THE OBJECTIVE OF A PRIVATELY OWNED FIRM UNDER IMPERFECT COMPETITION

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Abstract

This paper proposes a model of imperfect competition among privately owned firms that act in the best interest of their shareholders. The existence of a solution for the model is proved under weaker conditions than the ones generally used in the literature. In particular, the results did not require the existence of a continuous equilibrium price selection or concavity assumptions on the profit function.

Keywords Firm's objective, Shareholders' preferences, Imperfect competition, General equilibrium

JEL Classification D21, D43, D51, D70, L21

1 Introduction

A firm's production decisions are typically the result of a group decision process involving people with (possibly) conflicting interests. Modern corporations are generally organized as legal systems in which the ultimate authority belongs to their shareholders. It is to be expected then that a firm's production decisions reflect its shareholders' interests.

The general equilibrium literature abstracts from the internal organization of firms and models them as production entities whose decisions are driven by the profit maximization

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motive. While the necessity of simplification is perfectly understood in those models, it is still a legitimate question to ask whether profit maximization is indeed compatible with shareholders' interests.

The answer can be "yes" or "no", depending essentially on the economic environment in which that firm acts. Under complete markets and perfect competition, a privately owned firm that maximizes its profits taking the market prices as given does act in the best interest of its shareholders. In this case, the only effect that firm's choice has on shareholders' wealth is through the profits that they receive. Since higher profit relaxes every shareholder's budget constraint, the production plan that maximizes profits at given prices is ranked the best by all shareholders. Hence, if they are the ones that control the firm, profit maximization does represent their preferences.

However, if the firm has some market power, the above reasoning no longer applies, because a change in production plans now has two effects on shareholders' wealth: one is, as before, through the profits that they receive and the other is through the change in relative prices that the choice of different production plans generates. Since these two effects may work in opposite directions, profit maximization may no longer be unanimously supported by the firm's shareholders. In fact, if they are heterogeneous, shareholders may not have a common objective at all.

Another problem associated with profit maximization by an imperfectly competitive firm in a general equilibrium framework is its dependence on the price normalization chosen. The aggregate (residual) demand that each firm faces depends only on relative prices, while profit is evaluated at nominal prices. Since profit functions obtained under different normalization rules are not, in general, monotonic transformations of each other, different choices of the numeraire good lead to different solutions. Hence, profit maximization is an ill-defined mathematical problem to start with.

The existing literature on imperfect competition acknowledges these problems and then overcomes them at the expense of making strong assumptions about the market structure and/or the characteristics of market participants (see [2] for a survey).

Gabszewicz and Vial [8], were the first ones to propose a general equilibrium framework for the analysis of Cournot competition among firms. Their model, the Cournot-Walras model, designs an economy populated by a "small" number of firms and a "large" set of consumers. Firms are assumed to act strategically in the market via a quantity competition in the spirit of Cournot; their profits are distributed to consumers according to some preassigned shares. Consumers are then allowed to trade, as price takers, in a perfectly competitive market; Walrasian equilibrium prices resulting in this pure-exchange economy are used by firms to evaluate their profits. Whenever prices are not unique, a certain selection from the exchange equilibrium price correspondence is made. The authors themselves pointed out the above mentioned problems associated with their formulation of firm's objective.

Mas-Colell [9] and later on Dierker and Grodal [4] and Dierker et al. [5] refined Gabszewicz and Vial's model. Mas-Colell proposed a scenario in which profit maximization became a justified objective. He assumed the existence of a class of "capitalists" who own the firms and only care about some non-produced good, which is chosen as numeraire and its price is set to 1. In this situation, a change in firm's production plan influences its shareholder's wealth only through the profits received. Hence, under the imposed assumption, utility maximization by the owner results indeed in profit maximization by the firm, but the condition is too restrictive for most of the real life examples. Dierker and Grodal [4] argued that the normalization rule used to compute nominal prices should be intrinsic to the model and thus arise *endogenously*, rather than being chosen arbitrarily, a priori. They suggested that shareholders' demand should be taken into account when choosing the units to evaluate profits. According to their definition, a monopolistic firm's objective is that of choosing an input-output vector y^* such that profits, expressed in units of the shareholders' aggregate demand at y^* (i.e., the aggregate demand at the exchange equilibrium price, $p(y^*)$, that prevails in the market after the choice of y^*), are maximized at y^*. The authors called this objective *real wealth maximization.* In a subsequent paper, Dierker et al. [5] investigated to what extent real wealth maximization respects the interest of shareholders. They defined a production plan to be S-efficient (S stands for "shareholders") if no other plan existed such that shareholders' new aggregate demand could be redistributed in a way that all shareholders would be better off. Any firm that acts in the best interest of its shareholders should select $S-$efficient production plans. The authors showed that if shareholders have convex and smooth preferences and their aggregate compensated surplus function was concave, real wealth maximization selected S-efficient plans. The conclusion is no longer true once the assumption of concavity of the surplus function is dropped, but the authors argued that S-efficiency would then be too strong of a requirement, since S-efficient production plans may not exist.

This paper proposes an alternate objective for a privately owned monopolistic firm, called shareholders' wealth maximization (S-wealth maximization). S-wealth maximizing production plans maximize shareholders' wealth in the following sense: a plan y^* is S-wealth maximizing if shareholders' wealth at y^* is enough to buy, at prices prevailing in the market at y^*, any of the aggregate consumption bundles that they would have chosen if the firm had made a different choice y. S-wealth maximizing plans are shown to be S-efficient for all continuous, convex and locally non-satiated preferences. If profits, as a function of production plans, are differentiable and concave, S-wealth maximization is a stronger requirement than real wealth maximization, in the sense that every S-wealth maximizing production plan is real wealth maximizing, too. Under perfect competition, the two notions coincide with the familiar profit maximization. However, in general none of them implies the other.

To deal with the issue of existence of multiple exchange equilibrium prices, we do *not* use an exogenously given price selection. Instead, it is assumed that the firm holds some beliefs over the set of possible equilibrium prices (or inverse demands in the market). Each such belief generates a (possibly) different set of S-wealth maximizing production plans. An equilibrium for the oligopoly model consists of a system of beliefs together with a corresponding vector of S-wealth maximizing production plans for the firms. It is proved that such an equilibrium exists (possibly in mixed strategies).

The paper is organized as follows. Section 2 shows, by means of an example, that profit maximization does not respect shareholders' interests even when they have the same preferences and one can choose a normalization rule. Section 3 analyzes the case of a firm with a heterogeneous set of shareholders, defines the concept of S-efficiency and proposes an objective that selects only S-efficient production plans. Section 4 gives a characterization of S-wealth maximizing plans, section 5 proves existence of a solution for firm's problem and finally section 6 establishes the existence of an equilibrium in the oligopolistic game, in which firms choose S-wealth maximizing strategies. Section 7 concludes.

2 Profit Maximization under Imperfect Competition

Consider an $L-$good economy populated by a large number, I, of consumers and one firm. Let $(\mathbb{R}^L_+, u^i, \omega^i, \theta^i)$ be the vector of characteristics of consumer $i \in I$, where $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ is the utility function, $\omega^i \in \mathbb{R}^L_+$ is the endowment of goods and $\theta^i \in [0,1]$ represents shares in firm's profits.[1] Utility function u^i represents locally non-satiated preferences and it is continuous and strictly quasi-concave. The aggregate endowment $\sum_{i=1}^{I} \omega^i := \omega \in \mathbb{R}^L_{++}$ is strictly positive in every component and the total number of outstanding shares is normalized to 1: $\sum_{i \in I} \theta^i = 1$.

Let $Y \subseteq \mathbb{R}^L$ be firm's production set and $S = \left\{ i | \theta^i > 0 \right\} \subseteq I$ the set of firm's shareholders. Y is assumed to satisfy the standard conditions: (a) it is closed, convex and contains the origin, and (b) $Y \cap \mathbb{R}^L_+ = \{0\}$.

If both the firm and the consumers are price takers, profit maximization is unanimously supported by all shareholders. This ceases to be the case if firm is not a price taker. To see

[1] We are making the usual abuse of notation by letting $I = \{1, 2, ..., I\}$.

why, suppose that the firm is a (quantity setting) monopolist, while consumers continue to be price takers.

For every $y \in Y$ and $i \in I$ let $\widetilde{\omega}^i := \omega^i + \theta^i y$. Note that for $i \notin S$, $\widetilde{\omega}^i = \omega^i$. Define $\mathcal{E}_y$, as being the pure-exchange economy populated by I consumers with the same preferences as the original ones, but with endowments $\widetilde{\omega}^i$: $\mathcal{E}_y = \left\{(\mathbb{R}^L_+, u^i, \widetilde{\omega}^i)_{i \in I}\right\}$. For every consumer i of the economy $\mathcal{E}_y$, let $x^i(p, y)$ denote his/her demand at prices p and let $P(y) \subset \Delta^{L-1}$ be the set of Walrasian exchange equilibrium price vectors of $\mathcal{E}_y$, normalized to lie in the $(L-1)$-dimensional unit simplex.

For some production sets Y (in particular for those that exhibit free disposal) the economy $\mathcal{E}_y$ may have no Walras equilibrium. Certain lower bounds (or capacity constraints) need to be imposed on the firm's strategy set to avoid this occurrence and make the problem meaningful. It is sufficient, for example to restrict firm's choices to production plans y that satisfy: $y > -\min_{i \in S} \frac{\omega^i}{\theta^i}$ and $y \gg -\sum_{i=1}^{I} \omega^i$.[2] For such y, the main theorem in [10] implies that $P(y) \neq \emptyset$. However, these conditions are not necessary for the non-emptiness of $P(y)$ and therefore a strictly larger set than the one described above may still generate non-empty values for P. For the sake of generality, we define the *firm's strategy set* to be *some* subset $\widehat{Y}$ of Y such that $\widehat{Y} \subseteq \{y \in Y \mid P(y) \neq \emptyset\}$ and $0 \in \widehat{Y}$. As pointed out above, such subset always exists.

The non-empty valued correspondence $P : \widehat{Y} \rightrightarrows \Delta^{L-1}$ is called the *exchange equilibrium price correspondence*. Let $p : \widehat{Y} \to \Delta^{L-1}$ be an arbitrary selection from P.[3]

Shareholders preferences over production plans are represented by the indirect utility functions $V^i(y) := u^i\left(x^i(p(y), y)\right)$. In general, for heterogeneous economies, these utilities have no common maximizer on $\widehat{Y}$ which means that there is no unanimously supported production plan. Clearly, a common maximizer does exist if shareholders are identical. Still, as shown in the sequel, even in this case the unanimously supported production plan may not coincide with the profit maximizing plan.

Note that, when formulated in the context of a monopolistic economy, the profit maximization problem depends on the rule chosen to convert relative into nominal prices. For a given y, two vectors of absolute (or nominal) prices correspond to (or represent different normalizations of) the same vector of relative prices if and only if they are proportional, with the proportionality factor being a positive real number. Thus any vector of nominal prices that generates the same relative prices as $p(y) \in \Delta^{L-1}$ must be of the form: $\gamma(y)p(y)$, with $\gamma(y) \in \mathbb{R}_{++}$. A function $\gamma : \widehat{Y} \to \mathbb{R}_{++}$, is called a

[2] We are using the usual notation for vector inequalities. If $a, b \in \mathbb{R}^L$, then: $a \geq b$ if and only if $a_l \geq b_l$ for all $l = 1, .., L$; $a > b$ if and only if $a \geq b$ and $a \neq b$; $a \gg b$ if and only if $a_l > b_l$ for all $l = 1, .., L$.

[3] One could think of p as representing shareholders' belief about the realization of the equilibrium price. Section 5 elaborates on that.

price normalization rule. The profit function computed using the normalization rule γ is: $\Pi_\gamma(y) = \gamma(y)p(y)y$.

Different price normalization rules generate different profit functions *and* different profit maximizers. Therefore, a certain normalization rule must be specified from the beginning to make the problem meaningful. The following example shows, however, that *no matter what normalization rule is chosen*, profit maximizing production plan may not coincide with shareholders' most preferred plan even when such plan exists.

Example 1 *Consider an economy with 2 goods and 1 consumer. Let $u(x_1, x_2) = x_1x_2$, $\omega = (2, 1)$ and $\theta = 1$ be the characteristics of the consumer and*

$$Y = \{(y_1, y_2) \in \mathbb{R}_- \times \mathbb{R}_+ \, | y_1 + y_2 \leq 0\}$$

firm's production set.

The strategy space of the firm is $\widehat{Y} = \{(y_1, y_2) \in (-2, 0] \times [0, 2) \mid y_1 + y_2 \leq 0\}$ and the exchange equilibrium price correspondence is given by:

$$P(y) = \left(\frac{1 + y_2}{3 + y_1 + y_2}, \frac{2 + y_1}{3 + y_1 + y_2} \right) \in \Delta \subseteq \mathbb{R}^2_+.$$

The unique most preferred production plan[4] *from the firm owner's point of view is* $(-\frac{1}{2}, \frac{1}{2}) = \arg\max \left\{ (2 + y_1)(1 + y_2) \mid (y_1, y_2) \in \widehat{Y} \right\}$.

We show now that there is no price normalization rule under which $(-\frac{1}{2}, \frac{1}{2})$ maximizes profits.

Let $\gamma : \widehat{Y} \to \mathbb{R}_{++}$ be a normalization rule[5] *and suppose that $(-\frac{1}{2}, \frac{1}{2})$ is profit maximizer under γ. Then*

$$\gamma(y) \frac{y_1 + 2y_2 + 2y_1y_2}{3 + y_1 + y_2} \leq 0. \tag{1}$$

for every $y \in \widehat{Y}$. Since by definition $\gamma(y) > 0$, and $3 + y_1 + y_2 > 0$, inequality (1) implies $y_1 + 2y_2 + 2y_1y_2 \leq 0$, $\forall y \in \widehat{Y}$. However, this inequality fails at $(y_1, y_2) = \left(-\frac{1}{3}, \frac{1}{3}\right) \in \widehat{Y}$.

Thus the unique production plan that is unanimously ranked the best by all shareholders cannot arise as a solution of profit maximization under any normalization rule.

The example shows that the problem is not that of choosing the "right" price normalization, but rather revising the profit maximization objective itself.

[4] Note that this is independent of the choice of the numeraire.

[5] For example, the normalization rule corresponding to using the bundle $(\alpha, \beta) \in \mathbb{R}^2_+, \alpha^2 + \beta^2 \neq 0$ as the numeraire is: $\gamma(y) = \frac{\beta(1+y_1)+\alpha y_2}{1+y_1+y_2}$.

3 S-Wealth Maximization and S-Efficiency

As pointed out in the previous section, if shareholders are heterogeneous, they may not agree on a common most preferred production plan. Requiring that the firm's choice meet shareholders' unanimous approval would thus lead to an empty solution. A weaker and very natural requirement is that production plans chosen by the firm be Pareto undominated from the shareholders' point of view. However, this requirement alone is too weak because equilibrium allocations generated by different production plans may not be Pareto comparable and thus the class of solutions satisfying it may be very large. We therefore use a stronger criterion based on Pareto comparisons *accompanied by redistribution*, as in [4]: the firm should not select a production plan y if another one, say $\overline{y}$, is in firm's strategy set and is preferred by all shareholders after a potential redistribution of their consumption bundles. A firm's objective that fulfills this requirement is proposed here.

Consider the economy described in the previous section, with a single monopolistic firm and a large set of consumers. The set of firm's shareholders[6] is S. $\widehat{Y}$ denotes, as before, the firm's strategy set , $P : \widehat{Y} \rightrightarrows \Delta$ is the exchange equilibrium price correspondence and p is an arbitrary measurable selection[7] from P.

Given the price selection $p(\cdot)$ let $D_p^S(y) := \sum_{i \in S} x^i(p(y), y)$ be *shareholders' aggregate demand* when the firm chooses $y \in \widehat{Y}$ and let $W_p^S(q, y) := qD_p^S(p(y), y)$ be their *aggregate wealth* evaluated at some price vector q.

Definition 2 *Given the price selection p, a vector $y^* \in \widehat{Y}$ is called S-wealth maximizing if and only if $W_p^S(p(y^*), y^*) \geq W_p^S(p(y^*), y)$ for all $y \in \widehat{Y}$.*

The objective of a privately owned monopolistic firm is to choose an S-wealth maximizing production vector, y^*. If the firm chooses an S-wealth maximizing production plan, its shareholders' aggregate wealth suffices to buy, at the prices prevailing in the market, any of the other aggregate consumption bundles the group of shareholders would have had if the firm had made a different choice.

Note that S-wealth maximization is indeed independent of the price normalization chosen and it reduces to profit maximization if the firm is perfectly competitive (i.e., its

[6]Throughout the paper it is assumed that firm's decisions are governed by the interests of *all* shareholders and *only* theirs. However, this is not essential; all the results hold under the alternative assumption that there is a certain subset of consumers, called the *control group*, whose interests are reflected in the firm's decisions. Members of the control group could be shareholders as well as non-shareholders (for example, a non-shareholder representative of the firm's employees may be in the control group).

[7]Such selection exists (at least as long as $\widehat{Y}$ is compact), because P is upper hemicontinuous with compact values (see appendix) and thus weakly measurable. Kuratowski-Ryll-Nardzewski selection theorem implies then the existence of a measurable selection from P (see [1])

actions do not affect equilibrium prices). For the case of a monopolist, selecting S-wealth maximizing production plans respects the interests of the shareholders, in a sense that is made precise below.

Definition 3 *Let $y, y' \in \widehat{Y}$. We say that y is weakly S-dominated by y' (given the price selection p) if and only if $\exists (x^i)_{i\in S} \in \mathbb{R}_+^{LS}$ such that*

1. $\sum_{i\in S} x^i = D_p^S(y')$
2. $u^i(x^i) \geq u^i(x^i(p(y), y)), \forall i \in S$, *with at least one strict inequality for some* $j \in S$.

An element $y \in \widehat{Y}$ is called S-efficient if there is no $y' \in \widehat{Y}$ that weakly dominates it.

We say that a firm's objective respects its shareholders interests if it selects (a subset of the) S-efficient[8] production plans. The next proposition shows that S-wealth maximization satisfies this requirement.

Proposition 4 *Any S-wealth maximizing production plan is S-efficient. The converse is also true if $\left\{ D_p^S(y) - \omega^S \mid y \in \widehat{Y} \right\}$ is a convex set. If $S = I$ and $\widehat{Y}$ is convex, both S-wealth maximizing and S-efficient production plans coincide with the set of Walras equilibrium production vectors.*

Proof. The idea of the proof is the following: we construct an artificial production economy whose Pareto optimal and Walrasian equilibrium allocations are mapped to S-efficient and S-wealth maximizing production plans, respectively. We use then the standard welfare theorems to get the results.[9]

Let Z_p be the set of shareholders' net aggregate demands, i.e., $Z_p \overset{def}{=} \left\{ D_p^S(y) - \omega^S \mid y \in \widehat{Y} \right\}$, where $\omega^S = \sum_{i\in S} \omega^i$ is shareholders' aggregate endowment.

Consider the production economy $\mathcal{E}_p^S = \left\{ \left(\mathbb{R}_+^L, u^i, \omega^i, \theta^i \right)_{i\in S}, Z_p \right\}$ in which the set of consumers is S and the production set is Z_p.[10].

Lemma 5 *1. If $y^* \in \widehat{Y}$ is S-wealth maximizing for the original economy then*

$$\left(\left(x^i(p(y^*), y^*) \right)_{i\in S}, D_p^S(y^*) - \omega^S, p(y^*) \right)$$

[8] In case S is the set of all shareholders, this requirement could seem too strong, because the side payments involved in the definition are unlikely to be possible within a large group of shareholders. However, if S represents a (small) control group, it is to be expected that its members will bargain with each other over different proposals and agree on making certain transfers to achieve unanimity. On the other hand, it is important to understand that, in any case, the firm (or the group of its shareholders) is *not* required to be able to compute and/or make those transfers. If the firm follows the S-wealth maximization objective, its shareholders would not be able to improve upon the production plan chosen *even if* they had the ability to make side payments.

[9] A direct proof also works here; however, this construction will be useful later on.

[10] If S is not the set of all shareholders, consumers' shares in the artificial economy $\mathcal{E}_p^S$ have to be adjusted to $\frac{\theta^i}{\overline{\theta}}$, where $\overline{\theta} = \sum_{i\in S} \theta^i$.

is a Walrasian equilibrium in $\mathcal{E}_p^S$.

2. *If $y^* \in \widehat{Y}$ is S-efficient for the original economy then $\left((x^i(p(y^*),y^*))_{i\in S}, D^S(y^*)-\omega^S\right)$ is Pareto optimal in $\mathcal{E}_p^S$. Conversely, if the allocation $\left(\left(\overline{x^i}\right)_{i\in S}, \overline{z}\right)$ is Pareto efficient in $\mathcal{E}_p^S$ and there exists $\overline{y} \in \widehat{Y}$ such that $\overline{z} = D_p^S(\overline{y}) - \omega^S$ and $\overline{x^i} = x^i(p(\overline{y}), \overline{y})$. then $\overline{y}$ is S-efficient in the original economy.*

Proof of the lemma.

1. Assume $y^* \in \widehat{Y}$ is S-wealth maximizing.

Then $\left((x^i(p(y^*),y^*))_{i\in S}, D_p^S(y^*)-\omega^S\right)$ is a feasible allocation for the economy $\mathcal{E}_p^S$ and $D_p^S(y^*)-\omega^S$ maximizes profits in Z_p given the prices $p(y^*)$. Moreover, aggregation of budget constraints in the original economy gives: $p(y^*)\left(D_p^S(y^*)-\omega^S\right) = p(y^*)y^*$ i.e., shareholders' budget constraints are the same in the two economies. This proves that $\left((x^i(p(y^*),y^*))_{i\in S}, D_p^S(y^*)-\omega^S, p(y^*)\right)$ is a Walrasian equilibrium in $\mathcal{E}_p^S$.

2. Assume $y^* \in \widehat{Y}$ is S-efficient, but $\left((x^i(p(y^*),y^*))_{i\in S}, D_p^S(y^*)-\omega^S\right)$ is not Pareto optimal in $\mathcal{E}_p^S$. Then there exists a feasible allocation $((x^i)_{i\in S}, z)$ that Pareto dominates it. Hence, $\exists y \in \widehat{Y}$ such that $z = D_p^S(y) - \omega^S$, $\sum_{i\in S} x^i = D_p^S(y)$ and $u^i(x^i) \geq u^i(x^i(p(y^*),y^*)) \forall i \in S$, with at least one strict inequality. This contradicts S-efficiency and thus $\left((x^i(p(y^*),y^*))_{i\in S}, D_p^S(y^*)-\omega^S\right)$ is Pareto optimal in $\mathcal{E}_p^S$. The converse implication is also transparent from the above reasoning. ■

Since $\mathcal{E}_p^S$ is an economy without externalities, in which consumers have locally non-satiated preferences, the first welfare theorem holds and thus any Walrasian equilibrium allocation is Pareto optimal. Using then lemma 5 we can conclude that every S-wealth maximizing production plan is S-efficient. If, moreover, Z_p is a convex set, the second welfare theorem also holds for $\mathcal{E}_p^S$ and thus S-wealth maximizing and S-efficient production plans coincide.

If $S = I$ then $Z_p = \widehat{Y}$ and the conclusion is immediate. ■

4 Properties of S-Wealth Maximizing Production Plans

This section gives a characterization of the S-wealth maximizing production plans for smooth economies. For this class of economies, and the particular price selection, one can give a system of first order conditions that S-wealth maximizing production plans must satisfy.

For every $y \in \widehat{Y}$ let $W^S(y) := W_p^S(p(y), y)$. Assume that the economy is smooth and thus p is of class $\mathcal{C}^1$.

Proposition 6 *Assume that y^* is an S-wealth maximizing production plan. Then*

$$\nabla W^S(y^*) = J_p(y^*)\, D^S(y^*), \tag{2}$$

where $\nabla W^S(y^*)$ is the gradient of W^S evaluated at y^* and $J_p(y^*) \in \mathcal{M}_{L\times L}(\mathbb{R})$ is the Jacobian matrix of p at y^*.

Proof. The equality can be obtained as follows:

$$\frac{\partial W^S}{\partial y_l}(y^*) = \lim_{\substack{h\to 0\\ h>0}} \frac{p(y^*+he_l)D^S(y^*+he_l) - p(y^*)D^S(y^*)}{h} \leq \tag{3}$$

$$\leq \lim_{\substack{h\to 0\\ h>0}} \frac{[p(y^*+he_l) - p(y^*)]\, D^S(y^*+he_l)}{h} = \frac{\partial p}{\partial y_l}(y^*)D^S(y^*). \tag{4}$$

Similarly,

$$\frac{\partial W^S}{\partial y_l}(y^*) = \lim_{\substack{h\to 0\\ h<0}} \frac{p(y^*+he_l)D^S(y^*+he_l) - p(y^*)D^S(y^*)}{h} \geq \tag{5}$$

$$\geq \lim_{\substack{h\to 0\\ h>0}} \frac{[p(y^*+he_l) - p(y^*)]\, D^S(y^*+he_l)}{h} = \frac{\partial p}{\partial y_l}(y^*)D^S(y^*). \tag{6}$$

■

Formula (6) is exactly the first order condition that real wealth maximizing production plans, proposed by Dierker and Grodal in [4] have to satisfy[11]. According to Dierker and Grodal [4], $y^* \in \widehat{Y}$ is a *real wealth maximizer* if and only if $W_p^S(p(y), y^*) \geq W_p^S(p(y), y)$, $\forall y \in \widehat{Y}$. Although S-wealth maximization has the same real wealth maximization "flavor", the two concepts are, in general, different. According to proposition 4, S-wealth maximizing production plans are S-efficient for all continuous, convex and locally non-satiated preferences. Real wealth maximizing production plans fail to be S-efficient if shareholders' compensated surplus function is not concave[12] (see [5]). Hence, in general, the two objectives select different production plans.

The previous proposition shows that, for the case of a smooth economy, S-wealth maximizing and real wealth maximizing production plans satisfy the same system of first order conditions. One may wonder under which conditions the two concepts coincide.

[11]Dierker and Grodal work with a two-good economy, in which firm chooses a price. However, their analysis can be extended to multiple-good economies and firms that select production plans.

[12]General conditions (on the primitives of the model) that insure concavity of shareholders' social surplus function are, to the best of my knowledge, unknown.

Clearly, this happens under perfect competition, since in that case they both deliver the profit maximizing production plans. Other instances in which the two concepts coincide are described below.

For every $y \in \widehat{Y}$, define shareholders' aggregate budget set at y as

$$\mathcal{B}^S(y) = \left\{x \in \mathbb{R}_+^L \mid p(y)x \leq p(y)\omega^S + p(y)y\right\}.$$

The budget line at y, $BL^S(y)$, is $\left\{x \in \mathbb{R}_+^L \mid p(y)x = p(y)\omega^S + p(y)y\right\}$ and the aggregate budget set is

$$\mathcal{B}^S = \bigcup_{y \in \widehat{Y}} \mathcal{B}^S(y).$$

Note that the complement of $\mathcal{B}^S$ is a convex set, being an intersection of convex sets (the complements of budget sets at y, for different y-s).

The North-East part of the frontier of $\mathcal{B}^S$ is called the *aggregate budget curve*, and is denoted by $\mathcal{F}$. Since this is not, in general, a hyperplane, the budget set is a non-convex set, in general.

Proposition 7 *If $\mathcal{F}$ is smooth and both S-wealth maximizing and real wealth maximizing production plans exist, then they must be unique and coincide.*

Proof. Let y^S be an S-wealth maximizing production plan and y^R a real wealth maximizing production plan. Then $p(y^S)D^S(y^S) \geq p(y^S)D^S(y^R) \geq p(y^S)D^S(y^S)$ and thus $y^S, y^R \in BL^S(y^S)$. On the other hand, by definition, $D^S(y^R) \in \mathcal{F}$. Since $\mathcal{F}$ is smooth, $BL^S(y^R)$ is the unique hyperplane supporting $\left(\mathcal{B}^S\right)^C$ at $D^S(y^R)$. Therefore, $BL^S(y^S) = BL^S(y^R)$ and thus $p(y^S) = p(y^R)$. Since $W^S(y^S) = W^S(y^R)$ this implies $p(y^S)y^S = p(y^R)y^R$, which, together with the market clearing conditions in the corresponding exchange economies gives $y^S = y^R$. ■

The next proposition makes a first step towards analyzing the existence problem of S-wealth maximizing strategies.

Proposition 8 *If $\mathcal{F}$ is smooth, real wealth maximizing production plans exist and the set $\left\{D^S(y) \mid y \in \widehat{Y}\right\}$ is convex*[13]*, then an S-wealth maximizing plan exists, it is unique and coincides with the unique real wealth maximizing plan.*

Proof. Let y^R be a real wealth maximizing plan. Then $D^S(y^R) \in \mathcal{F}$ and $BL^S(y^R)$ is the unique hyperplane supporting $\left(\mathcal{B}^S\right)^C$ at $D^S(y^R)$. Moreover, $\left\{D^S(y) \mid y \in \widehat{Y}\right\} \subseteq \mathcal{B}^S$,

[13] This condition is met, for example, in linear economies (see [7]).

it is convex and contains $D^S(y^R)$. The separating hyperplane theorem implies then that $BL^S(y^R)$ must separate $\left\{D^S(y) \mid y \in \widehat{Y}\right\}$ and $\left(\mathcal{B}^S\right)^C$. This means that y^R is also S-wealth maximizing, which proves existence. Uniqueness is implied by the previous proposition. ■

Note that if $S = I$ (i.e., every consumer is a shareholder), $\left\{D^S(y) \mid y \in \widehat{Y}\right\} = \widehat{Y}$. Hence, as long as $\widehat{Y}$ is convex, real wealth maximization and S-wealth maximization coincide and they both deliver the Walrasian production plans.

Example 9 *Consider a two-good economy with two consumers and one monopolistic firm. Consumers preferences are represented by the utility functions* $u^1(c_1, c_2) = \log c_1 + \log c_2$, $u^2(c_1, c_2) = \log c_1 + 2\log c_2$. *Their endowments are* $\omega^1 = \omega^2 = (3, 1)$, $\theta^1 = \theta^2 = \frac{1}{2}$.

Firm's production set is $Y = \left\{(y_1, y_2) \in \mathbb{R}_- \times {\mathbb{R}_+}^2 \, | y_1 + y_2 \leq 0\right\}$.

Simple computations show that $\widehat{Y} = Y \cap ((-6, 0] \times [0, 6))$ *and* $P(y) = \frac{10+5y_2}{52+7y_1+5y_2}$, *for all* $(y_1, y_2) \in \widehat{Y}$. *Consumers' demands are* $D^1(y) = \left(\frac{6}{5}\left(3 + \frac{1}{2}y_1\right), \frac{6}{7}\left(1 + \frac{1}{2}y_2\right)\right)$ *and* $D^2(y) = \left(\frac{4}{5}\left(3 + \frac{1}{2}y_1\right), \frac{8}{7}\left(1 + \frac{1}{2}y_2\right)\right)$. *Their utilities are:* $\widetilde{u}^1(y) = \ln\frac{6}{5}\left(3 + \frac{1}{2}y_1\right) + \ln\frac{6}{7}\left(1 + \frac{1}{2}y_2\right)$ *and* $\widetilde{u}^2(y) = \ln\frac{4}{5}\left(3 + \frac{1}{2}y_1\right) + 2\ln\frac{8}{7}\left(1 + \frac{1}{2}y_2\right)$.

The utility possibility set is $\{(\widetilde{u}^1(y), \widetilde{u}^2(y)) | y \in \widehat{Y}\}$. *Its frontier is depicted below:*

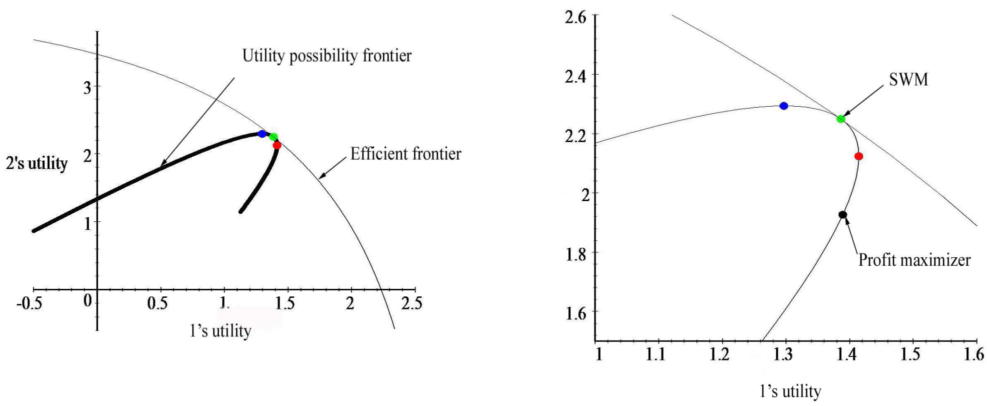

Figure 1: Utility Possibility Frontier

It is clear from the picture that the two shareholders do not agree on their most preferred production plan. Consumer 1's most preferred plan is $(-2, 2)$, *which gives him an utility of* 1.41; *consumer 2's most preferred plan is* $\left(-\frac{10}{3}, \frac{10}{3}\right)$ *which gives her an utility of* 2.3. *If both shareholders are in the control group then the unique* S*-wealth maximizing production plan is* $\left(-\frac{8}{3}, \frac{8}{3}\right)$, *which is exactly the Walras equilibrium production plan.*

The thin line on the graph represents the Pareto frontier in the artificial pure-exchange economy. Its intersection with the utility frontier is the S-efficient set. In this case it consists of only one point: the Walras equilibrium. Hence $SE = \left\{ \left(-\frac{8}{3}, \frac{8}{3} \right) \right\}$.

The real wealth maximizing plan is $\left(-\frac{8}{3}, \frac{8}{3} \right)$, *too and thus all three concepts (real wealth maximization, S-wealth maximization and S-efficiency) deliver the same solution in this example.*

Note also that $\left\{ D^S(y) - \omega^S \mid y \in \widehat{Y} \right\}$ *is a convex set for every possible structure of the control group (i.e., for* $S = \{1\}$, $S = \{2\}$, *or* $S = \{1,2\}$*). Hence, in particular, if only one of the consumers is in the firm's control group, the corresponding S-wealth maximizing production plan coincides with his/her most preferred plan.*

If prices are normalized to lie in the unit simplex, the profit maximizing plan is $(-1.3694, 1.3694)$, *which gives consumers strictly lower utilities than than at the S-wealth maximizing plan:* 1.3893 *and respectively* 1.9266 .

5 Existence of S-Wealth Maximizing Production Plans

In section 3 it was proved that S-wealth maximizing plans are related to Walrasian equilibrium allocations of the economy $\mathcal{E}_p^S$. Note that, even if $\widehat{Y}$ is compact and convex and the price selection is a smooth function, Z_p may not be a convex set (although it is compact and connected). Hence $\mathcal{E}_p^S$ is in general a non-convex production economy and a Walrasian equilibrium allocation may not exist for this economy. Consequently, an S-wealth maximizing production plan may not exist either.[14] [15]

In the sequel we show that, if instead of choosing only deterministic production plans the firm can also randomize over the input-output vectors in its strategy set (i.e., the firm can use mixed strategies), then the set of shareholders' net demands is convexified and an equilibrium does exist.

Note that the S-wealth maximizing production plans are in a bijective correspondence with the pure strategy Nash equilibria of the two-player simultaneous move game $\Gamma_p = \left(\widehat{Y}, \widehat{Y}, \Phi_p^1, \Phi_p^2 \right)$, where $\Phi_p^1, \Phi_p^2 : \widehat{Y} \times \widehat{Y} \to \mathbb{R}$,

$$\Phi_p^1(y_1, y_2) = p(y_2) D_p^S(y_1), \Phi_p^2(y_1, y_2) = -\left\| y_1 - y_2 \right\|^2,$$

and $\|y_1 - y_2\|$ is the Euclidian norm of $(y_1 - y_2)$.

Without claiming that the game is a mechanism that implements the solution of the model, but rather for the sake of exposition, we call the two players "the assembly of

[14] If Z_p^S is a convex set, pure strategy S-wealth maximizing production plans *do* exist and they coincide with the set of S-efficient production plans.

[15] Note also that firm's preferences over production plans need not be acyclic or convex.

shareholders" and "the manager". The assembly of shareholders and the manager simultaneously choose a production plan. The assembly of shareholders chooses the production plan that maximizes their aggregate wealth, while the manager's objective is to match shareholders' choice.

A pure-strategy Nash equilibrium of Γ_p is (y^*, y^*), such that y^* is an S-wealth maximizing plan. Reciprocally, if y^* is an S-wealth maximizing plan given the price selection p, then (y^*, y^*) is a pure-strategy Nash equilibrium of the game Γ_p.

If p were a continuous price selection, both payoff functions would be continuous. However, the payoff of the shareholders' assembly may not be quasi-concave and thus existence of a pure strategy Nash equilibrium of Γ_p is not guaranteed. Conditions (on the primitives of the model) that would insure quasi-concavity of Φ_p^1 are not known. However, as long as p is continuous, the game Γ_p has a Nash equilibrium in mixed strategies.

Nevertheless, it is well known that P may not admit *any* continuous price selection. Thus Φ_p^1 may be discontinuous and, in this case, even the existence of mixed-strategy equilibria becomes non-trivial.

Since, in general, P is not single valued, a selection has to be made. Any such selection can be interpreted as representing the firm's *belief* about the prevailing market prices. We see no compelling reason to restrict the firm's (or shareholders') beliefs to those that put the entire mass on a single point (i.e., to Dirac measures). Instead, the firm can have *any* belief over the possible market prices, i.e., any probability measure over the set of measurable selections from P. Given such belief, the shareholders' objective is that of maximizing their *expected* aggregate wealth. As with single price selections, different beliefs generate different S-wealth maximizing plans, but rather than being chosen arbitrarily, the firm's beliefs will be part of the solution of the model. The details of these refinements of the model are presented in the sequel.

Let $\mathcal{M}$ be the space of all bounded and measurable functions from $\widehat{Y}$ to $\mathbb{R}_+^L$ endowed with the pointwise convergence topology (or product topology). Let $\mathcal{B}(\mathcal{M})$ denote the Borel sigma-algebra of $\mathcal{M}$.

Define $\mathcal{P}$ to be the set of all measurable selections from P. If $\widehat{Y}$ is compact, P has compact values and thus $\mathcal{P} \subseteq \mathcal{M}$ (see the appendix for a proof). On $\mathcal{P}$ consider the σ-algebra induced by $\mathcal{B}(\mathcal{M})$.

Definition 10 *A belief is any probability measure μ over $\mathcal{P}$.*

A mixed strategy for the firm is any probability measure σ over $\widehat{Y}$.

If σ is a mixed strategy for the firm and μ a belief, let $p(\sigma) := p(\int_{\widehat{Y}} y d\sigma(y))$ and E_μ be the expectation operator with respect to the probability μ.

Definition 11 *An equilibrium for the monopolistic economy $\mathcal{E}$ consists of a system of beliefs μ^*, together with a (mixed) strategy σ^* for the firm such that, for every $y^* \in$ supp(σ^*),*

$$E_\mu(W_p^S(p(\sigma^*), y^*)) \geq E_\mu(W_p(p(\sigma^*), y), \tag{7}$$

for all $y \in \widehat{Y}$.

Note that the two values of the expectation operator in (7) are well defined, because, given the pointwise convergence topology on $\mathcal{P}$, the mapping $\mathcal{P} \ni p(\cdot) \longmapsto p(y) \in \mathbb{R}^L$ is continuous, and thus integrable, for every y.

Theorem 12 *If $\widehat{Y}$ is closed and convex, and consumers' utilities are strictly increasing in every component, then a monopolistic equilibrium exists.*

Proof. S-wealth maximizing production plans corresponding to a certain belief μ are Nash equilibrium strategies of the game $\Gamma_\mu = \left(\widehat{Y}, \widehat{Y}, \varphi_\mu^1, \varphi^2\right)$, where $\varphi_\mu^1(y_1, y_2) = \int_{\mathcal{P}} p(y_2) D^S(p(y_1), y_1) d\mu(p)$ and $\varphi^2(y_1, y_2) = -\|y_1 - y_2\|^2$. An equilibrium of the monopolistic economy $\mathcal{E}$ exists if and only if there exists a probability μ such that the game Γ_μ has a Nash equilibrium.

Define the discontinuous game with endogenous sharing rule (see [11] for a definition), $\Gamma := \left(\widehat{Y}, \widehat{Y}, \Phi^1, \varphi^2\right)$ where $\Phi^1 : \widehat{Y} \times \widehat{Y} \rightrightarrows \mathbb{R}$, is defined by

$$\Phi^1(y_1, y_2) = \left\{\varphi_\mu^1(y_1, y_2) \mid \mu = \text{probability measure over } \mathcal{P}\right\}. \tag{8}$$

Note that every belief μ generates the measurable selection φ_μ^1 from Φ^1 and, reciprocally, for every measurable selection φ^1 from Φ^1 there exists a probability measure μ on $\mathcal{P}$ such that $\varphi^1 = \varphi_\mu^1$. Thus the monopolistic economy $\mathcal{E}$ has an equilibrium if and only if the game Γ has a solution.[16] We show that all hypotheses of the main theorem in [11] are satisfied.

1. $\widehat{Y}$ is compact:

Since the set is closed we only need to prove that it is bounded. Suppose it is not. Then there exists a sequence $(y^n)_n \subseteq \widehat{Y}$ such that $\|y^n\| > n$, $\forall n \geq 1$. Convexity of $\widehat{Y}$ together with $0 \in \widehat{Y}$ implies [17] $\frac{1}{\|y^n\|} y^n + \left(1 - \frac{1}{\|y^n\|}\right) 0 \in \widehat{Y}$, for all $n \geq 2$.

[16] According to Simon and Zame [11] the game Γ is said to have a solution if and only if there exists a measurable selection φ^1 from Φ^1 such that the normal-form game $\left(\widehat{Y}, \widehat{Y}, \varphi^1, \varphi^2\right)$ has a Nash equilibrium.

[17] $0 \in \widehat{Y}$ holds because $\omega^i \in \mathbb{R}_+^L$ for every i, and $\sum_{i \in I} \omega^i \gg 0$ (see, for example [10]).

Since $\left\|\frac{1}{\|y^n\|}y^n\right\| = 1$, we can assume, without loss of generality, that $\frac{1}{\|y^n\|}y^n \to y^0 \in \mathbb{R}^L$, with $\|y^0\| = 1$. $\widehat{Y}$ closed implies then that $y^0 \in \widehat{Y}$. On the other hand, since $P(y) \neq \emptyset$ $\forall y \in \widehat{Y}$, $\widehat{Y} \subseteq \{y \geq -\omega\}$. Thus $\lim_{n\to\infty} \frac{1}{\|y^n\|}y^n \geq -lim_{n\to\infty}\frac{\omega}{\|y^n\|} = 0$ and $y^0 = 0$, which contradicts $\|y^0\| = 1$.

2. Φ^1 is upper hemicontinuous with compact and convex values.

To show this we prove that

$$\Phi^1(y_1, y_2) = \text{co } \left\{qD^S(r, y_1) \mid q \in P(y_2), r \in P(y_1)\right\}, \tag{9}$$

and the correspondence $Q : \widehat{Y} \times \widehat{Y} \rightrightarrows \mathbb{R}$,

$$Q(y_1, y_2) := \left\{qD^S(r, y_1) \mid q \in P(y_2), r \in P(y_1)\right\}$$

is upper hemicontinuous with compact values. We denote by co$X$ the convex hull of the set X (i.e., the smallest convex set that contains X).

2a. $\Phi^1(y_1, y_2) = \text{co } \left\{qD^S(r, y_1) \mid q \in P(y_2), r \in P(y_1)\right\}$

We prove that the double inclusion holds.

Take $\varphi^1_\mu(y_1, y_2) \in \Phi^1(y_1, y_2)$ arbitrary. On $P(y_1) \times P(y_2)$ with the product σ-algebra define the measure $\mu_{(y_1,y_2)}$ such that

$$\mu_{(y_1,y_2)}(A \times B) := \mu\{p(\cdot) \in \mathcal{P} \mid (p(y_1), p(y_2)) \in A \times B\}$$

for every A, B measurable sets in the σ-algebra of $P(y_1)$, respectively $P(y_2)$. Then

$$\varphi^1_\mu(y_1, y_2) = \int_{P(y_1)\times P(y_2)} qD^S(r, y_1)d\mu_{(y_1,y_2)}(r, q)$$

and clearly $\varphi^1_\mu(y_1, y_2) \in \text{co } Q(y_1, y_2)$.

To prove the opposite inclusion is enough to show that $Q(y_1, y_2) \subseteq \Phi^1(y_1, y_2)$, because Φ^1 is convex-valued. Let then $qD^S(r, y_1) \in Q(y_1, y_2)$, with $q \in P(y_2), r \in P(y_1)$. Notice that there exists a measurable selection p from P such that $p(y_1) = r$ and $p(y_2) = q$.[18] Define $\mu = \delta_p$ to be the Dirac measure that puts its entire mass on p. Then $qD^S(r, y_1) = \varphi^1_\mu(y_1, y_2) \in \Phi^1(y_1, y_2)$.

2b. Q is upper hemicontinuous with compact values.

To show this we use the following lemma whose proof is given in the appendix:

[18] Indeed, to construct such a selection is enough to start with an arbitrary one and modify its values in y_1 and y_2 to become r and respectively q. The transformed function will still be measurable, because its inverse images differ from the inverse images of the original measurable selection only by a subset (possibly $\emptyset$) of $\{y_1, y_2\}$.

Lemma 13 *Under the assumptions of the theorem, P is upper hemicontinuous and has compact values.*

Since P is upper hemicontinuous and compact valued and the mapping $(q, r, y_1) \longmapsto qD^S(r, y_1)$ is continuous, Q is upper hemicontinuous too. Since Q is upper hemicontinuous and compact valued, so is co$Q$ and thus Φ^1.

On the other hand, $\varphi^2(y_1, y_2) = -\|y_1 - y_2\|^2$ is continuous and single-valued (and thus convex and compact valued). Hence all the hypotheses of the main theorem in [11] are satisfied, and thus there exists a belief μ such that the game Γ_μ has a (mixed strategy) Nash equilibrium. In consequence, an equilibrium for the monopolistic economy exists. ■

6 The Oligopoly Game with S-wealth Maximizing Firms: Existence of Equilibria

Consider now an economy with J monopolistic firms and I consumers. Let Y_j be firm j's production set $(j \in \{1, .., J\})$, and $(\mathbb{R}^L_+, u^i, \omega^i, \left(\theta^i_j\right)_{j=1..J})$ be consumers i's characteristics $(i \in \{1, .., I\})$. u^i and ω^i are defined as before and $\theta^i_j \in [0, 1]$ denotes consumer $i's$ share in firm $j's$ profits, where $\sum_{i\in I} \theta^i_j = 1$, $\forall j \in \{1, .., J\}$. For every $j \in \{1, .., J\}$ production set Y_j is assumed to satisfy conditions (a) and (b) of section 2. Define S^j to be the set of firm $j's$ shareholders, i.e., $S^j = \left\{i|\theta^i_j > 0\right\}$, and let $Y = \prod_{j\in J} Y_j$.

Let $\widehat{Y}$ be the set of those production plans $y = (y_1, y_2, ...y_J) \in Y$ for which a competitive equilibrium of the pure exchange economy $\mathcal{E}_y = \left\{(\mathbb{R}^L_+, u^i, \widetilde{\omega}^i)_{i\in I}\right\}$ with endowments $\widetilde{\omega^i} := \omega^i + \sum_j \theta^i_j y_j$ exists. Let also $P(\cdot)$ be the equilibrium price correspondence; P is defined on Y but it is non-empty valued only on $\widehat{Y} \subseteq Y$.

Let $\mathcal{P}$ be the set of all measurable selections from P endowed, as before, with Borel sigma-algebra generated by the topology of pointwise convergence. Firms' (common[19]) beliefs over equilibrium prices are described by a probability measure μ on $\mathcal{P}$.

Denote by $\widehat{Y_j}$ the projection of $\widehat{Y}$ on the $j's$ component. Depending on the production sets, it may be that $\widehat{Y} \neq \prod_{j\in J} \widehat{Y_j}$ and thus the strategy space of each player may depend on the actions chosen by the others. For example, given y_{-j} (i.e. actions chosen by all firms except j), the strategy space of firm j is $\widehat{Y_j}(y_{-j}) = \left\{y_j \in \widehat{Y_j} \mid (y_j, y_{-j}) \in \widehat{Y}\right\}$.

For different vectors y_{-j} these sets are in general different, and they can even be empty. Defining a pure-strategy Nash equilibrium for such a generalized game poses no problems. However, as seen before, even a single player's problem may not have a solution

[19]The assumption of identical beliefs is imposed only because it seems natural. However, the analysis of this section carries through even if firms hold different beliefs.

in pure strategies. Since players' strategy sets are not independent, it is not clear what a mixed strategy would mean for such a game. Hence, the standard Nash equilibrium concept is not appropriate in this situation; one has to allow for some coordination among firms' actions and maybe to design a dynamic game in order to accommodate for firms' dependent strategy spaces. We will comment more on this in the conclusions. For now, we are going to adopt the (implicit) assumption used in the rest of the literature on oligopolistic competition and avoid this difficulty by requiring that $\widehat{Y} = \prod_{j \in J} \widehat{Y}_j$. This could happen if, for example, firms have bounded below production sets, with bounds tight enough to ensure $\omega^i + \sum_j \theta_j^i y_j > 0$ for all i and j.

Definition 14 *Given the belief μ and choices y_{-j}^* of all the firms except j, an $S-$wealth maximizing production plan for firm j is $y_j^* \in \widehat{Y}_j$ that satisfies*

$$\int_{\mathcal{P}} p(y^*) D^{Sj}(p(y^*), y^*) d\mu(p) \geq \int_{\mathcal{P}} p(y^*) D^{Sj}(p(y_j, y_{-j}^*) y_j, y_{-j}^*) d\mu(p), \quad \forall y_j \in \widehat{Y}_j,$$

where $y^* = \left(y_j^*, y_{-j}^*\right)$.

Hence firms compete in a Cournot fashion, each of them taking the actions of the others as given when making their own production decisions.

Following the same reasoning as in the previous section, the equilibria of the Cournot game among the J S-wealth maximizing firms are in a bijective correspondence with the Nash equilibria of the $(2J)$-player normal form game, Γ_μ in which:

1. all players have the same action set, $\widehat{Y}_j$,
2. the players' payoff functions are:

$$\begin{aligned} \varphi_\mu^{2j}(y, z) &= \int p(z_j, y_{-j}) D^S(p(y_j, y_{-j}), y_j, y_{-j}) d\mu(p) \\ \varphi^{2j+1}(y, z) &= -\|z_j - y_j\|^2. \end{aligned}$$

Theorem 15 *If $\widehat{Y}$ is closed and convex and consumers' utilities are strictly increasing in every component, then there exists a system of beliefs μ and an associated Nash equilibrium for the Cournot game among firms.*

The proof of this theorem is very similar with that of theorem (12) and will be omitted.

7 Conclusions

This paper has proposed a model of imperfect competition among privately owned firms that act in the best interest of their shareholders. The existence of a solution for the model was proved under weaker conditions than the ones generally used in the literature. In particular, the results did not require the existence of a continuous equilibrium price perception or concavity assumptions on the profit function. However, the solution may exist *only* in mixed strategies.

The model also pointed out an interesting relation between the equilibria of these oligopolistic markets and Walrasian equilibria of some non-convex production economies. It can be shown (following a construction procedure similar to the one developed in Section 3) that oligopolistic markets in which firms choose $S-$wealth maximizing strategies are in fact "equivalent" to competitive economies that exhibit some sort of production externalities (and thus non-convexities).

A difficulty that was assumed away here, as well as in the rest of the literature, is the non-Cartesian structure of the aggregate strategy space, or, in other words, the dependence of one firm's strategy space on the actions taken by the other firms. We think this calls for a more sophisticated structure of the model, and maybe a dynamic one. To keep the static framework, one may get some help from the mechanism design literature. One way out seems to be to "design" a different market mechanism that firms follow. Say that each firm j has an "abstract" strategy space S^j and there is an outcome function

$$\tau : \prod S^j \to \widehat{Y}$$

that associates to each $J-$tuple of strategies an equilibrium compatible production plan. Then, as long as the strategy spaces S^j are compact metric spaces and τ is continuous, the existence of an equilibrium in mixed strategies can be proved as above.

A Appendix

Proof of lemma 13. The idea of the proof is to use upper hemicontinuity of the demand correspondence. It is known, however, that this correspondence may fail to be upper hemicontinuous at pairs (y, p) that generate zero wealth. To overcome this difficulty, we define for each consumer a smoothed demand correspondence (as in Debreu [3]) that coincides with his/her demand correspondence whenever the zero wealth situation does not arise, and is everywhere upper hemicontinuous. A notion of equilibrium consistent with the smoothed demand correspondence is that of a quasi-equilibrium. It is shown that under the hypotheses of the theorem, for each economy $\mathcal{E}_y$ the set of quasi-equilibria coincides with the set of Walras equilibria.

Let $y \in Y$ and $\mathcal{P}_y := \left\{p \in \Delta^{L-1} \mid p\omega^i + \theta^i py \geq 0 \;\; \forall i = 1..I\right\}$ be the set of price systems that generate non-negative wealth for all consumers. Clearly $P(y) \subseteq \mathcal{P}_y$, $\forall y \in \widehat{Y}$. Denote by $\mathcal{R}$ the correspondence $\mathcal{R} : \widehat{Y} \rightrightarrows \Delta^{L-1}$, $\mathcal{R}(y) = \mathcal{P}_y$ and note that $\mathcal{R}$ is closed, and non-empty, compact and convex valued.

The proof will be done in 4 steps.

Step 1: The set of Walras equilibria coincides with the set of quasi-equilibria for $\mathcal{E}_y$.

Following [3] we say that:

Definition 16 $\left(\overline{p}, \left(\overline{x}^i\right)_{i=1}^I\right) \in \mathcal{P}_y \times \mathbb{R}_+^{LI}$ *is a quasi-equilibrium for* $\mathcal{E}_y$ *if and only if:*

1. $\forall i = 1..I$, $\overline{x}^i \in \arg\max\left\{u^i(x) \mid x \in \mathbb{R}_+^L,\ \overline{p}x \leq \overline{p}\omega^i + \theta^i \overline{p}y\right\}$ *or* $\overline{p}\overline{x}^i = \overline{p}\omega^i + \theta^i\overline{p}y = 0$,

2. $\sum_{i=1}^I \overline{x}^i = y + \sum_{i=1}^I \omega^i$.

It is clear that every Walras equilibrium is a quasi-equilibrium. Let now $\left(\overline{p}, \left(\overline{x}^i\right)_{i=1}^I\right)$ be a quasi-equilibrium for $\mathcal{E}_y$. Clearly, if $\overline{p}\omega^i + \theta^i\overline{p}y > 0$ for every $i \in I$, $\left(\overline{p}, \left(\overline{x}^i\right)_{i=1}^I\right)$ is also a Walras equilibrium. If, $\overline{p}\omega^i + \theta^i\overline{p}y = 0$ for some i, we distinguish two cases.

Case 1: $\overline{p} \in Int\left(\Delta^{L-1}\right)$

In this case, $\left\{x \in \mathbb{R}_+^L \mid \overline{p}x = 0\right\} = \{0\}$ and thus

$$\overline{x}^i = 0 = \arg\max\left\{u^i(x) \mid x \in \mathbb{R}_+^L,\ \overline{p}x \leq \overline{p}\omega^i + \theta^i\overline{p}y\right\}.$$

Case 2: $\overline{p} \in \partial\left(\Delta^{L-1}\right)$.

In this case, $\overline{p}\omega^j + \theta^j\overline{p}y = 0 \;\; \forall j \in I$, for otherwise any consumer j with $\overline{p}\omega^j + \theta^j\overline{p}y > 0$ would demand infinite amounts of the goods whose prices are zero, and thus no quasi-equilibrium could exist. But then, $\overline{p}\left(\sum_{j=1}^I \omega^j + y\right) = 0$ and since $\sum_{j=1}^I \omega^j + y \gg 0$, we get $p = 0$ which contradicts $p \in \Delta^{L-1}$.

Next, we introduce a family of truncated economies whose quasi-equilibria coincide with the quasi-equilibria of $\left(\mathcal{E}_y\right)_{y \in Y}$ and whose aggregate smoothed demand correspondences are upper hemicontinuous everywhere with non-empty, convex and compact values.

Definition 17 *For every $y \in Y$ define the truncated economy* $\widehat{\mathcal{E}_y} = \left(\left(\widehat{X}, u^i, \omega^i + \theta^i y\right)_{i=1}^{I}\right)$, *where* $\widehat{X} = \left\{x \in R^L \mid 0 \leq x \ll \omega + y + 1\right\}$. *For every $(y,p) \in Graph(\mathcal{R})$ let $\widehat{B^i_y}$ and $\widehat{x^i_y}$ be the budget constraint and , respectively, the smoothed demand correspondence of the i^{th} consumer in $\widehat{\mathcal{E}_y}$:*

$$\begin{aligned}\widehat{B^i_y}(p) &= \left\{x \in \widehat{X} \mid px \leq p\omega^i + \theta^i py\right\} \\ \widehat{x^i_y}(p) &= \begin{cases} \arg\max\left\{u^i(x) \mid x \in \widehat{B^i_y}(p)\right\} & \text{if } p\omega^i + \theta^i py > 0 \\ \left\{x \in \widehat{X} \mid px = 0\right\} & \text{if } p\omega^i + \theta^i py = 0 \end{cases}.\end{aligned}$$

Step 2: Every quasi-equilibrium of $\widehat{\mathcal{E}_y}$ is a quasi-equilibrium of $\mathcal{E}_y$. Suppose the statement is not true. Let then $\left(\overline{p}, \left(\overline{x}^i\right)_{i=1}^{I}\right)$ be a quasi-equilibrium of $\widehat{\mathcal{E}_y}$ that is not quasi-equilibrium of $\mathcal{E}_y$. Then $\exists i$ and $\exists \overline{\overline{x}}^i \in \mathbb{R}^L_+$ such that $\overline{p}\overline{\overline{x}}^i \leq \overline{p}\omega^i + \theta^i \overline{p}y$ and $u^i(\overline{\overline{x}}^i) > u^i(\overline{x}^i)$. Strict quasi-concavity of u^i implies then that $u^i(\alpha\overline{\overline{x}}^i + (1-\alpha)\overline{x}^i) > u^i(\overline{x}^i)$, $\forall \alpha \in (0,1)$. But $\overline{p}\left(\alpha\overline{\overline{x}}^i + (1-\alpha)\overline{x}^i\right) \leq \overline{p}\omega^i + \theta^i\overline{p}y$ and, for α sufficiently close to 0, $\alpha\overline{\overline{x}}^i + (1-\alpha)\overline{x}^i \ll \omega + y + 1$. These inequalities contradict the maximality of $\overline{x}^i$ in $\widehat{B^i_y}(\overline{p})$.

Step 3: $\widehat{x^i} : Graph(\mathcal{R}) \rightrightarrows \widehat{X}$, $\widehat{x^i}(y,p) \overset{def}{=} \widehat{x^i_y}(p)$ *is upper hemicontinuous with non-empty, compact and convex values.*

If $(y,p) \in Graph(\mathcal{R})$ are such that $p\omega^i + \theta^i py = 0$, $\widehat{x^i_y}(p)$ is obviously non-empty, convex and compact. If $(y,p) \in Graph(\mathcal{R})$ are such that $p\omega^i + \theta^i py > 0$, $\widehat{B^i_y}(p)$ is a non-empty, convex and compact set and, since u^i is continuous and strictly quasi-concave, $\widehat{x^i_y}(p)$ will be a singleton (thus convex and compact).

Since $\left\{x \in R^L \mid 0 \leq x \leq \omega + 1\right\}$ is a compact set, to prove upper hemicontinuity it is enough to show that $\widehat{x^i}$ has closed graph.

Let then $(y^n, p^n, x^n) \to (y^0, p^0, x^0)$ in $Graph(\mathcal{R}) \times \left\{x \in R^L \mid 0 \leq x \leq \omega + 1\right\}$, such that $x^n \in \widehat{x^i}(y^n, p^n)$. We need to prove that $x^0 \in \widehat{x^i}(y^0, p^0)$. If $(y^0, p^0) \in Graph(\mathcal{R})$ is such that $p^0\omega^i + \theta^i p^0 y^0 = 0$, the conclusion is trivial. If $p^0\omega^i + \theta^i p^0 y^0 > 0$ the conclusion follows from upper hemicontinuity of $\widehat{x^i}$ at (y^0, p^0) (see, for example [3]).

Step 4: P is upper hemicontinuous with compact values

It is enough to show that P has closed graph. Let then $(y^n, p^n) \to (y^0, p^0)$ in $Y \times \Delta^{L-1}$ with $p^n \in P(y^n)$ and $x^n = (x^n_1, x^n_2, ..., x^n_I)$ be the equilibrium allocation corresponding to (y^n, p^n). Then (p^n, x^n) is a quasi-equilibrium for the truncated economy $\widehat{\mathcal{E}_{y^n}}$ and thus $x^n_i \in \widehat{x^i}(y^n, p^n)$.

Upper hemicontinuity of $\widehat{x^i}$ implies that there exists a convergent subsequence of x^n_i, say $x^{n_k}_i \to x_i \in \widehat{x^i}(y^0, p^0)$.

Since $\sum_{i=1}^{I} x_i^{n_k} = y^{n_k} + \sum_{i=1}^{I} \omega^i$, $\lim_{k\to\infty} \sum_{i=1}^{I} x_i^{n_k} = \lim_{k\to\infty} y^{n_k} + \sum_{i=1}^{I} \omega^i$ and thus $\sum_{i=1}^{I} x_i = y^0 + \sum_{i=1}^{I} \omega^i$, which proves that $(x_i)_{i=1}^{I}$is a quasi-equilibrium for $\widehat{\mathcal{E}_{y^0}}$ and thus a Walras equilibrium allocation for $\mathcal{E}_{y^0}$, corresponding to the prices p^0. Therefore $p^0 \in P\left(y^0\right)$ and P has closed graph. Since Δ^{L-1} is compact, P is also upper hemicontinuous. On the other hand, P closed implies that P has closed values and thus compact (since they are subsets of Δ^{L-1}).

■

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