}$.[10]

Critical-level generalized utilitarianism uses the sum of the differences between transformed individual utilities and a transformed fixed critical level to rank utility vectors. Again, representative-utility functions are fixed-population generalized utilitarian, and the value functions are given by

$$W_{CLGU}(n,\xi) = n\big(g(\xi) - g(\alpha)\big) = \sum_{i=1}^{n}\big(g(u_i) - g(\alpha)\big) \tag{41}$$

for all $(n,\xi) \in \mathcal{Z}_{++} \times \mathcal{R}$, where α is the fixed critical level. The classical generalized-utilitarian family is the subfamily of CLGU with $\alpha = 0$.

A population principle implies the repugnant conclusion if, for any population size $n \in \mathcal{Z}_{++}$, any positive utility level ξ, and any utility level $\varepsilon \in \mathcal{R}_{++}$, there exists a population size $m > n$ such that an m-person alternative in which every individual experiences utility level ε is ranked as better than an n-person society in which every individual's utility level is ξ. ε is above neutrality but can be arbitrarily close to it. Formally,

Repugnant Conclusion: For all $\xi \in \mathcal{R}_{++}$, for all $n \in \mathcal{Z}_{++}$, and for all $\varepsilon \in \mathcal{R}_{++}$, there exists $m > n$ such that $\varepsilon \mathbf{1}_m P \xi \mathbf{1}_n$.

7.3. Independence Axioms and their Consequences

The same-number version of independence of the utilities of unconcerned individuals requires that, for a fixed population size, the relative ranking of two utility vectors is independent of the utilities of those individuals whose utility levels are the same in both.

Independence of the Utilities of Unconcerned Individuals: For all $n, m \in \mathcal{Z}_{++}$, for all $u, v \in \mathcal{R}^n$, for all $w, s \in \mathcal{R}^m$,

$$(u,w)R(v,w) \Longleftrightarrow (u,s)R(v,s). \tag{42}$$

Together with continuity, strong Pareto, and anonymity, independence of the utilities of unconcerned individuals implies that the restrictions of R to $\mathcal{R}^n \times \mathcal{R}^n$ must be generalized utilitarian for all $n \geq 3$. See Debreu [1960] and Fleming [1952]; a proof can be found in Blackorby, Bossert, and Donaldson [2000].

[10] An ordinally equivalent function $\hat{W}$ is given by $\hat{W}(n,\xi) = \xi$.

Theorem 1: *If R satisfies continuity, strong Pareto, anonymity, and independence of the utilities of unconcerned individuals, then, for all $n \geq 3$, there exists a continuous and increasing function $g^n\colon \mathcal{R} \longrightarrow \mathcal{R}$ with $g^n(0) = 0$ such that, for all $u, v \in \mathcal{R}^n$,*

$$uRv \Longleftrightarrow \sum_{i=1}^{n} g^n(u_i) \geq \sum_{i=1}^{n} g^n(v_i). \tag{43}$$

An extended version of the above independence axiom applies to different-number comparisons as well.

Extended Independence of the Utilities of Unconcerned Individuals: For all $u, v, w \in \Omega$,

$$(u, w)R(v, w) \Longleftrightarrow uRv. \tag{44}$$

Expansion equivalence requires the existence of critical levels for all utility vectors $u \in \Omega$. Note that the critical levels may be different for different utility vectors.

Expansion Equivalence: For all $u \in \Omega$, there exists $c \in \mathcal{R}$ such that $(u, c)Iu$.

The following weakening of this axiom requires the existence of at least one critical level.

Weak Expansion Equivalence: There exist $\bar{u} \in \Omega$ and $\bar{c} \in \mathcal{R}$ such that $(\bar{u}, \bar{c})I\bar{u}$.

If R satisfies strong Pareto and expansion equivalence, all critical levels exist and are unique. Thus, there exist critical-level functions $C^n\colon \mathcal{R}^n \longrightarrow \mathcal{R}$ for all $n \in \mathcal{Z}_{++}$ such that, for all $n \in \mathcal{Z}_{++}$ and for all $u \in \mathcal{R}^n$, $(u, C^n(u))Iu$.

Weak expansion equivalence and extended independence of the utilities of unconcerned individuals together imply expansion equivalence and, furthermore, all critical levels must be the same (see, for example, Blackorby, Bossert, and Donaldson [1995, 1998, 2000]). This observation is useful in proving the following characterization result for critical-level generalized utilitarianism (see Blackorby, Bossert, and Donaldson [1998]).

Theorem 2: *R satisfies continuity, strong Pareto, anonymity, extended independence of the utilities of unconcerned individuals, and weak expansion equivalence if and only if there exist a continuous and increasing function $g\colon \mathcal{R} \longrightarrow \mathcal{R}$ with $g(0) = 0$ and $\alpha \in \mathcal{R}$ such that, for all $n, m \in \mathcal{Z}_{++}$, for all $u \in \mathcal{R}^n$, for all $v \in \mathcal{R}^m$,*

$$uRv \Longleftrightarrow \sum_{i=1}^{n} \big(g(u_i) - g(\alpha)\big) \geq \sum_{i=1}^{m} \big(g(v_i) - g(\alpha)\big). \tag{45}$$

The following independence axiom is intermediate in strength between independence of the utilities of unconcerned individuals and its extended counterpart.

Intermediate Independence of the Utilities of Unconcerned Individuals: For all $u, v \in \Omega_\emptyset$, for all $r \in \mathcal{Z}_{++}$, for all $w, s \in \mathcal{R}^r$,

$$(u, w)R(v, w) \Longleftrightarrow (u, s)R(v, s). \tag{46}$$

Analogously, there is an axiom that is intermediate in strength between weak expansion equivalence and expansion equivalence.

Intermediate Expansion Equivalence: For all $n \in \mathcal{Z}_{++}$, there exist $\bar{u} \in \mathcal{R}^n$ and $c_n \in \mathcal{R}$ such that $(\bar{u}, c_n)I\bar{u}$.

The value function for number-sensitive critical-level generalized utilitarianism is given by

$$W(n, \xi) = ng(\xi) - A_g(n) \tag{47}$$

for all $(n, \xi) \in \mathcal{Z}_{++} \times \mathcal{R}$, where

$$\xi = \Xi^n(u) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(u_i)\right) \tag{48}$$

and $A_g: \mathcal{Z}_{++} \longrightarrow \mathcal{R}$ is a function that depends on the transformation g. Defining the function $\bar{A}_g: \mathcal{Z}_{++} \longrightarrow \mathcal{R}$ as $\bar{A}_g(n) = g^{-1}\big(A_g(n)/n\big)$ for all $n \in \mathcal{Z}_{++}$, the value function for NCLGU can be rewritten as

$$W(n, \xi) = n\Big(g(\xi) - g\big(\bar{A}_g(n)\big)\Big). \tag{49}$$

The following theorem is a characterization of number-sensitive critical-level generalized utilitarianism.

Theorem 3: *R satisfies continuity, strong Pareto, anonymity, intermediate independence of the utilities of unconcerned individuals, and intermediate expansion equivalence if and only if there exist a continuous and increasing function $g: \mathcal{R} \longrightarrow \mathcal{R}$ with $g(0) = 0$ and a function $A_g: \mathcal{Z}_{++} \longrightarrow \mathcal{R}$ such that, for all $n, m \in \mathcal{Z}_{++}$, for all $u \in \mathcal{R}^n$, for all $v \in \mathcal{R}^m$,*

$$uRv \Longleftrightarrow \sum_{i=1}^{n} g(u_i) - A_g(n) \geq \sum_{i=1}^{m} g(v_i) - A_g(m). \tag{50}$$

Proof. That the number-sensitive critical-level generalized-utilitarian principles satisfy the axioms is straightforward to verify. Now suppose R satisfies continuity, strong Pareto, anonymity, intermediate independence of the utilities of unconcerned individuals and intermediate expansion equivalence. Because intermediate independence of the utilities of unconcerned individuals implies independence of the utilities of unconcerned individuals, Theorem 1 implies that, for all $n \geq 3$, there exists a continuous and increasing function $g^n : \mathcal{R} \longrightarrow \mathcal{R}$ with $g^n(0) = 0$ such that, for all $u, v \in \mathcal{R}^n$,

$$uRv \Longleftrightarrow \sum_{i=1}^{n} g^n(u_i) \geq \sum_{i=1}^{n} g^n(v_i). \tag{51}$$

Let $n \in \mathcal{Z}_{++}$. By intermediate expansion equivalence, there exist $\bar{u} \in \mathcal{R}^n$ and $c_n \in \mathcal{R}$ such that

$$(\bar{u}, c_n) I \bar{u}. \tag{52}$$

Let $r = n$ and $u = (c_n)$, $v = u_\emptyset$, $w = \bar{u}$, and $s \in \mathcal{R}^n$ in the definition of intermediate independence of the utilities of unconcerned individuals. By (52) and intermediate independence of the utilities of unconcerned individuals, $(s, c_n) I s$ and, because $s \in \mathcal{R}^n$ was chosen arbitrarily, c_n must be a critical level for any n-dimensional vector.

Next, we prove that, for all $n \geq 3$, the functions g^n and g^{n+1} can be chosen to be the same. Let $u, v \in \mathcal{R}^n$. Because c_n is a critical level for u and v, it follows that

$$uRv \Longleftrightarrow (u, c_n) R (v, c_n). \tag{53}$$

By (51),

$$uRv \Longleftrightarrow \sum_{i=1}^{n} g^n(u_i) \geq \sum_{i=1}^{n} g^n(v_i) \tag{54}$$

and

$$\begin{aligned} (u, c_n) R (v, c_n) &\Longleftrightarrow \sum_{i=1}^{n} g^{n+1}(u_i) + g^{n+1}(c_n) \geq \sum_{i=1}^{n} g^{n+1}(v_i) + g^{n+1}(c_n) \\ &\Longleftrightarrow \sum_{i=1}^{n} g^{n+1}(u_i) \geq \sum_{i=1}^{n} g^{n+1}(v_i). \end{aligned} \tag{55}$$

Therefore, using (53),

$$\sum_{i=1}^{n} g^n(u_i) \geq \sum_{i=1}^{n} g^n(v_i) \Longleftrightarrow \sum_{i=1}^{n} g^{n+1}(u_i) \geq \sum_{i=1}^{n} g^{n+1}(v_i) \tag{56}$$

which means that the same function can be used for g^n and for g^{n+1}. Because this is true for all $n \geq 3$, it follows that the functions g^n can be chosen independently of n, and we write $g = g^n$ for all $n \geq 3$. Together with (51), it follows that, for all $n \geq 3$ and for all $u, v \in \mathcal{R}^n$,

$$uRv \iff \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i). \tag{57}$$

Next, we prove that (57) must be true for $n \in \{1, 2\}$ as well. Suppose $u, v \in \mathcal{R}^1$. By strong Pareto and the increasingness of g,

$$uRv \iff u_1 \geq v_1 \iff g(u_1) \geq g(v_1). \tag{58}$$

If $n = 2$ and $u, v \in \mathcal{R}^n$, intermediate independence of the utilities of unconcerned individuals and (57) together imply

$$\begin{aligned} uRv \iff (u, c_2)R(v, c_2) &\iff \sum_{i=1}^{2} g(u_i) + g(c_2) \geq \sum_{i=1}^{2} g(v_i) + g(c_2) \\ &\iff \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{n} g(v_i). \end{aligned} \tag{59}$$

To complete the proof, let $n, m \in \mathcal{Z}_{++}$ with $n \neq m$, $u \in \mathcal{R}^n$, and $v \in \mathcal{R}^m$. Without loss of generality, suppose $n > m$. By definition of the critical levels and letting $c_0 \in \mathcal{R}$ be arbitrary,

$$\begin{aligned} uRv &\iff uR(v, c_m, \ldots, c_{n-1}) \\ &\iff \sum_{i=1}^{n} g(u_i) \geq \sum_{i=1}^{m} g(v_i) + \sum_{i=m+1}^{n} g(c_{i-1}) \\ &\iff \sum_{i=1}^{n} g(u_i) - \sum_{i=1}^{n} g(c_{i-1}) \geq \sum_{i=1}^{m} g(v_i) - \sum_{i=1}^{m} g(c_{i-1}) \end{aligned} \tag{60}$$

and, defining $A_g(n) = \sum_{i=1}^{n} g(c_{i-1})$ for all $n \in \mathcal{Z}_{++}$, this completes the proof. ■

c_0 is arbitrary when alternatives with positive population sizes only are ranked, but when NCLU is extended to cover the null alternative, c_0 is its critical level. For that reason, we refer to c_0 as a critical level.

7.4. Number-Sensitive Critical-Level Generalized Utilitarianism

The functions A_g in Theorem 3 can be chosen arbitrarily. Because they determine the critical levels, with $g(c_n) = A_g(n+1) - A_g(n)$, restrictions on the function A_g are implied by restrictions on critical levels. The following theorems provide necessary and sufficient conditions on A_g for several ethically attractive restrictions.

Theorem 4: *Let R be a number-sensitive critical-level generalized-utilitarian social-evaluation ordering. All critical levels are nonnegative if and only if A_g is nondecreasing.*

Proof. By definition, $A_g(n) = \sum_{i=1}^{n} g(c_{i-1})$ for all $n \in \mathcal{Z}_{++}$. Therefore, for all $n \in \mathcal{Z}_{++}$, $g(c_n) = A_g(n+1) - A_g(n)$ and hence

$$c_n = g^{-1}[A_g(n+1) - A_g(n)] \tag{61}$$

for all $n \in \mathcal{Z}_{++}$ (note that g^{-1} is well-defined and increasing because g is increasing). Because $g(0) = 0$,

$$\begin{aligned} c_n \geq 0 &\Longleftrightarrow g^{-1}[A_g(n+1) - A_g(n)] \geq 0 \\ &\Longleftrightarrow A_g(n+1) - A_g(n) \geq 0 \\ &\Longleftrightarrow A_g(n+1) \geq A_g(n) \end{aligned} \tag{62}$$

for all $n \in \mathcal{Z}_{++}$. ■

If the null alternative is present, nonnegativity of all critical levels also implies that $A_g(n)$ is nonnegative because $A_g(1) = g(c_0) \geq 0$.

Theorem 5: *Let R be a number-sensitive critical-level generalized-utilitarian social-evaluation ordering. Critical levels are nondecreasing if and only if A_g is convex.*

Proof. Using (61) and the increasingness of g^{-1}, it follows that

$$\begin{aligned} c_{n+1} \geq c_n &\Longleftrightarrow g^{-1}[A_g(n+2) - A_g(n+1)] \geq g^{-1}[A_g(n+1) - A_g(n)] \\ &\Longleftrightarrow A_g(n+2) - A_g(n+1) \geq A_g(n+1) - A_g(n) \\ &\Longleftrightarrow A_g(n+1) \leq \frac{1}{2}[A_g(n) + A_g(n+2)] \end{aligned} \tag{63}$$

for all $n \in \mathcal{Z}_{++}$. ■

Theorem 6: *Let R be a number-sensitive critical-level generalized-utilitarian social-evaluation ordering. R implies the repugnant conclusion if and only there exists an increasing sequence $\langle m_j \rangle_{j \in \mathcal{Z}_{++}}$ in $\mathcal{Z}_{++}$ such that, for all $\delta \in \mathcal{R}_{++}$, there exists $\tilde{j} \in \mathcal{Z}_{++}$ such that $\bar{A}_g(m_j) < \delta$ for all $j \geq \tilde{j}$.*

Proof. 'Only if.' Suppose a number-sensitive critical-level generalized-utilitarian social-evaluation ordering R implies the repugnant conclusion, which means that, for all $\xi \in \mathcal{R}_{++}$, for all $n \in \mathcal{Z}_{++}$, for all $\varepsilon \in \mathcal{R}_{++}$, there exists $m > n$ such that

$$m\Big[g(\varepsilon) - g\big(\bar{A}_g(m)\big)\Big] > n\Big[g(\xi) - g\big(\bar{A}_g(n)\big)\Big]. \tag{64}$$

Let $\langle \varepsilon_j \rangle_{j \in \mathcal{Z}_{++}}$ be a sequence in $\mathcal{R}_{++}$ such that $\lim_{j \to \infty} \varepsilon_j = 0$.

Let $n = 1$. By the repugnant conclusion, for all $\xi \in \mathcal{R}_{++}$, there exists $m_1 > 1$ such that

$$m_1\Big[g(\varepsilon_1) - g\big(\bar{A}_g(m_1)\big)\Big] > \Big[g(\xi) - g\big(\bar{A}_g(1)\big)\Big]. \tag{65}$$

Choosing $\xi > c_0 = \bar{A}_g(1)$, the right side of (65) is positive and, consequently, (65) implies $g\big(\bar{A}_g(m_1)\big) < g(\varepsilon_1)$.

For any $j \in \mathcal{Z}_{++}$, set $n = m_j$. By the repugnant conclusion, there exists $m_{j+1} > m_j$ such that

$$m_{j+1}\Big[g\big(\varepsilon_{j+1}\big) - g\big(\bar{A}_g(m_{j+1})\big)\Big] > m_j\Big[g\big(\xi\big) - g\big(\bar{A}_g(m_j)\big)\Big] \tag{66}$$

for all $\xi \in \mathcal{R}_{++}$. Choosing $\xi > \max\{c_{i-1} \mid i \in \{1, \ldots, m_j\}\}$, the right side of (66) is positive and, consequently, the inequality requires $g\big(\bar{A}_g(m_{j+1})\big) < g(\varepsilon_{j+1})$. Therefore, we have proven that $g\big(\bar{A}_g(m_j)\big) < g(\varepsilon_j)$ and, because g is increasing,

$$\bar{A}_g(m_j) < \varepsilon_j \tag{67}$$

for all $j \in \mathcal{Z}_{++}$. Because $\lim_{j\to\infty} \varepsilon_j = 0$, for any $\delta \in \mathcal{R}_{++}$, there exists $\tilde{\jmath} \in \mathcal{Z}_{++}$ such that $\varepsilon_j < \delta$ for all $j \geq \tilde{\jmath}$. Hence, using (67), $\bar{A}_g(m_j) < \delta$ for all $j \geq \tilde{\jmath}$.

'If.' Suppose there exists an increasing sequence $\langle m_j\rangle_{j\in\mathcal{Z}_{++}}$ in $\mathcal{Z}_{++}$ with the properties stated in the theorem. Let $\xi \in \mathcal{R}_{++}$, $n \in \mathcal{Z}_{++}$, and $\varepsilon \in \mathcal{R}_{++}$. Letting $\delta = g(\varepsilon/2)$, it follows that there exists $\tilde{\jmath} \in \mathcal{Z}_{++}$ such that $\bar{A}_g(m_j) < \varepsilon/2$ and hence $g\big(\bar{A}_g(m_j)\big) < g(\varepsilon/2)$ for all $j \geq \tilde{\jmath}$. Choose $\hat{\jmath} \in \mathcal{Z}_{++}$ so that $m_{\hat{\jmath}} > n$ (this is possible because $\langle m_j\rangle_{j\in\mathcal{Z}_{++}}$ is an increasing sequence in $\mathcal{Z}_{++}$ and, thus, unbounded), and let $\mathring{\jmath} = \max\{\hat{\jmath}, \tilde{\jmath}\}$. By definition, $m_j > n$ and $g\big(\bar{A}_g(m_j)\big) < g(\varepsilon/2)$ for all $j \geq \mathring{\jmath}$. It follows that

$$m_{\mathring{\jmath}}\Big[g\big(\varepsilon\big) - g\big(\bar{A}_g(m_{\mathring{\jmath}})\big)\Big] > m_{\mathring{\jmath}}\Big[g\big(\varepsilon\big) - g\big(\varepsilon/2\big)\Big] > 0 \tag{68}$$

and hence $g(\varepsilon) - g\big(\bar{A}_g(m_{\mathring{\jmath}})\big) > g(\varepsilon) - g(\varepsilon/2) > 0$. Thus, $g(\varepsilon) - g\big(\bar{A}_g(m_j)\big) > g(\varepsilon) - g(\varepsilon/2)$ and hence

$$m_j\Big[g\big(\varepsilon\big) - g\big(\bar{A}_g(m_j)\big)\Big] > m_j\Big[g\big(\varepsilon\big) - g\big(\varepsilon/2\big)\Big] > 0 \tag{69}$$

for all $j \geq \mathring{\jmath}$. Because $\langle m_j\rangle_{j\in\mathcal{Z}_{++}}$ is an increasing sequence in $\mathcal{Z}_{++}$, $m_j[g(\varepsilon) - g(\varepsilon/2)] \longrightarrow \infty$ as $j \longrightarrow \infty$ and, hence, j can be chosen sufficiently large so that

$$m_j\Big[g\big(\varepsilon\big) - g\big(\bar{A}_g(m_j)\big)\Big] > m_j\Big[g\big(\varepsilon\big) - g\big(\varepsilon/2\big)\Big] > n\Big[g\big(\xi\big) - g\big(\bar{A}_g(n)\big)\Big] \tag{70}$$

which is the repugnant conclusion. ■

If critical levels are nonnegative or nondecreasing, the necessary and sufficient condition for NCLGU to imply the repugnant conclusion can be simplified as stated in the following two theorems. The proofs are similar to the proof of Theorem 6 and are omitted.

Theorem 7: *Let R be a number-sensitive critical-level generalized-utilitarian social-evaluation ordering with nonnegative critical levels. R implies the repugnant conclusion if and only there exists an increasing sequence $\langle m_j\rangle_{j\in\mathcal{Z}_{++}}$ in $\mathcal{Z}_{++}$ such that* $\lim_{j\to\infty} \bar{A}_g(m_j) = 0$.

Theorem 8: *Let R be a number-sensitive critical-level generalized-utilitarian social-evaluation ordering with nondecreasing critical levels. R implies the repugnant conclusion if and only if all critical levels are nonpositive.*

The next theorem examines the behaviour of number-sensitive critical-level generalized utilitarianism in choice problems. Given a feasible set of utility vectors $\mathcal{U}$, suppose we choose the best utility vectors according to number-sensitive critical-level generalized utilitarianism. We assume that the set of feasible utility vectors $\mathcal{U}$ is such that best elements exist. Let $c = \langle c_n \rangle_{n \in \mathcal{Z}_{++}}$ be the sequence of critical levels, and let $\Phi(c)$ denote the set of chosen utility vectors for the critical levels c. We obtain

Theorem 9: *Let c and c' be two sequences of critical levels such that there exists $r \in \mathcal{Z}_{++}$ with $c'_r > c_r$ and $c'_j = c_j$ for all $j \in \mathcal{Z}_{++} \setminus \{r\}$. Let $n, m \in \mathcal{Z}_{++}$, $u \in \mathcal{R}^n$, and $v \in \mathcal{R}^m$. If $u \in \Phi(c)$ and $v \in \Phi(c')$ and $\Phi(c) \neq \Phi(c')$, then $n > m$.*

Proof. Suppose $u \in \Phi(c)$ and $v \in \Phi(c')$. Because the chosen utility vectors are best elements in $\mathcal{U}$ according to number-sensitive critical-level generalized utilitarianism, it follows that

$$\sum_{i=1}^{n} g(u_i) - \sum_{i=1}^{n} g(c_{i-1}) \geq \sum_{i=1}^{m} g(v_i) - \sum_{i=1}^{m} g(c_{i-1}) \tag{71}$$

and

$$\sum_{i=1}^{m} g(v_i) - \sum_{i=1}^{m} g(c'_{i-1}) \geq \sum_{i=1}^{n} g(u_i) - \sum_{i=1}^{n} g(c'_{i-1}). \tag{72}$$

Adding these inequalities and rearranging, we obtain

$$\sum_{i=1}^{n} g(c'_{i-1}) - \sum_{i=1}^{m} g(c'_{i-1}) \geq \sum_{i=1}^{n} g(c_{i-1}) - \sum_{i=1}^{m} g(c_{i-1}). \tag{73}$$

If $r \geq \max\{n, m\}$ or $r < \min\{n, m\}$, (73) is trivially satisfied and both u and v are solutions to both problems, which contradicts the assumption that $\Phi(c) \neq \Phi(c')$.

If $\min\{n, m\} \leq r < \max\{n, m\}$, it follows that $n \neq m$. Because $m > n$ is incompatible with (73), it follows that we must have $n > m$. ■

7.5. Number-Dampened Generalized Utilitarianism

Ng [1986] suggests number-dampened utilitarianism (NDU) as an alternative to critical-level utilitarianism in order to avoid the repugnant conclusion (see also Hurka [1983]). NDU can be made consistent with same-number generalized utilitarianism and we investigate number-dampened generalized utilitarianism in this subsection. The value function for NDGU is given by

$$W(n,\xi)=f(n)g(\xi)=\frac{f(n)}{n}\sum_{i=1}^{n}g(u_i) \tag{74}$$

for all $(n,\xi)\in\mathcal{Z}_{++}\times\mathcal{R}$, where $f\colon\mathcal{Z}_{++}\longrightarrow\mathcal{R}_{++}$, $g\colon\mathcal{R}\longrightarrow\mathcal{R}$ is continuous and increasing, $g(0)=0$ and

$$\xi=\Xi^n(u)=g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}g(u_i)\right). \tag{75}$$

NDU results from setting g equal to the identity map. For future reference, we define the average function $\bar{f}\colon\mathcal{Z}_{++}\longrightarrow\mathcal{R}_{++}$ by defining $\bar{f}(n)=f(n)/n$ for all $n\in\mathcal{Z}_{++}$.

Critical-level functions for NDGU can be computed from the value function. Let c be the critical level for $u\in\mathcal{R}^n$. Then

$$f(n)g(\xi)=f(n+1)\frac{ng(\xi)+g(c)}{n+1}. \tag{76}$$

Solving for $g(c)$, we get

$$g(c)=g\big(C^n(u)\big)=\frac{(n+1)f(n)-nf(n+1)}{f(n+1)}g(\xi) \tag{77}$$

or

$$c=C^n(u)=g^{-1}\left(\frac{(n+1)f(n)-nf(n+1)}{f(n+1)}g(\xi)\right) \tag{78}$$

Defining

$$h(n)=\frac{(n+1)f(n)-nf(n+1)}{f(n+1)}, \tag{79}$$

(77) can be rewritten as

$$g(c)=g\big(C^n(u)\big)=h(n)g(\xi). \tag{80}$$

If the same-number principles are utilitarian, then

$$C^n(u)=\frac{(n+1)f(n)-nf(n+1)}{f(n+1)}\mu=h(n)\mu. \tag{81}$$

Number-dampened generalized utilitarianism can be characterized by adding expansion equivalence and a requirement on the structure of the critical-level functions C^n to the assumption that R is same-number generalized utilitarian.

Theorem 10: *Let R be a social-evaluation ordering whose same-number subprinciples are generalized utilitarian with $g^n=g$ for all $n\in\mathcal{Z}_{++}$. R satisfies expansion equivalence and there exists a function $h\colon\mathcal{Z}_{++}\longrightarrow\mathcal{R}$ with $h(n)>-n$ for all $n\in\mathcal{Z}_{++}$ such that $g\big(C^n(u)\big)=h(n)g(\Xi^n(u))$ if and only if R is a number-dampened generalized-utilitarian social-evaluation ordering.*

Proof. 'If.' Suppose R is NDGU. The argument in the text above shows that $g\big(C^n(u)\big) = h(n)g(\Xi^n(u))$ where $h\colon \longrightarrow \mathcal{R}$ is given by (79). Because $f(n) > 0$ for all $n \in \mathcal{Z}_{++}$,

$$\begin{aligned} &(n+1)f(n) > 0 \\ &\qquad \Longrightarrow (n+1)f(n) - nf(n+1) > -nf(n+1) \\ &\qquad \Longrightarrow \frac{(n+1)f(n) - nf(n+1)}{f(n+1)} > -n \\ &\qquad \Longrightarrow h(n) > -n. \end{aligned} \tag{82}$$

'Only if.' Suppose $g\big(C^n(u)\big) = h(n)g(\xi)$ with $h(n) > -n$ for all $n \in \mathcal{Z}_{++}$ and all $u \in \mathcal{R}^n$, where $\xi = \Xi^n(u)$. Again, we construct f explicitly and show that it has the required properties. Define the function $H\colon \mathcal{Z}_{++} \longrightarrow \mathcal{R}$ by letting $H(1) = 1$ and

$$H(n) = \frac{h(n-1)}{n-1}\left[\sum_{j=1}^{n-1} H(j)\right] \tag{83}$$

for all $n \geq 2$. Let $w \in \mathcal{R}$ be such that $uI(w)$. By definition of H,

$$(w)Iw\,(H(1))\,I \ldots Iw\,(H(1), \ldots, H(n)) \tag{84}$$

and, therefore, using same-number GU,

$$\begin{aligned} uI(w) &\Longleftrightarrow \frac{g(w)}{n}\left[H(1) + \ldots H(n)\right] = g(\xi) \\ &\Longleftrightarrow w = \frac{ng(\xi)}{\sum_{j=1}^{n} H(j)}. \end{aligned} \tag{85}$$

Now define $f(n) = n/\sum_{j=1}^{n} H(j)$ for all $n \in \mathcal{Z}_{++}$. By definition, $W(n,\xi) = f(n)g(\xi)$ for all $n \in \mathcal{Z}_{++}$ and all $\xi \in \mathcal{R}$. It remains to be shown that $f(n) > 0$ for all $n \in \mathcal{Z}_{++}$. We proceed by induction. For $n = 1$, we obtain $f(1) = 1/H(1) = 1 > 0$. Now suppose $n \geq 2$ and $f(m) > 0$ for all $m \in \{1, \ldots, n-1\}$. Clearly, $f(n) > 0$ if and only if $\sum_{j=1}^{n} H(j) > 0$. By definition,

$$\begin{aligned} \sum_{j=1}^{n} H(j) &= \sum_{j=1}^{n-1} H(j) + H(n) \\ &= \sum_{j=1}^{n-1} H(j) + \frac{h(n-1)}{n-1}\sum_{j=1}^{n-1} H(j) \\ &= \left[1 + \frac{h(n-1)}{n-1}\right]\sum_{j=1}^{n-1} H(j). \end{aligned} \tag{86}$$

Because $h(n-1) > -(n-1)$ by assumption and $\sum_{j=1}^{n-1} H(j) > 0$ by the induction hypothesis, this implies $f(n) > 0$. ■

Next, we examine some properties of number-dampened generalized utilitarianism.

Theorem 11: *Let R be an NDGU social-evaluation ordering. All critical levels are nonnegative if and only if R is classical generalized utilitarian.*

Proof. Clearly, CGU implies that all critical levels are zero and hence nonnegative. Conversely, nonnegativity of the critical levels according to NDGU requires that $h(n)g(\xi) \geq 0$ for all $n \in \mathcal{Z}_{++}$ and all $\xi \in \mathcal{R}$. Because ξ and, hence, $g(\xi)$, may be positive or negative for all $n \in \mathcal{Z}_{++}$, this is only possible if $h(n) = 0$ for all $n \in \mathcal{Z}_{++}$. By definition of h, this requires

$$(n+1)f(n) - nf(n+1) = 0 \tag{87}$$

or, equivalently,

$$\frac{f(n)}{n} = \frac{f(n+1)}{n+1} \tag{88}$$

which implies that $f(n) = an$ for all $n \in \mathcal{Z}_{++}$, where $a \in \mathcal{R}_{++}$ is a constant. The resulting value function is ordinally equivalent to CGU. ■

Theorem 12: *Let R be an NDGU social-evaluation ordering. The function f is nondecreasing if and only if $h(n) \leq 1$ for all $n \in \mathcal{Z}_{++}$.*

Proof. By definition of h,

$$\begin{aligned} h(n) \leq 1 &\iff (n+1)f(n) - nf(n+1) \leq f(n+1) \\ &\iff (n+1)f(n) - (n+1)f(n+1) \leq 0 \\ &\iff f(n) \leq f(n+1) \end{aligned} \tag{89}$$

for all $n \in \mathcal{Z}_{++}$. ■

Theorem 13: *Let R be an NDGU social-evaluation ordering. The function h is nonnegative-valued if and only if $\bar{f}$ is nonincreasing.*

Proof. By definition of h,

$$\begin{aligned} h(n) \geq 0 &\iff \frac{(n+1)f(n) - nf(n+1)}{f(n+1)} \geq 0 \\ &\iff (n+1)f(n) \geq nf(n+1) \\ &\iff \frac{f(n)}{n} \geq \frac{f(n+1)}{n+1} \end{aligned} \tag{90}$$

for all $n \in \mathcal{Z}_{++}$. ■

Nonincreasingness of $\bar{f}$ implies concavity of f but the reverse implication is not true. Suppose $\bar{f}$ is nonincreasing. This implies

$$\frac{f(n+2)}{n+2} \leq \frac{f(n+1)}{n+1} \leq \frac{f(n)}{n} \tag{91}$$

for all $n \in \mathcal{Z}_{++}$. This implies

$$f(n+2) - f(n+1) \leq \frac{n+2}{n+1} f(n+1) - f(n+1) = \frac{f(n+1)}{n+1} \tag{92}$$

and

$$f(n+1) - f(n) \geq -\frac{n}{n+1} f(n+1) + f(n+1) = \frac{f(n+1)}{n+1} \tag{93}$$

for all $n \in \mathcal{Z}_{++}$. Combining these inequalities and rearranging, we obtain

$$f(n+1) \geq \frac{1}{2}[f(n) + f(n+2)] \tag{94}$$

for all $n \in \mathcal{Z}_{++}$.

The following example shows that concavity of f is not sufficient for nonincreasingness of $\bar{f}$ (and, thus, nonnegative-valuedness of h). Let $f(1) = 1$ and $f(n) = n+1$ for all $n \geq 2$. Clearly, f is concave, but we obtain $h(1) = -1/3 < 0$ (and $h(n) = 0$ for all $n \geq 2$).

Theorem 14: *Let R be an NDGU social-evaluation ordering. The function h is nondecreasing if and only if*

$$(n+2)f(n+1)f(n+1) - f(n+1)f(n+2) - (n+1)f(n)f(n+2) \geq 0 \tag{95}$$

for all $n \in \mathcal{Z}_{++}$.

Proof. h is nondecreasing if and only if $h(n+1) \geq h(n)$ for all $n \in \mathcal{Z}_{++}$, and

$$\begin{aligned} & h(n+1) \geq h(n) \\ \iff & \frac{(n+2)f(n+1) - (n+1)f(n+2)}{f(n+2)} \geq \frac{(n+1)f(n) - nf(n+1)}{f(n+1)} \\ \iff & (n+2)f(n+1)f(n+1) - f(n+1)f(n+2) - (n+1)f(n)f(n+2) \geq 0. \end{aligned} \tag{96}$$

■

Theorem 15: *Let R be an NDGU social-evaluation ordering. R implies the repugnant conclusion if and only the function f is unbounded.*

Proof. 'If.' Suppose f is unbounded. Let $\xi \in \mathcal{R}_{++}$, $n \in \mathcal{Z}_{++}$, and $\varepsilon \in \mathcal{R}_{++}$. Because f is unbounded, there exists $m > n$ such that $f(m) > f(n)g(\xi)/g(\varepsilon)$ which implies the repugnant conclusion.

'Only if.' Suppose f is bounded by $\delta \in \mathcal{R}_{++}$, that is, $f(n) \leq \delta$ for all $n \in \mathcal{Z}_{++}$. Let $g(\xi) = \delta$, $n = 1$, and $g(\varepsilon) = f(1)$. The repugnant conclusion requires that there exists $m > n = 1$ such that $f(m) > \delta$ which, by assumption, is ruled out. ■

It is possible to derive a formula for the function f in terms of h. For any utility vector $u \in \Omega$ with representative utility $\xi \neq 0$, if $g(c) = h(n)g(\xi)$, then

$$f(n)g(\xi) = f(n+1)\frac{ng(\xi) + h(n)g(\xi)}{n+1}. \tag{97}$$

Dividing both sides by $g(\xi)$ and rearranging,

$$f(n+1) = \left(\frac{n+1}{n+h(n)}\right)f(n) \tag{98}$$

for all $n \in \mathcal{Z}_{++}$. If $f(1)$ is normalized to one, then (98) implies that

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ \left(\frac{2}{1+h(1)}\right) \cdots \left(\frac{n}{n-1+h(n-1)}\right) & \text{if } n \geq 2. \end{cases} \tag{99}$$

Now consider the special case where the function h is a positive fraction with $h(n) = h_\gamma(n) = \gamma \in (0,1)$ for all $n \in \mathcal{Z}_{++}$. Writing f_γ as the function f corresponding to h_γ,

$$f_\gamma(n) = \begin{cases} 1 & \text{if } n = 1, \\ \left(\frac{2}{1+\gamma}\right) \cdots \left(\frac{n}{n-1+\gamma}\right) & \text{if } n \geq 2. \end{cases} \tag{100}$$

Theorem 16 shows that all members of the subfamily of the number-dampened generalized utilitarian family with $f = f_\gamma$ for some $\gamma \in (0,1)$ imply the repugnant conclusion. In the proof of this result, we use Abel's theorem (Brand [1955, p.52]) which states that if $\langle a_n \rangle_{n \in \mathcal{Z}_{++}}$ is a positive and decreasing sequence and $(a_1 + \ldots + a_n)$ converges, then $\lim_{n\to\infty} na_n = 0$.

Theorem 16: *If R is an NDGU social-evaluation ordering with $f = f_\gamma$ for some $\gamma \in (0,1)$, it implies the repugnant conclusion.*

Proof. Suppose that f_γ is bounded. Because f_γ is increasing in n, it is bounded if and only if it converges and, because $i/(i-1+\gamma) > 1$ for all $i \geq 2$, the limit is at least one. Write f_γ as $f_\gamma(1) = 1$ and, for $n \geq 2$,

$$f_\gamma(n) = \prod_{i=1}^{n-1} \frac{i+1}{i+\gamma}. \tag{101}$$

Then the product

$$\prod_{i=1}^{n} \frac{i+1}{i+\gamma} \tag{102}$$

converges to the same limit. Therefore, the logarithm

$$\sum_{i=1}^{n} \ln\left(\frac{i+1}{i+\gamma}\right) \tag{103}$$

converges as well. The individual terms $\ln\big((i+1)/(i+\gamma)\big)$ are positive and decreasing in i because $\gamma < 1$. By Abel's theorem,

$$\lim_{n\to\infty} n \ln\left(\frac{n+1}{n+\gamma}\right) = 0 \tag{104}$$

and, therefore,

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n+\gamma}\right)^n = 0. \tag{105}$$

Because $\ln\big((n+1)/(n+\gamma)\big)$ converges to zero,

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n+\gamma}\right)^n + \lim_{n\to\infty} \ln\left(\frac{n+1}{n+\gamma}\right) = \lim_{n\to\infty} \ln\left(\frac{n+1}{n+\gamma}\right)^{n+1} = 0, \tag{106}$$

which is equivalent to

$$\lim_{n\to\infty} \left(\frac{n+1}{n+\gamma}\right)^{n+1} = 1. \tag{107}$$

Inverting both sides,

$$\lim_{n\to\infty} \left(\frac{n+\gamma}{n+1}\right)^{n+1} = \lim_{n\to\infty} \left(1 + \frac{\gamma-1}{n+1}\right)^{n+1} = 1. \tag{108}$$

But

$$\lim_{n\to\infty} \left(1 + \frac{\gamma-1}{n+1}\right)^{n+1} = e^{\gamma-1}, \tag{109}$$

which is less than one for all $\gamma \in (0,1)$ and a contradiction is obtained. Consequently, f_γ is not bounded and, by Theorem 15, R implies the repugnant conclusion. ■

Finally, we discuss the properties of number-dampened generalized utilitarianism in choice problems. Again, suppose we choose the best utility vectors from a feasible set $\mathcal{U}$, this time according to NDGU. Let $\Psi(\gamma)$ be the set of chosen utility vectors for the parameter $\gamma \in (0,1)$ when $f = f_\gamma$.

Theorem 17: *Let $\gamma, \gamma' \in (0,1)$ be such that $\gamma' > \gamma$. Let $n, m \in \mathcal{Z}_{++}$, $u \in \mathcal{R}^n$, and $v \in \mathcal{R}^m$. If $u \in \Psi(\gamma)$ and $v \in \Psi(\gamma')$ and $\Psi(\gamma) \neq \Psi(\gamma')$, then $n > m$.*

Proof. Suppose $u \in \Psi(\gamma)$ and $v \in \Psi(\gamma')$. Because the chosen vectors are best elements according to number-dampened generalized utilitarianism where h is a fraction, this implies

$$\frac{f_\gamma(n)}{n}\sum_{i=1}^{n} g(u_i) \geq \frac{f_\gamma(m)}{m}\sum_{i=1}^{m} g(v_i) \tag{110}$$

and

$$\frac{f_{\gamma'}(m)}{m}\sum_{i=1}^{m} g(v_i) \geq \frac{f_{\gamma'}(n)}{n}\sum_{i=1}^{n} g(u_i). \tag{111}$$

This implies that $\sum_{i=1}^{n} g(u_i)$ and $\sum_{i=1}^{m} g(v_i)$ must have the same sign. Multiplying (110) and (111) and simplifying yields

$$f_\gamma(n) f_{\gamma'}(m) \geq f_\gamma(m) f_{\gamma'}(n), \tag{112}$$

which implies

$$\frac{f_{\gamma'}(m)}{f_{\gamma'}(n)} \geq \frac{f_\gamma(m)}{f_\gamma(n)}. \tag{113}$$

If $n = m$ it follows that $\Psi(\gamma) = \Psi(\gamma')$, contradicting our assumption. If $n < m$, (112) implies

$$\left(\frac{n+1}{n+\gamma'}\right)\cdots\left(\frac{m}{m-1+\gamma'}\right) \geq \left(\frac{n+1}{n+\gamma}\right)\cdots\left(\frac{m}{m-1+\gamma}\right), \tag{114}$$

which is a contradiction to $\gamma' > \gamma$. Therefore, we must have $n > m$. ■

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