

# No Profitable Decompositions in Quasi-Linear Allocation Problems\*

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(PRELIMINARY AND INCOMPLETE)

## Abstract

We study the problem of allocating a bundle of perfectly divisible private goods from an axiomatic point of view, in situations where compensations can be made through monetary transfers. The key property we impose on the allocation rule requires that no agent should be able to gain by decomposing the problem into sequences of subproblems. Combined with additional standard and rather weak properties, it leads to a characterization of the rule that shares the total surplus equally. Hence a welfarist rule emerges as the unique consequence of our axioms phrased in a natural economic environment.

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## 1. INTRODUCTION

We consider situations where a group of people have to share a bundle of perfectly divisible private goods. We assume in most of the paper that compensations can be

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achieved through monetary transfers (quasi-linear framework). As often, instead of solving each specific problem in isolation, we study allocation rules that may be applied in many different situations. For most allocation problems and most rules, some participants can gain by decomposing the stakes in some way, requesting for instance to allocate good  $l$  before  $l'$ , or to share a proportion of the total amount of goods available before allocating what remains. Of course, such decompositions often lead to an efficiency loss, and even if not the case, a gain for one participant must result in a loss for another one when the allocation rule selects efficient outcomes. The purpose of our main axiom - No Profitable Decompositions (NPD) - is to suppress the possibility of such profitable manipulations of the agenda, a potential source of conflict within the society. Allocation rules will thus be required to be such that no individual can gain by decomposing the problem into sequences of subproblems.

The main result of the paper establishes that NPD, once combined with other standard and rather weak axioms, characterizes the allocation rule that corresponds to an equal split of the maximal total surplus among the participants. Equal surplus sharing being probably the simplest notion of microeconomic justice, one would think that there exist numerous axiomatic characterizations of this solution in bargaining and social choice theory. In reality there are only few such results. The reason is that most contributions in axiomatic bargaining and social choice are phrased in utilities. Equal surplus sharing follows trivially from the properties of anonymity and efficiency in quasi-linear environments under this welfarist assumption. Most of the literature focuses instead on finding extensions of the equal surplus sharing solution to environments that are not quasi-linear. Unfortunately, the welfarist assumption lacks a clear normative and/or positive content, and is thus hard to accept as an axiom or postulate (see Roemer (1986, 1988)). The existence of appealing contextual solutions (e.g. fairness or egalitarian equivalence) also shows that the welfarist assumption is far from being innocuous. To be precise, we are not arguing that a solution is unappealing because it is welfarist. Instead, we suggest that the axiomatic approach should be applied more systematically to explicit economic and social environments. Some properties that were incompatible in the utility space may lead to the characterization of new (necessarily contextual) solutions. In other cases, welfarism will come as a consequence of axioms, hence giving us a deeper understanding of traditional solutions. Our main result belongs to this second category. It is worth noting that NPD cannot even be phrased under the welfarist assumption, since the set of utilities that are feasible in the second step of a decomposition depends on the economic

description of the problem. This set may be strictly smaller than, and unrelated to, the set of utilities that are achievable when solving the problem in its entirety.

Beyond usual properties of anonymity, efficiency, and continuity, the result will require an axiom of independence with respect to preferences over non-feasible allocations (IND). As hinted by its name, IND requires that the solution of two allocation problems that differ only in the participants' preferences over outcomes that are not feasible coincide. As far as we can tell, this type of property was first mentioned explicitly in Karni and Schmeidler (1975).<sup>1</sup> It has been rather often invoked since then (see e.g. Plott, 1976; Grether and Plott, 1982; Campbell, 1992; Dutta et al., 2001; Ehlers and Weymark, 2003; Fleurbaey, 2003; Chambers, 2005; Fleurbaey and Tadenuma, 2007; de Clippel, 2008).<sup>2</sup> Though IND may appear completely innocuous at first sight, we observe that it rules out Pazner and Schmeidler's (1978) egalitarian equivalent solution.

We can now provide some intuition for our main characterization result. Consider various countries that have an equal claim over a newly-discovered field of natural gas. A total quantity  $Q$  is available to share. Let  $v_i$  be the function that measures the net social surplus for country  $i$ , as a function of the share it receives.<sup>3</sup> These functions are most likely to vary across countries because of different transportation costs and different needs (e.g. existence of alternative sources, and use of different technologies that make the resource more or less productive). NPD is more restrictive when it applies to many decompositions of the original problem. Consider for instance the case where the division of  $Q$  is tested against the iteration cubic meter by cubic meter of the solution. Suppose that  $Q' < Q$  cubic meters have already been shared (combined with some monetary transfers). Given the possibility of monetary compensations, the efficient allocation prescribed by the solution must equalize the marginal social surplus across countries (assuming for simplicity that we have an interior solution). When considering the additional cubic meter to be shared in the next iteration of the decomposition, all the countries look identical, because a cubic meter is essentially an infinitesimal quantity when compared

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<sup>1</sup>Karni and Schmeidler themselves refer to a 1969 *mimeo* written by A. Gibbard.

<sup>2</sup>The list of references aims at illustrating various formulations of the same idea, and various contexts where the property has been applied, but it is far from being exhaustive.

<sup>3</sup>The story is of course rather stylized, the objective being to emphasize the argument behind the main result of the paper. For instance, we overlook the costs of extraction. As a first approximation, we may express them in terms of the energy required to extract the gas, which itself can be obtained from a fraction of the natural gas extracted.  $Q$  can then be interpreted as the net quantity available in the field. Also, our story does not incorporate time explicitly, but the functions  $v_i$  can be reinterpreted as the net present value of streams of resources to be extracted.

to  $Q$ , and the countries' preferences over quantities that are larger than this infinitesimal amount are treated as irrelevant, by IND. In order to be anonymous (a minimal requirement for equitability), the solution should give an equal share to each country of the additional total surplus generated by the additional cubic meter to allocate. Iterating the process, it follows that the total surplus associated to  $Q$  should be shared equally across countries. The formal reasoning is more general (e.g. multiple goods, and without restricting attention to functions  $v_i$  that guarantee interior solutions), but also requires to focus on solutions that are regular (formalized in an axiom of continuity) in order to make the argument at the margin complete.

The paper unfolds as follows. Section 2 presents the model. The axioms and the main result are included in Section 3, while its proof is postponed to Section 5. Section 4 offers a review of the related literature.

## 2. MODEL

A set  $\mathcal{I}$  of  $I \geq 2$  individuals have to allocate a bundle  $\omega$  of  $L$  perfectly divisible goods ( $\omega \in \mathbb{R}_+^L$ ). Some compensation can be achieved through monetary transfers. An *allocation* is a couple  $(x, t) \in \mathbb{R}_+^{LL} \times \mathbb{R}^I$  where, for each  $i \in \mathcal{I}$ ,  $t_i$  (resp.  $x_i$ ) represents the net amount of money (resp. bundle of goods) that individual  $i$  receives. It is *feasible* if  $\sum_{i \in \mathcal{I}} x_i \leq \omega$  and  $\sum_{i \in \mathcal{I}} t_i \leq 0$ . The set of feasible allocations will be denoted by  $\mathcal{F}(\omega)$ .

Utilities are quasi-linear. The *utility function*  $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}_+$  determines the maximal (or reservation) price  $u_i(x)$  that individual  $i$  is ready to pay to consume each bundle  $x \in \mathbb{R}_+^L$ . The utility functions are assumed to be non-decreasing, continuous and such that  $u(0) = 0$ .<sup>4</sup> The set of all such functions is denoted by  $\mathcal{U}$ . Occasionally, we will also use the *concave domain*  $\mathcal{C}$ , namely the subset of  $\mathcal{U}$  consisting of concave functions. Agent  $i$ 's utility associated to the allocation  $(x, t)$  is  $u_i(x_i) + t_i$ . An *utility profile* is a vector  $\mathbf{u}$  in  $\mathbb{R}^I$ . It is *feasible* if there exists a feasible allocation  $(x, t)$  such that  $\mathbf{u}_i = u_i(x_i) + t_i$ , for each  $i \in I$ .

An *allocation problem*  $P$  is a couple  $(\omega, u)$ , where  $\omega$  is the bundle of  $L$  goods to share, and  $u = (u_i)_{i \in \mathcal{I}} \in \mathcal{U}^I$  is the list of utility functions. The set of all allocation problems is denoted by  $\mathcal{P}$ . The set of concave allocation problems is denoted by  $\mathcal{P}_{\mathcal{C}}$ .

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<sup>4</sup>It is natural to assume that an individual's reservation price for consuming nothing is zero. Dropping this assumption would not change the substance of our argument (see discussion in Section ...).

An *allocation rule* is a correspondence  $\mathcal{R} : \mathcal{P} \rightrightarrows \mathbb{R}_+^{IL} \times \mathbb{R}^I$ , which associates to each allocation problem a nonempty set of feasible allocations. We will assume throughout the paper that the allocation rules are “exhaustive,” meaning that

$$\{(x, t) \in \mathcal{R}(P) \text{ and } u_i(x'_i) + t'_i = u_i(x_i) + t_i, \forall i\} \Rightarrow \{(x', t') \in \mathcal{R}(P)\}, \quad (1)$$

for each  $P \in \mathcal{P}$ , and each pair  $((x, t), (x', t'))$  of feasible allocations. That is, if  $(x, t)$  is an allocation that solves the problem  $P$ , and if  $(x', t')$  is a feasible allocation that generates the same utility profile, then  $(x', t')$  solves  $P$  as well. We also assume that the allocation rules determine a single utility profile:

$$\{(x, t) \in \mathcal{R}(P) \text{ and } (x', t') \in \mathcal{R}(P)\} \Rightarrow \{u_i(x_i) + t_i = u_i(x'_i) + t'_i, \forall i\}, \quad (2)$$

for each  $P \in \mathcal{P}$ , and each pair  $((x, t), (x', t'))$  of allocations.

A *solution* is a function  $\sigma : \mathcal{P} \rightarrow \mathbb{R}^I$  that associates a utility profile to each allocation problem. There exists a natural bijection between the set of solutions and the set of allocation rules that satisfy (1) and (2). Let  $P = (\omega, u) \in \mathcal{P}$ . The solution associated to an allocation rule  $\mathcal{R}$  is given by:

$$\sigma_i^{\mathcal{R}}(P) = u_i(x_i) + t_i, \forall i,$$

for some (or each by (2))  $(x, t) \in \mathcal{R}(P)$ . The allocation rule associated to a solution  $\sigma$  is given by:

$$\mathcal{R}\sigma(P) = \{(x, t) \in \mathcal{F}(\omega) \mid u_i(x_i) + t_i = \sigma_i(P), \forall i \in \mathcal{I}\}.$$

For each allocation problem  $P = (\omega, u)$ ,

$$s(P) = \max_{x \in \mathbb{R}_+^{IL}} \left\{ \sum_{i \in \mathcal{I}} u_i(x_i) \mid \sum_{i \in \mathcal{I}} x_i \leq \omega \right\}.$$

denotes the maximal total surplus achievable. The *equal surplus sharing solution*<sup>5</sup>  $\sigma^{ESS}$

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<sup>5</sup>One could argue that  $\sigma^{ESS}$  is actually the egalitarian solution. We refrain from using this terminology for the moment, because it also coincides with many other solutions such as the Nash or the Kalai-Smorodinsky solutions applied to the bargaining problem  $(U(P), d(P))$ , where  $d(P) = 0$  and  $U(P) = \{u \in \mathbb{R}^I \mid \sum_{i \in \mathcal{I}} u_i \leq s(P)\}$ , for each  $P \in \mathcal{P}$ . The problems being quasi-linear,  $\sigma^{ESS}$  actually coincides with any solution that is welfarist, and satisfies EFF and ETE (cf. definitions below in the main text).

is then given by:

$$\sigma_i^{ESS}(P) = \frac{s(P)}{I},$$

for each  $i \in \mathcal{I}$ , and each  $P = (\omega, u) \in \mathcal{P}$ . The associated allocation rule is denoted  $\mathcal{R}^{ESS}$ .

### 3. MAIN RESULT

Here are the axioms that we will impose on the allocation rule.

**No Profitable Decompositions (NPD)** Let  $P = (\omega, u) \in \mathcal{P}$ , let  $\tilde{\omega} \in \mathbb{R}_+^L$  be such that  $\tilde{\omega} \geq \omega$ , and let  $\tilde{P} = (\tilde{\omega}, u)$ . Then

$$u_i(\tilde{x}_i) + \tilde{t}_i \geq u_i(x_i + y_i) + t_i + r_i, \quad (3)$$

for each  $i \in \mathcal{I}$ , each  $(x, t) \in \mathcal{R}(P)$ , each  $(\tilde{x}, \tilde{t}) \in \mathcal{R}(\tilde{P})$ , and each  $(y, r) \in \mathcal{R}(P_x)$ , where  $P_x = (\tilde{\omega} - \omega, u^x)$  and  $u_i^x(y_i) = u_i(x_i + y_i) - u_i(x_i)$ , for each  $y_i \in \mathbb{R}_+^L$  and each  $i \in \mathcal{I}$ .

**Efficiency (EFF)**  $\sum_{i \in \mathcal{I}} \sigma_i^{\mathcal{R}}(P) = s(P)$ , for each  $P \in \mathcal{P}$ .

**Equal Treatment of Equals (ETE)**  $\sigma_i^{\mathcal{R}}(P) = \sigma_j^{\mathcal{R}}(P)$ , for each  $P = (\omega, u) \in \mathcal{P}$ , and each  $i, j$  in  $\mathcal{I}$  such that  $u_i = u_j$ .

**Independence of Preferences over Non-Feasible Allocations (IND)** Let  $P = (\omega, u) \in \mathcal{P}$  and  $\tilde{P} = (\omega, \tilde{u}) \in \mathcal{P}$  be such that  $u_i(x) = \tilde{u}_i(x)$ , for each  $i \in \mathcal{I}$  and each  $x \in \mathbb{R}_+^L$  with  $x \leq \omega$ . Then  $\sigma_i^{\mathcal{R}}(P) = \sigma_i^{\mathcal{R}}(\tilde{P})$ .

**Continuity (CONT)** a) Let  $\omega \in \mathbb{R}_+^L$  and let  $(\omega_k)_{k \in \mathbb{N}}$  be a sequence in  $\mathbb{R}_+^L$  that converges to  $\omega$ . Then the sequence  $(\sigma^{\mathcal{R}}(\omega_k, u))_{k \in \mathbb{N}}$  converges to  $\sigma^{\mathcal{R}}(\omega, u)$ , for each  $u \in \mathcal{U}^I$ .

b) For every compact set  $K \subseteq \mathbb{R}_+^L$ , there exist  $M > 0$  such that<sup>6</sup>

$$\|\sigma^{\mathcal{R}}(\omega, u) - \sigma^{\mathcal{R}}(\omega, \tilde{u})\| \leq Md(u, \tilde{u}),$$

for every  $\omega \in K$  and  $u, \tilde{u} \in \mathcal{U}^I$ .

As explained in the introduction, NPD guarantees that no participant can have an interest in manipulating the allocation rule through some decomposition of the stakes. EFF

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<sup>6</sup> $d(u, \tilde{u}) = \max_{i \in \mathcal{I}} \sup_{x \in \mathbb{R}_+^L} |u_i(x) - \tilde{u}_i(x)|$ . Note that since functions in  $\mathcal{U}_0$  need not be bounded on  $\mathbb{R}_+^L$ ,  $d(\cdot, \cdot)$  is an *extended* metric.

simply imposes on the rule to specify allocations that are Pareto efficient. It should not be possible to find another feasible allocation that would make all the individuals happier. ETE guarantees some form of equity, in that two individuals with the same utility functions are treated identically. An allocation rule must specify feasible allocations, and hence no individual can ever receive more than the amounts that are available for division. It is natural to assume that the individuals' reservation prices for bundles that are not feasible should be irrelevant in the determination of the final allocation, as required by IND (see references in the introduction). It is also natural to require some form of continuity on the allocation rule. CONT formalizes the idea that small measurement mistakes should not trigger a major difference when computing the solution. Part (a) applies this principle to the total resources available, while part (b) requires the stronger property of Lipschitz continuity in utilities.

**Theorem**  $\mathcal{R}^{ESS}$  is the only rule that satisfies NPD, EFF, ETE, IND and CONT on  $\mathcal{P}$  (resp.  $\mathcal{P}_C$ ).

We already gave some intuition for this theorem in the introduction, and we defer the complete proof to Section 5.

(To be added: examples to show the independence of the axioms appearing in the Theorem)

#### 4. RELATED LITERATURE

While studying collusive behavior in auctions, Graham and Marshall (1987) (see also Graham et al. (1990)) derived an interesting rule to allocate a single unit of an indivisible object, and design monetary compensations. Its definition is reminiscent of the principle of serial cost sharing to allocate, and it happens to coincide with the Shapley value of some characteristic function derived from the problem (see also Moulin (1992)). There seems to be a natural procedure to adapt the solution to the allocation of some quantity  $Q$  of a divisible good: decompose the problem into the iteration of allocation problems with infinitesimal quantities; treat these infinitesimal quantities as indivisible, and solve each of these problems via the previous solution; integrate to obtain a solution for the divisible case. One may wonder what is the resulting solution. Of course, the procedure works only for problems with decreasing marginal utilities, as otherwise the resulting solution is not necessarily efficient. Suppose also that the utility functions are regular, that is

differentiable and such that the efficient allocation of any positive  $Q$  gives a positive amount of the good to each participant (interior solution). In such cases, the resulting solution actually coincides with equal surplus sharing. This is true not only when starting with Graham and Marshall’s solution, but also with any solution that guarantees to each agent a payoff that is larger than or equal to his valuation for the indivisible good to be allocated divided by the number of participants, a rather weak axiom introduced by Moulin and Thomson (1988) (and that plays a central role in Moulin (1992) as well).<sup>7</sup> It is interesting to note that Graham and Marshall’s solution, as well as many of the rules satisfying this property, are not welfarist. Yet, once iterated to obtain a solution for the divisible case, they all result in the same welfarist solution.

NPD is somewhat reminiscent of Kalai’s axiom of *step-by-step negotiation*. In fact, the two axioms are rather different. Indeed, NPD cannot even be phrased in Kalai’s welfarist framework, because the set of utility profiles that are feasible when sharing the bundle  $\omega - \omega'$  after having solved for  $\omega' < \omega$  may be strictly smaller than the set of utility profiles that are feasible when sharing the bundle  $\omega$ . Kalai assumes instead that the solution for the problem of dividing  $\omega'$  is a partial agreement that serves as a disagreement point in a new bargaining problem where any division of the bundle  $\omega$  can be agreed upon. NPD, on the contrary, assumes that any partial agreement is final and non-renegotiable. If one replaces NPD with Kalai’s axiom rephrased in our economic environments (with an additional exogenous variable to represent partial agreements), one does not obtain anymore a characterization of the equal surplus sharing solution. In addition, our problems being quasi-linear, equal sharing of the surplus follows immediately from the properties of efficiency and anonymity if one works in Kalai’s welfarist framework. The purpose of Kalai’s argument instead is to characterize proportional solutions in a welfarist framework with non-transferable utility.

O’Neill et al. (2004) introduce a new welfarist model of bargaining, where the set of feasible utility profiles expands over time according to a differentiable function. Our argument and theirs are thus similar, in that both solutions are determined by integrating its local behavior which can be determined by rather weak properties. A first difference

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<sup>7</sup>We omit the proof, because it is very similar to step 1 in Section 5. The idea is that the lower bound on the participant’s final utility is binding, and equal to their common marginal utility (which is also equal to the derivative of the total surplus, by the envelope theorem) at every step of the continuous summation. Notice that requiring the efficient allocations to be interior is important. If the first participant’s utility function equals the quantity he consumes, while the second participant’s utility function equals twice the quantity he consumes, then the solution obtained by iterating Graham and Marshall’s solution for the single-unit case does not coincide with an equal split of the total surplus.

is that there is no given bargaining agenda in our model. The integration step follows from the NPD property instead. More importantly, the arguments bear on different objects in our two papers. Working in the space of utilities, equal surplus sharing is not derived in O' Neill et al., but instead assumed by their symmetry property. The key ingredient in their result is that the efficient frontier of the expanded set of feasible utilities at time  $t + \Delta t$  that lies above the agreement reached at time  $t$  is essentially linear when  $\Delta t$  is infinitesimal. Scale covariance then leads to a problem in the space of utilities that can be solved by direct application of the symmetry axiom. The key ingredient in our result is that the participants' preferences are essentially identical when an infinitesimal quantity  $\Delta\omega$  has to be divided after a strictly positive quantity  $\omega$  has already been distributed (assuming that we have an interior solution). Notice how the set of feasible utilities at time  $t$  does not depend on previous agreements in O'Neill et al.'s model. Rephrased in an economic environment like ours, this implies that the whole quantity of all the goods that have been bargained in the past must be renegotiated at every  $t$ , as in Kalai's interpretation of the property of step-by-step decomposition. In our case, to integrate the solution of local problems that follow a path from 0 to  $\omega$  often leads to an inefficient solution because past agreements are assumed to be non-renegotiable (except when  $L = 1$ , as in the first paragraph of the present section).

NPD is related to the CONRAD property that Roemer (1988) introduced to recover most classical results in bargaining theory with axioms phrased in economic environments. Though weaker, the CONRAD property is far more cumbersome than NPD, because it restricts in a rather ad-hoc way the set of decompositions over which it applies (adding goods in which at most one agent is interested, provided the set of feasible utility profiles remains the same). If somebody likes Roemer's idea of consistency in CONRAD, then he would probably prefer to go all the way to NPD. Notice that Roemer's proof cannot be adapted to our framework because he makes crucial use of preferences that are not quasi-linear.

(Add comparison with Moulin (1992).)

(Add comparison with the linear bargaining solutions of Chambers and Green (2005).)

(Add comparison with the axiom 'Composition Up' in bankruptcy problems - see Thomson (2003).)

(Add comparison with the results of Moulin (1985), Chen and Maskin (1999), Gines and Marhuenda (2000), Yoshihara (2003), and other papers that derive welfare egalitarianism on economic environments.)

(Add comparison with the (welfarist) literature on bargaining over multiple issues, see e.g. Fershtman (1990), John and Raith (1997), and Watson and Ponsati (1997).)

## 5. PROOF OF THE THEOREM

It is clear that  $\mathcal{R}^{ESS}$  satisfies EFF, ETE and IND. Part (a) of CONT is an immediate consequence of Berge's maximum principle. As for part (b), let  $x \in \mathbb{R}_+^{IL}$  be such that  $\sum_{i \in \mathcal{I}} x_i \leq \omega$  and  $\sum_{i \in \mathcal{I}} u_i(x_i) = s(\omega, u)$ . Then  $\sum_{i \in \mathcal{I}} \tilde{u}_i(x_i) \leq s(\omega, \tilde{u})$ , and hence  $s(\omega, u) - s(\omega, \tilde{u}) \leq Id(u, \tilde{u})$ . A similar argument also implies that  $s(\omega, \tilde{u}) - s(\omega, u) \leq Id(u, \tilde{u})$ . Hence  $\|\sigma_i^{ESS}(\omega, u) - \sigma_i^{ESS}(\omega, \tilde{u})\| \leq \sqrt{I} Id(u, \tilde{u})$ , for every  $u, \tilde{u} \in \mathcal{U}^I$  (independently of  $\omega$ ), and thus  $\mathcal{R}^{ESS}$  satisfies part (b) of CONT. Finally, to check that it satisfies NPD, it is enough to observe that

$$s(\omega', u) \geq \max \left\{ \sum_{i \in \mathcal{I}} u_i(x_i^* + y_i) \mid y \in \mathbb{R}_+^{IL} \text{ and } \sum_{i \in \mathcal{I}} y_i \leq \omega' - \omega \right\},$$

for any  $x^*$  that maximizes  $\sum_{i \in \mathcal{I}} u_i(x_i)$  over the set of vectors  $x \in \mathbb{R}_+^{IL}$  such that  $\sum_{i \in \mathcal{I}} x_i \leq \omega$ .

Let now  $\mathcal{R}$  be a rule that satisfies the five axioms on  $\mathcal{P}$  (resp.  $\mathcal{P}_C$ ). Let  $\mathcal{V}$  be defined as

$$\mathcal{V} = \left\{ u \in \mathcal{C}^1(\mathbb{R}_{++}^L) \mid \forall d \in \partial \mathbb{R}_+^L, \exists 1 \leq l \leq L \text{ such that } \lim_{x \rightarrow d} \frac{\partial u}{\partial x_l}(x) = +\infty \right\},$$

where  $\mathcal{C}^1(\mathbb{R}_{++}^L)$  denotes the set of continuously differentiable functions on  $\mathbb{R}_{++}^L$ .

The proof will proceed in four steps.

**Step 1:** *Let  $u \in (\mathcal{V} \cap \mathcal{U})^I$  (resp.  $u \in (\mathcal{V} \cap \mathcal{C})^I$ ). Then  $\sigma_i^{\mathcal{R}}(\cdot, u)$  admits a right directional derivative along any vector  $d \in \mathbb{R}_+^L \setminus \{0\}$ , at any point  $\omega \in \mathbb{R}_{++}^L$ . In addition, this derivative is equal to  $\frac{1}{I} \nabla \omega s(\cdot, u) \cdot d$ , i.e., for every  $\omega \in \mathbb{R}_{++}^L$  and  $d \in \mathbb{R}_+^L \setminus \{0\}$*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} = \frac{1}{I} \nabla \omega s(\omega, u) \cdot d.$$

Let  $\omega, d$  as above,  $\varepsilon \in (0, 1]$  and  $(x, t) \in \mathcal{R}(\omega, u)$ . NPD implies that

$$\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u) \geq \sigma_i^{\mathcal{R}}(\varepsilon d, u_i^x). \quad (4)$$

For each  $i \in \mathcal{I}$ , let

$$\alpha_i(\varepsilon) = \max_{0 \leq y \leq \varepsilon d} |\nabla u_i(x_i) \cdot y - u_i^x(y)|,$$

and let  $u_i^\varepsilon \in \mathcal{U}$  be the utility function defined as follows:

$$u_i^\varepsilon(y) = \begin{cases} \nabla u_i(x_i) \cdot y & \text{if } |\nabla u_i(x_i) \cdot y - u_i^x(y)| \leq \alpha_i(\varepsilon) \\ u_i^x(y) + \alpha_i(\varepsilon) & \text{if } \nabla u_i(x_i) \cdot y - u_i^x(y) > \alpha_i(\varepsilon) \\ u_i^x(y) - \alpha_i(\varepsilon) & \text{if } u_i^x(y) - \nabla u_i(x_i) \cdot y > \alpha_i(\varepsilon) \end{cases}$$

for each  $y \in \mathbb{R}_+^L$ .

Clearly, if  $u \in (\mathcal{V} \cap \mathcal{C})^I$  then  $\alpha_i(\varepsilon) = \nabla u_i(x_i) \cdot \varepsilon d - u_i^x(\varepsilon d)$  and  $u_i^\varepsilon(y) = \min\{\nabla u_i(x_i) \cdot y, u_i(x_i + y) + \nabla u_i(x_i) \cdot \varepsilon d - u_i(x_i + \varepsilon d)\}$ .

Let  $K$  be the compact set  $K = \{y \in \mathbb{R}_+^L | y \leq d\}$ . Part (b) of CONT implies that there exists  $M > 0$  such that

$$\sigma_i(\varepsilon d, u_i^x) \geq \sigma_i(\varepsilon d, u_i^\varepsilon) - Md(u_i^x, u_i^\varepsilon) \quad (5)$$

for each  $i \in \mathcal{I}$  and each  $\varepsilon \in (0, 1]$ . Since  $u \in \mathcal{V}^I$  and  $x$  is an efficient split of  $\omega$ , it must be interior, and thus  $\nabla u_i(x_i) = \nabla u_j(x_j)$ , for every  $i \neq j$ . Then IND, ETE, and EFF imply  $\sigma_i(\varepsilon d, u_i^\varepsilon) = \frac{\varepsilon}{I} \nabla u_i(x_i) \cdot d$ . This in turn equals  $\frac{\varepsilon}{I} \nabla_\omega s(\omega, u) \cdot d$ , by the envelope theorem. Observe also that the uniform distance between  $u_i^x$  and  $u_i^\varepsilon$  is equal to  $\alpha(\varepsilon)$ . Hence (4) and (5) imply that

$$\frac{\sigma_i^{\mathcal{R}}(\omega + \varepsilon d, u) - \sigma_i^{\mathcal{R}}(\omega, u)}{\varepsilon} \geq \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d - M \frac{\alpha(\varepsilon)}{\varepsilon}. \quad (6)$$

We are now ready to prove by contradiction that the ratio on the left-hand side of (6) converges to  $\frac{1}{I} \nabla_\omega s(\omega, u) \cdot d$  when  $\varepsilon$  converges to 0, for each  $i \in \mathcal{I}$ . For simplicity, let's refer to this ratio as  $r_i(\varepsilon)$ . If the property is not true, then we can find  $j \in \mathcal{I}$ ,  $\beta > 0$ , and a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  of strictly positive numbers that converges to 0 such that  $|r_j(\varepsilon_k) - \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d| \geq \beta$  for all  $k$ . Taylor's theorem implies that  $M \frac{\alpha(\varepsilon_k)}{\varepsilon_k}$  converges to 0 when  $k$  goes to infinity, and hence there exists  $k_0 \in \mathbb{N}$  such that  $M \frac{\alpha(\varepsilon_k)}{\varepsilon_k} < \beta$ , for all  $k \geq k_0$ . Hence, for all those  $k$ 's, we must have  $r_j(\varepsilon_k) - \frac{1}{I} \nabla_\omega s(\omega, u) \cdot d \geq \beta$ . Combining this with (6) for  $i \in \mathcal{I} \setminus \{j\}$ , we obtain:

$$\frac{s(\omega + \varepsilon_k d, u) - s(\omega, u)}{\varepsilon_k} = \sum_{i \in \mathcal{I}} r_i(\varepsilon_k) \geq \nabla_\omega s(\omega, u) \cdot d + \beta + M(I - 1) \frac{\alpha(\varepsilon_k)}{\varepsilon_k},$$

for all  $k \geq k_0$ . Taking the limit when  $k$  tends to infinity, we get a contradiction:  $\nabla_\omega s(\omega, u) \cdot d \geq \nabla_\omega s(\omega, u) \cdot d + \beta$ .

**Step 2.** Let  $\omega \in \mathbb{R}_{++}^L$  and  $u \in (\mathcal{V} \cap \mathcal{U})^I$  (resp.  $u \in (\mathcal{V} \cap \mathcal{C})^I$ ). Then  $\sigma_i^{\mathcal{R}}(\omega, u) = \sigma_i^{ESS}(\omega, u)$ , for all  $i \in \mathcal{I}$ .

Fix  $i \in \mathcal{I}$ , and define the function  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(t) = \sigma_i^{\mathcal{R}}(t\omega, u) - \frac{1}{7}s(t\omega, u)$ . Part (a) of CONT implies that  $f$  is continuous and, according to step 1,  $f$  also has a right derivative with  $f'_+(t) = 0$  for all  $t \in ]0, 1[$ . Then  $f$  must be a constant function (for a proof see for example Knight, 1980) and thus,  $\sigma_i^{\mathcal{R}}(\omega, u) - \sigma_i^{\mathcal{R}}(0, u) = \frac{1}{7}s(\omega, u)$ . IND implies that  $\sigma_i^{\mathcal{R}}(0, u) = \sigma_i^{\mathcal{R}}(0, v)$  for any utility profile  $v$ . In particular, one can take a utility profile in which all agents are identical. Then ETE together with  $s(0, u) = 0$  implies that  $\sigma_i^{\mathcal{R}}(0, u) = 0$  and thus  $\sigma_i^{\mathcal{R}}(\omega, u) = \frac{1}{7}s(\omega, u)$ .

**Step 3.** Let  $\omega \in \mathbb{R}_{++}^L$  and  $\bar{X}(\omega) = \{x \in \mathbb{R}_+^L | x \leq \omega\}$ . Every  $u \in \mathcal{U}$  (resp.  $u \in \mathcal{C}$ ) can be uniformly approximated on  $\bar{X}(\omega)$  by a sequence  $(u^n)_{n \in \mathbb{N}}$  of functions in  $\mathcal{V} \cap \mathcal{U}$  (resp.  $\mathcal{V} \cap \mathcal{C}$ ).

Let  $u \in \mathcal{U}$ . According to Stone-Weierstrass theorem, there exists a sequence of polynomials  $(Q^n)_{n \in \mathbb{N}}$  that converges uniformly to  $u$  on  $\bar{X}(\omega)$ . Moreover, it can be proved that the polynomials can be chosen to be monotonic (see Proposition 1 in the Appendix for a proof). For each  $n \in \mathbb{N}$ , define the function  $v^n : \mathbb{R}_+^L \rightarrow \mathbb{R}$  by

$$u^n(x) = (1 - \frac{1}{n})(Q^n(x) - Q^n(0)) + \frac{1}{n}(e^{\sum_{i=1}^L \sqrt{x_i}} - 1),$$

for each  $x \in \mathbb{R}_+^L$ . Then  $u^n \in \mathcal{V} \cap \mathcal{U}$  and  $u^n$  is increasing in each argument if  $Q^n$  is.

Moreover,

$$\sup_{x \in \bar{X}(\omega)} |u(x) - u^n(x)| \leq (1 - \frac{1}{n})|Q^n(0)| + \sup_{x \in \bar{X}(\omega)} |u(x) - Q^n(x)| + \frac{1}{n} \sup_{x \in \bar{X}(\omega)} |Q^n(x) + e^{\sum_{i=1}^L \sqrt{x_i}} - 1|.$$

Each of the three terms on the right-hand side converges to 0 when  $n$  tends to infinity. Indeed,  $\lim_{n \rightarrow \infty} Q^n(0) = u(0) = 0$ , and the sequence  $(Q^n)_n$  is uniformly bounded on  $\bar{X}(\omega)$ , since it is uniformly convergent. Hence  $u^n$  is uniformly convergent to  $u$ .

The proof for  $u \in \mathcal{C}$  is delegated to the Appendix (Proposition 2).

**Step 4.**  $\mathcal{R} = \mathcal{R}^{ESS}$ .

Let  $\omega \in \mathbb{R}_{++}^L$  and  $u \in \mathcal{U}^I$  (resp.  $u \in \mathcal{C}^I$ ). For each  $i \in \mathcal{I}$ , construct a sequence  $(u_i^n)_{n \in \mathbb{N}}$  of functions in  $\mathcal{V} \cap \mathcal{U}$  (resp.  $\mathcal{V} \cap \mathcal{C}$ ) that converges uniformly to  $u_i$  on  $\bar{X}(\omega)$ , and let  $u^n = (u_1^n, \dots, u_I^n)$ . We have:

$$\sigma_i^{\mathcal{R}}(\omega, u) = \lim_{n \rightarrow \infty} \sigma_i^{\mathcal{R}}(\omega, u^n) = \lim_{n \rightarrow \infty} \frac{s(\omega, u^n)}{I} = \frac{s(\omega, u)}{I},$$

for each  $i \in \mathcal{I}$ , where the first equality follows from part (b) of CONT, the second equality follows from step 2, and the third equality follows from the fact that  $\mathcal{R}^{ESS}$  satisfies part (b) of CONT.

Let now  $\omega \in \mathbb{R}_{++}^L$  and  $u \in \mathcal{U}^I$  (resp.  $u \in \mathcal{C}^I$ ). We can construct a sequence  $(\omega^n)_{n \in \mathbb{N}}$  in  $\mathbb{R}_{++}^L$  that converges to  $\omega$ . We have:

$$\sigma_i^{\mathcal{R}}(\omega, u) = \lim_{n \rightarrow \infty} \sigma_i^{\mathcal{R}}(\omega^n, u) = \lim_{n \rightarrow \infty} \frac{s(\omega^n, u)}{I} = \frac{s(\omega, u)}{I},$$

for each  $i \in \mathcal{I}$ , where the first equality follows from part (a) of CONT, the second equality follows from the previous paragraph, and the third equality follows from the fact that  $\mathcal{R}^{ESS}$  satisfies part (a) of CONT. ■

## APPENDIX

**Proposition 1.** *Let  $f : K \rightarrow \mathbb{R}$  be a continuous and increasing function defined on a compact cube  $K \subset \mathbb{R}^L$ . There exists a sequence of increasing polynomials that converges uniformly to  $f$  on  $K$ .*

*Proof.* This proof is a generalization to multi-variable functions of the well-known convergence of Bernstein polynomials to a single-variable continuous function (see Lorentz (1953), for example). Note first that using a linear (and increasing) change of variable, the problem can be reduced, without loss of generality, to functions defined on  $K = [0, 1]^L$ . For such a function and natural numbers  $r_1, \dots, r_L$  define the generalized Bernstein polynomial:

$$B_{r_1, \dots, r_L}(x; f) := \sum_{k_1=0}^{r_1} \sum_{k_2=0}^{r_2} \dots \sum_{k_L=0}^{r_L} f\left(\frac{k_1}{r_1}, \dots, \frac{k_L}{r_L}\right) \prod_{l=1}^L \binom{r_l}{k_l} (x_l)^{k_l} (1 - x_l)^{r_l - k_l}.$$

Polynomials  $B_{r_1, \dots, r_L}(x; f)$  are increasing in every variable if  $f$  is, because

$$\frac{\partial B_{r_1, \dots, r_L}(x; f)}{\partial x_l} = r_l \sum_{k_1=0}^{r_1} \dots \sum_{k_{l-1}=0}^{r_{l-1}} \dots \sum_{k_L=0}^{r_L} \left( f\left(\frac{k_1}{r_1}, \dots, \frac{k_{l-1}+1}{r_{l-1}}, \dots, \frac{k_L}{r_L}\right) - f\left(\frac{k_1}{r_1}, \dots, \frac{k_l}{r_l}, \dots, \frac{k_L}{r_L}\right) \right) \cdot \binom{r_l-1}{k_l} (x_l)^{k_l} (1-x_l)^{r_l-k_l-1} \cdot \prod_{s \neq l} \binom{r_s}{k_s} (x_s)^{k_s} (1-x_s)^{r_s-k_s}.$$

Let  $\varepsilon > 0$  arbitrary. Since  $f$  is continuous on a compact set, it is uniformly continuous and thus  $\exists \delta_\varepsilon > 0$  such that  $\|x - \xi\| < \delta_\varepsilon$  implies  $|f(x) - f(\xi)| < \frac{\varepsilon}{2}$ . On the other hand,  $f$  is bounded and thus  $\exists M > 0$  such that  $\|f(x)\| \leq M$  for all  $x \in [0, 1]^L$ . Then, for  $x, \xi \in [0, 1]^L$  such that  $\|x - \xi\| \geq \delta_\varepsilon$ ,  $|f(x) - f(\xi)| \leq 2M \frac{\|x - \xi\|^2}{\delta_\varepsilon^2}$ . Hence, for every  $x, \xi \in [0, 1]^L$ ,

$$|f(x) - f(\xi)| \leq 2M \frac{\|x - \xi\|^2}{\delta_\varepsilon^2} + \frac{\varepsilon}{2}. \quad (7)$$

Let  $g(x) = 2M \frac{\|x - \xi\|^2}{\delta_\varepsilon^2} + \frac{\varepsilon}{2}$ .

Note that  $B_{r_1, \dots, r_L}(x; f) - f(\xi) = B_{r_1, \dots, r_L}(x; f - f(\xi))$  and thus  $|B_{r_1, \dots, r_L}(x; f) - f(\xi)| \leq B_{r_1, \dots, r_L}(x; g)$ , according to (7). Elementary computations show that  $B_{r_1, \dots, r_L}(x; g) = \prod_{l=1}^L \left[ (x_l - \xi_l)^2 + \frac{x_l(1-x_l)}{r_l} \right]$  and thus evaluating everything at  $\xi$  we obtain:

$$|B_{r_1, \dots, r_L}(\xi; f) - f(\xi)| \leq \prod_{l=1}^L \frac{\xi_l(1-\xi_l)}{r_l} \quad (8)$$

Since the right hand side of (8) converges to 0 as  $r_1, \dots, r_L \rightarrow +\infty$ , the sequence of polynomials  $(B_{r_1, \dots, r_L}(\cdot; f))_{r_1, \dots, r_L}$  converges uniformly to  $f$  on  $[0, 1]^L$ . ■

**Proposition 2.** *Every  $u \in \mathcal{C}$  can be uniformly approximated by a sequence  $(u^n)_{n \in \mathbb{N}}$  of functions in  $\mathcal{V} \cap \mathcal{C}$ .*

*Proof.* Let  $u \in \mathcal{C}$  and define its extension  $\bar{u} : \mathbb{R}^L \rightarrow \overline{\mathbb{R}}$  as

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}_+^L \\ -\infty & \text{otherwise.} \end{cases}$$

Clearly,  $\bar{u}$  is no longer continuous (only upper semi-continuous), but it is still concave.

Let  $u^* : \mathbb{R}^L \rightarrow \overline{\mathbb{R}}$  be the conjugate function of  $\bar{u}$ , defined as

$$u^*(y) = \min_{x \in \mathbb{R}^L} \{xy - \bar{u}(x)\}. \quad (9)$$

Then  $u^*(y) = \bar{x}_y y - u(\bar{x}_y)$  for some  $\bar{x}_y \in \mathbb{R}_+^L$  and  $u^*$  is differentiable (whenever finite), non-decreasing and concave (see, for example, Rockafellar (1970)).

For every  $n \in \mathbb{N}$ , define the function  $v_n : \mathbb{R}^L \rightarrow \overline{\mathbb{R}}$  by

$$v_n(y) = u^*(y) - \frac{1}{n}(e^{-\sum_{l=1}^L \sqrt{y_l}} - 1).$$

The function  $v_n$  is increasing and concave and, for every  $y \in \mathbb{R}^L$ ,

$$v_n(y) - \frac{1}{n} \leq u^*(y) \leq v_n(y). \quad (10)$$

Passing to conjugates in (10) and using the fact that  $(u^*)^* = \bar{u}$  we obtain

$$v_n^*(x) + \frac{1}{n} \geq \bar{u}(x) \geq v_n^*(x), \quad (11)$$

for every  $x \in \mathbb{R}^L$  and, in particular,

$$v_n^*(x) + \frac{1}{n} \geq u(x) \geq v_n^*(x), \quad (12)$$

for every  $x \in \mathbb{R}_+^L$ . Equation (12) implies that  $v_n^*(x)$  is finite for every  $x \in \mathbb{R}_+^L$ ,  $v_n^*(0) \xrightarrow{n \rightarrow \infty} u(0) = 0$  and the sequence of concave functions  $(v_n^* - v_n^*(0))_n$  converges uniformly to  $u$ .

For each  $n \in \mathbb{N}$ , define the function  $u^n : \mathbb{R}_+^L \rightarrow \mathbb{R}$  by

$$u^n(x) = (v_n^*(x) - v_n^*(0)) - \frac{1}{n}(e^{\sum_{l=1}^L -\sqrt{x_l}} - 1),$$

for each  $x \in \mathbb{R}_+^L$ . Then  $u^n \in \mathcal{V}$  and

$$\|u^n - u\| \leq \|v_n^* - u\| + \|v_n^*(0)\| + \frac{1}{n}. \quad (13)$$

The three terms on the right-hand side converge to 0 when  $n$  tends to infinity and thus  $u^n$  is uniformly convergent to  $u$ . ■

## References

- Campbell, D. E., 1992. Implementation of Social Welfare Functions. *International Economic Review* **33**, 525-533.
- Chambers, C. P., 2005. Allocation Rules for Land Division. *Games and Economic Behavior* **121**, 236-258.
- Chambers, C. P., and J. Green, 2005. Additive rules for the quasi-linear bargaining problem. Caltech and Harvard University, *mimeo*.
- Chen M. A., and E. S. Maskin, 1999. Bargaining, Production, and Monotonicity in Economic Environments. *Journal of Economic Theory* **89**, 140-147.
- de Clippel, G., 2008. Axiomatic Bargaining on Economic Environments with Lotteries. Brown University, *mimeo*.
- Dutta, B., M. O. Jackson, and M. Le Breton, 2001. Strategic Candidacy and Voting Procedures. *Econometrica* **69**, 1013-1037.
- Ehlers, L., and J. Weymark, 2003. Candidate Stability in Nonbinary Social Choice. *Economic Theory* **22**, 233-243.
- Fershtman, C., 1990. The importance of the agenda in bargaining. *Games and Economic Behavior* **2**, 224-238.
- Fleurbaey, M., 2003. On the Informational Basis of Social Choice. *Social Choice and Welfare* **21**, 347-384.
- Fleurbaey, M., and K. Tadenuma, 2007. Do Irrelevant Commodities Matter? *Econometrica* **75**, 1143-1174.
- Ginés, M., and F. Marhuenda, 2000. Welfarism in Economic Domains. *Journal of Economic Theory* **93**, 191-204.
- Graham, D. A., and R. Marshall, 1987. Collusive Bidder Behavior at Single-Object Second-Price and English Auctions. *Journal of Political Economy* **95**, 1217-1239.
- Graham, D. A., R. Marshall, and J.-F. Richard, 1990. Differential Payments within a Bidder Coalition and the Shapley Value. *American Economic Review* **80**, 493-510.
- Grether, D. M., and C. R. Plott, 1982. Nonbinary Social Choice: an Impossibility Theorem. *Review of Economic Studies* **49**, 143-149.
- John, R., and M. G. Raith, 1997. Optimizing multi-stage negotiations. *Journal of Economic Behavior and Organization* **45**, 155-173.
- Kalai, E., 1977. Proportional solutions to bargaining situations: interpersonal utility comparisons. *Econometrica* **45**, 1623-1630.

- Karni, E., and D. Schmeidler, 1976. Independence of Nonfeasible Alternatives, and Independence of Nonoptimal Alternatives. *Journal of Economic Theory* **12**, 488-493.
- Lorentz, G. G., 1953. Bernstein Polynomials. Toronto: University of Toronto Press.
- Moulin, H., 1985. Egalitarianism and utilitarianism in quasi-linear bargaining. *Econometrica* **53**, 49-67.
- Moulin, H., 1992. An application of the Shapley value to fair division with money. *Econometrica* **60**, 1331-1349.
- Moulin, H., and W. Thomson, 1988. Can Everyone Benefit from Growth?: two Difficulties. *Journal of Mathematical Economics* **17**, 339-345.
- O'Neill, B., D. Samet, Z. Wiener, and E. Winter, 2004. Bargaining with an agenda. *Games and Economic Behavior* **48**, 139-153.
- Pazner, E., and D. Schmeidler, 1978. Egalitarian equivalent allocations: a new concept of economic equity. *Quarterly Journal of Economics* **92**, 671-687.
- Plott, C. R., 1976. Axiomatic Choice Theory: an Overview and Interpretation. *American Journal of Political Science* **20**, 511-596.
- Ponsati, C., and J. Watson, 1997. Multiple-issue bargaining and axiomatic solutions. *International Journal of Game Theory* **26**, 501-524.
- Rockafellar, R. T., 1970. Convex Analysis. Princeton: Princeton University Press.
- Roemer, J. E., 1986. The mismatch of bargaining theory and distributive justice. *Ethics* **97**, 88-110.
- Roemer, J. E., 1988. Axiomatic bargaining theory on economic environments. *Journal of Economic Theory* **45**, 1-31.
- Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences* **45**, 249-297.
- Yoshihara, N., 2003. Characterizations of Bargaining Solutions in Production Economies with Unequal Skills. *Journal of Economic Theory* **108**, 256-285.