Innovation and horizontal differentiation in a continuous-time Hotelling model

Borghan Narajabad
Department of Economics
Rice University
borghan@rice.edu

Randal Watson
Department of Economics
University of Texas at Austin
watson@eco.utexas.edu

28 June 2008

Abstract

We study the relationship between innovation and horizontal differentiation in a continuous-time dynamic duopoly. At any instant firms compete in prices in a Hotelling game. They may be differentiated both in terms of product quality (for which consumers have homogeneous tastes) and product location (for which tastes are heterogeneous). We consider in turn: (a) a game in which firms invest to improve quality (with fixed locations), and (b) a game in which they invest to relocate products (with an exogenous rate of quality improvement). Spillovers keep the interfirm quality differential from becoming too large. Symmetric Markov-perfect equilibria in the game with fixed locations show a relationship between taste heterogeneity and quality improvement that is U-shaped, in contrast with previous literature. In the game of endogenous product locations we show conditions for uniqueness of the symmetric MPE, and use this result to examine long-run impacts of changes in parameters, e.g., in the costs of product relocation.

1 Introduction

In this paper we study the relationship between competition and innovation in a continuous-time dynamic duopoly in which the firms' products are differentiated in more than one dimension. Specifically we allow products to be distinguished both vertically, by quality (which is viewed homogeneously by all consumers), and horizontally, by a location in a space of idiosyncratic consumer tastes. Firms may invest in changing their products' characteristics over time. We examine the interaction between such investments and exogenous features of the environment, asking, for example, how the heterogeneity in consumer tastes affects the rate at which firms improve product quality.

The large theoretical literature on the relationship between competition and innovation does not devote much explicit attention to the issue of horizontal differentiation.[1] Our model is most closely related to the frameworks used in Aghion and Howitt (1997), and Aghion, Harris, Howitt and Vickers (2001). Those papers studied innovation, interpreted as investment in cost reduction, in Markov-perfect equilibria of continuous-time infinite-horizon dynamic duopolies. At any instant firms receive a flow of profits from instantaneous price competition, taking as given the current state of their relative costs. Simultaneously firms may invest in cost reduction, in the hope of improving their future relative cost position. In the former paper the authors considered innovation by producers of homogeneous goods, comparing for example outcomes under conditions of Cournot and Bertrand competition. In the latter paper Aghion et al. extended the framework to incorporate product differentiation, using a representative-agent CES model of consumer preferences. That extension in particular considered the relationship between the endogenous rate of innovation and the exogenous elasticity of substitution. They found that (under certain conditions) as the two products become more substitutable the innovation rate could either increase monotonically, or first increase and then decrease, exhibiting an inverse-U shape.

Our basic model is a variation on these frameworks. We assume that the industry is a duopoly in which each firm sites its single product at either end of a Hotelling interval, over which consumer tastes are uniformly distributed. Products are horizontally differentiated if they are sited at different endpoints and if consumers face non-negligible transport costs. These

[1] For general surveys of this literature, see Van Cayseele (1998) and Gilbert(2006).

exogenous transport costs are our measure of 'taste heterogeneity', or 'product substitutability'. Firms may also be differentiated by a quality indicator, for which consumers by assumption have homogeneous tastes. If the market is covered the profits from Hotelling price competition depend only on the difference in these qualities, not their absolute magnitudes. This difference then becomes the industry's quality state. As in Aghion and Howitt (1997), for tractability we use the possibility of R&D spillovers to restrict this quality differential to just three values: a firm can be one step ahead, neck-and-neck (equal qualities), or one step behind.[2]

The difference between our model and the differentiated-product framework in Aghion et al. (2001) lies in the specification of consumer preferences. Their paper uses representative-agent CES preferences, whereas here we adopt a heterogeneous-agent Hotelling specification. This approach allows us to explicitly locate each firm's product in the space of consumer tastes. It is then possible to allow endogenous changes in these locations, and to study how such product relocations interact with exogenous features of the environment. Ideally we would like to endogenize investment in both quality and product location simultaneously. Since this proves to be analytically intractable, we split the analysis into two parts, analyzing investment in each dimension separately.[3]

First we fix firms' locations at opposite endpoints of the Hotelling line, and allow them to invest just in improving quality. Solving for the symmetric Markov-perfect equilibrium in state-conditional investment choices, we compare the resulting patterns of innovation (defined as the long-run average rate of quality improvement) with those in Aghion et al. (2001). We find distinct differences with respect to the relationship between innovation and product substitutability. As noted above, in the CES model this relationship is monotonic increasing or inverse-U shaped as substitutability increases. Essentially the opposite result holds in our Hotelling model. As products become more substitutable (i.e., as consumer transport costs fall), the innovation rate always exhibits a strict U shape, never monotonically increasing or inverse-U shaped.

To see the reason for this pattern, consider increases in substitutability

[2] Aghion and Howitt actually work with cost differentials rather than quality differentials, but the two specifications are functionally equivalent.

[3] In a companion paper (Narajabad and Watson 2008) we use numerical techniques to analyze a dynamic duopoly with simultaneous investment in quality and horizontal differentiation. Time is discrete in that study, not continuous as here.

starting from a situation of highly differentiated products. Such increases create more aggregate surplus in the market, through falling transport costs. All of this extra surplus initially goes to consumers, due to the intensified price competition between firms. Although the incentives for investment by neck-and-neck firms increase somewhat, those for investment by quality laggards fall. This leads the industry to spend less time over the long run in the high-innovation state of neck-and-neck quality, and long-run average innovation thus falls. Investment incentives change significantly when substitutability rises to the point where the quality leader gets the whole market. Any further increases in substitutability then induce no competitive price response from a low-quality firm (whose equilibrium price is effectively stuck at zero), and therefore allow a high-quality firm to start extracting more surplus from consumers. As a consequence investment by both quality laggards and neck-and-neck firms turns up sharply, and long-run innovation starts to increase. Thus the overall nature of the substitutability-innovation relationship appears to be somewhat sensitive to the specification of consumer tastes.[4]

In the second part of our analysis we make the evolution of firms' qualities exogenous, and focus instead on endogenous choices of horizontal differentiation. For simplicity firms are still restricted to locating at the endpoints of the Hotelling line, but they may now invest in relocating their product from one endpoint to the other (at some cost). Their incentive to relocate depends on the current quality differential and the location of the rival. Bernard, Redding and Schott (2006) have recently documented the empirical relevance of such product relocations in the overall economy. Theoretical studies of their dynamic aspects are scarce. Static models can provide some intuition, by showing for example the range of quality differentials over which the leader seeks to duplicate the rival's product. However such models are silent about the industry's long-run distribution over differentiation states when firms can keep responding to their rivals' relocation decisions, which is inherently a dynamic question.

To analyze this long-run distribution we work once again in continuous time with an infinite horizon. We allow firms' relative product qualities to

[4]Quality leaders invest nothing in equilibrium – by assumption any further quality improvements they make would be dissipated as spillovers to the rival firm. Investment by laggards makes no direct contribution to innovation, because by assumption it does not advance the frontier quality level. Instead it advances innovation by sending the industry back to the neck-and-neck state sooner.

cycle randomly through the three states introduced previously, according to a predetermined hazard rate – a special case arises when qualities are exogenously fixed.[5] A firm's investment now determines a hazard rate with which it is successfully able to move locations to the opposite endpoint. Costs of such switches are allowed to be 'directional'. That is, we distinguish the costs of 'moving away', i.e., of developing a new product niche, from the costs of 'joining', i.e., of duplicating a rival's product. Such directional costs could for example reflect the impact of patent policies, which would affect the costs of moving away and the costs of joining differently.

The key contribution is then to show that under reasonable conditions this game has a unique symmetric MPE in state-conditional investments. (*Ex ante* it may not be obvious that this should be true – for example firms' relocation investments are strategic complements in some dimensions, and it is well known that such games often have multiple equilibria.) Sufficient conditions for uniqueness are that: (a) the exogenous hazard of a change in the quality state is not too high, and (b) the directional costs of location switching (for moving away and joining) are not too different from each other. Given uniqueness, effects of parameter changes on the industry's long-run distribution over location states are then in many cases straightforward extensions of intuitions from the static model. Suppose for example that there is sufficient taste heterogeneity to allow separated firms with asymmetric qualities to both make positive sales. As tastes become more heterogeneous a quality laggard's incentive to differentiate its product increases and a leader's incentive to chase the laggard's product is reduced. Then in equilibrium the long-run proportion of time spent in states of separation is increasing in taste heterogeneity.

Similarly, under certain conditions the long-run proportion of time spent at separate locations increases if the costs of moving away are reduced, or if the costs of joining are increased. To illustrate the use of such results we briefly consider the effects of such cost changes on consumer welfare. All else equal, consumers prefer a horizontally-differentiated duopoly over one with duplicate products if and only if taste heterogeneity (i.e., the cost of transport) is not too high. (Otherwise the local market power that firms exploit when they are differentiated leads to a loss in consumer surplus.)

[5]In line with the comments in footnote 2, such random fluctuations in qualities can alternatively be viewed as random fluctuations in cost, which naturally arise from, e.g., exogenous firm-specific variation over time in factor prices.

Then shifts in costs in the directions described above (perhaps as a result of changes in patent policy) increase consumer welfare if and only if taste heterogeneity is not too high.

The next section describes the underlying Hotelling model which generates firms' flow profits. Section 3 then analyzes a model of quality improvement, with fixed product locations, while section 4 treats the case of endogenous location choices with stochastic quality reversals. Some analysis of effects on consumer welfare is in section 5, and concluding comments are in section 6. The appendix contains proofs of the two principal propositions – proofs of other results are available on request.

2 Instantaneous flow profits

We start by explaining firms' instantaneous payoffs conditional on any state. These payoffs are the translation into continuous time of the stage-game payoffs from the discrete-time model of Narajabad and Watson (2008). For completeness we repeat the essential details of those state-conditional payoffs here. In later sections we introduce the laws of motion for state transitions, which depend on the particular specification of firms' investment technologies.

Two firms compete in a continuous-time Hotelling game with infinite horizon and discount rate $r > 0$. Each firm $i = 1, 2$ produces a single perishable good, characterized at any point in time by a location a_i on the unit interval (i.e., a position in horizontal characteristics space) and an idiosyncratic quality q_i. Locations and qualities may change over time. There is a unit mass of consumers uniformly distributed on $[0, 1]$. If a consumer located at $\tau \in [0, 1]$ purchases a good of quality q at price p from a firm located at $a \in [0, 1]$, he realizes an instantaneous net utility flow of

$$u = -\alpha(a - \tau)^2 + q - p \; .$$

Transport costs (a.k.a. 'taste heterogeneity') are represented by the parameter $\alpha > 0$.[6] Consumers' locations are held fixed, and they face no costs of switching between firms, so at any instant a consumer purchases the product which offers the highest net flow utility. To ensure that the market is always covered in equilibrium, we assume throughout that $q_i > 3\alpha$, $i = 1, 2$.

[6]The utility specification assumes quadratic transport costs – unless otherwise stated all results continue to hold with linear transport costs.

Firms' flow profits at any instant arise from the equilibrium of competition in prices, given current qualities and locations. For simplicity we restrict firms to locating at either end of the unit interval, i.e., $a_i \in \{0, 1\}$, $i = 1, 2$. The symmetry in the distribution of consumers then implies that the only locational factor directly affecting flow profits is whether or not the firms are currently operating at the same endpoint. We therefore invoke the 'payoff-relevant' state restriction inherent in the MPE concept (Maskin and Tirole 1988a,b) and summarize the industry's location state as $s \in S \equiv \{0, 1\}$, representing cases where firms are 'co-located' ($s = 0$, sharing the same endpoint) or 'separated' ($s = 1$, operating at different endpoints).

Let $d_i \equiv q_i - q_j$, $i, j = 1, 2$, $i \neq j$ denote firm i's advantage in product quality (which could be negative or positive). As a notational convention we will write each firm's actions and payoffs as functions of the quality differential *viewed from its own perspective*. Thus firm i's payoffs are a function of d_i, while firm j's payoffs are a function of $d_j \equiv -d_i$. Since we focus on symmetric equilibria we will often drop the firm subscript on the quality differential unless there is some risk of confusion.

Given a location state $s \equiv \{0, 1\}$, let π_d^s denote the flow profits from the Nash equilibrium of the simultaneous pricing game for a firm with a quality advantage of d units. For example π_{+k}^1 with $k > 0$ denotes the flow profits of a firm leading by k units in the quality race and operating at the opposite endpoint to its rival. Solving for the Nash equilibrium at all parameter values we arrive at:

a. if $d > 3\alpha$, $\pi_d^0 = d$, $\pi_d^1 = d - \alpha$

b. if $-3\alpha \leq d \leq 3\alpha$, $\pi_d^0 = \max(d, 0)$, $\pi_d^1 = \frac{\alpha}{2}[1 + \frac{d}{3\alpha}]^2$

c. if $d < -3\alpha$, $\pi_d^0 = 0$, $\pi_d^1 = 0$.

If $s = 0$ then there is no horizontal product differentiation and these expressions represent Bertrand profit outcomes. Cases (a) and (c) indicate that if $s = 1$ and $\mid d \mid > 3\alpha$ then the quality leader's advantage is such that it gets the whole market in equilibrium, even if the products are horizontally differentiated. Case (b) represents the situation of a smaller quality asymmetry, where if $s = 1$ then both firms get positive sales. A key property of these profit expressions is that firms' absolute quality levels q_i do not enter directly – since the market is covered in equilibrium, flow profits depend only on the *difference* in qualities.

From now on we restrict a firm's quality advantage to one of three states: where $k > 0$, we let $d \in \Delta \equiv \{-k, 0, +k\}$, respectively denoting a firm that is lagging, neck-and-neck, or leading in the quality race. Thus the industry's full state space, incorporating both horizontal differentiation and quality differences, comprises the six elements in $S \times \Delta$. It will be useful for the subsequent analysis to define expressions for between-state differences in flow profits. These differences determine firms' best responses in the dynamic games of investment, whether in quality improvement or location changes. For changes in flow profits between location states:

$$
\begin{aligned}
a^{01} &\equiv \pi^0_{+k} - \pi^1_{+k} = \begin{cases} \alpha & \text{if } k \geq 3\alpha \\ k - \frac{\alpha}{2}\left[1 + \frac{k}{3\alpha}\right]^2 & \text{if } 0 < k < 3\alpha \end{cases} \\
b^{10} &\equiv \pi^1_0 - \pi^0_0 = \alpha/2 \quad \text{for all } \alpha \\
c^{10} &\equiv \pi^1_{-k} - \pi^0_{-k} = \begin{cases} 0 & \text{if } k \geq 3\alpha \\ \frac{\alpha}{2}\left[1 - \frac{k}{3\alpha}\right]^2 & \text{if } 0 < k < 3\alpha \end{cases}
\end{aligned}
$$

The first of these expressions represents the gain in flow profits realized by a quality leader who succeeds in joining his rival's location. The latter two expressions, b^{10} and c^{10}, respectively represent the gain in profits realized by neck-and-neck and lagging firms when they move to separate locations. Note that we have $b^{10} > 0$ for all $\alpha > 0$, and $a^{01} > 0$ if and only if $k > 3(2 - \sqrt{3})\alpha \approx 0.804\alpha$. We have $c^{10} > 0$ if $0 < k < 3\alpha$, and $c^{10} = 0$ if $k \geq 3\alpha$; in the latter case the leader's quality differential is such that the laggard makes zero profits regardless of its product location.

Whereas b^{10} and c^{10} are strictly increasing or constant in α at all α, a^{01} is increasing for $k > 3\alpha$ and decreasing for $k < 3\alpha$. This reflects the different margins faced by a horizontally differentiated quality leader when it gets the whole market ($k > 3\alpha$), compared to when it shares the market with its rival ($k < 3\alpha$). In the former case increases in α (i.e., greater taste heterogeneity) cause the leader to reduce price in order to retain its most distant customer. In contrast in the latter case increases in α lead to a higher equilibrium price (and higher profits) – the leader loses its most distant customers to the rival, but since the rival's price increases the leader is able to extract more surplus from its inframarginal consumers.

For changes in the quality state, fix $s = 1$, and consider the incremental flow profits that are realized when a firm catches up one rung on the quality ladder. If a laggard advances its quality differential from $d = -k < 0$ to

$d = 0$, it receives incremental profits of

$$K_L = \begin{cases} \frac{\alpha}{2} & \text{if } k > 3\alpha, \text{ or} \\ \frac{k}{6}\left[2 - \frac{k}{3\alpha}\right] & \text{if } k \leq 3\alpha. \end{cases}$$

If a neck-and-neck firm advances its quality differential from $d = 0$ to $k > 0$ it receives incremental profits of

$$K_H = \begin{cases} k - \frac{3\alpha}{2} & \text{if } k > 3\alpha, \text{ or} \\ \frac{k}{6}\left[2 + \frac{k}{3\alpha}\right] & \text{if } k \leq 3\alpha. \end{cases}$$

The incremental profits of the quality leader are decreasing in α, while those of the quality laggard are increasing in α, i.e., $dK_H/d\alpha < 0$ and $dK_L/d\alpha > 0$ for all α and $k > 0$. As noted in our companion paper and explained further below, the sum of these incremental profits plays an important role in determining long-run innovation rates. We term this sum the 'leader's surplus', equal to the difference between the maximum and minimum flow profits given the location state, and defined as:

$$\begin{aligned} LS^1 &= \pi^1_{+k} - \pi^1_{-k} \\ &= K_L + K_H = \begin{cases} \frac{2}{3}k & \text{if } k \leq 3\alpha, \text{ or} \\ k - \alpha & \text{if } k > 3\alpha. \end{cases} \end{aligned}$$

Note that LS^1 is independent of the taste heterogeneity α if $k \leq 3\alpha$, and decreasing in α otherwise.

3 Competition in quality improvement

In this section we allow firms to invest in improving quality, while holding the location state fixed at $s = 1$. Our quality investment game follows Aghion et al. (1997), with the distinction that our flow profits come from a Hotelling game (rather than Cournot or Bertrand), in which consumers have heterogeneous tastes for the firms' products. By varying α we study the relationship between that taste heterogeneity and the rate of quality improvement.

Improvements in each firm's quality q_i happen in incremental steps of size $k > 0$. The industry's state at any time t is the quality differential $d_i(t) \equiv q_i(t) - q_j(t)$. For simplicity we assume that this quality gap can never exceed one step. That is, if a firm that is already one step ahead makes a further quality improvement, the laggard automatically keeps up by copying the leader's previous technology, and the leader's quality advantage is then unchanged at the level of $+k$. Thus the state $d_i(t)$ is restricted to the 3-element set Δ defined previously.

At time t incremental improvements in i's quality arrive with Poisson hazard rate $w_i(t)$, which is chosen by the firm. The flow cost of a hazard rate w is $h(w) \equiv (\phi/2)w^2$, where $\phi > 0$ is a parameter indexing the difficulty of quality improvement. We look for a symmetric equilibrium in Markov strategies. Each firm's strategy is a state-conditional choice of hazard rate $w_i(d_i)$ (or $w_j(d_j)$). Given a strategy $w_j(d_j)$ for the rival, the Bellman equation for firm i's continuation value $V_i(d_i)$ may be derived as in Aghion and Howitt (1997). Where $k > 0$ we have, at any $d_i \in \Delta$:

$$\begin{aligned} rV_i(d_i) = \max_{w \geq 0} \{ & \pi^1_{d_i} - h(w) + w[V_i(\min(d_i + k, +k)) - V_i(d_i)] \\ & -w_j(-d_i))[V_i(d_i) - V_i(\max(d_i - k, -k))]\} \ . \end{aligned} \tag{1}$$

Thus a firm's continuation value is comprised of current flow profits, less the cost of investment, plus the expected gain in value arising from own quality investment, minus the expected loss as a result of the rival's investment $w_j(-d_i)$. The min and max operators take account of the upper and lower bounds on the quality differential d_i.

The value function $V_i(.)$ solving (1) is unique at any $w_j(d_j)$, and non-decreasing in d_i. Firm i's best response to $w_j(d_j)$ is the policy function $w_i(d_i)$ generated by this solution. This best response is unique at any $w_j(d_j)$ (because of the strict convexity of $h(w_i)$), and satisfies:

$$w_i(d_i) = \begin{cases} (V_i(d_i + k) - V_i(d_i))/\phi & \text{if } d_i = -k \text{ or } 0 \\ 0 & \text{if } d_i = +k \end{cases} \tag{2}$$

Note that in this continuous-time model the leader has no incentive to invest in improving quality, since the knowledge spillovers do not allow it to further increase its advantage over the laggard.

Henceforth we normalize the quality step k to $k = 1$, since increases in this quality step are equivalent to reductions in ϕ, the difficulty of quality improvement. In a symmetric MPE we require $w_i(d_i) = w_j(-d_i)$, at all $d_i \in \Delta$, $i, j = 1, 2$. That is, firms choose the same actions at any given level of quality advantage. It can be shown that a symmetric MPE exists and is unique at any parameter values. For brevity write $w_{-1} \equiv w_i(-k) \equiv w_j(-k)$

and $w_0 \equiv w_i(0) \equiv w_j(0)$ for the equilibrium hazard rates chosen by laggards and neck-and-neck firms respectively, in this symmetric MPE. (We know from (2) that leaders choose hazard rates of $w_{+1} \equiv 0$.)

Applying symmetry in the three equations defined in (1) (one for each state), and substituting from (2), we arrive after some additional working at expressions for w_0 and w_{-1} in terms of parameters:

$$w_0 = -r + \sqrt{r^2 + \frac{2}{\phi} K_H} \tag{3}$$

$$w_{-1} = -(w_0 + r) + \sqrt{(w_0 + r)^2 + w_0^2 + \frac{2}{\phi} K_L} \; . \tag{4}$$

These equilibrium hazard rates are equivalent to equations (10) and (11) in Aghion and Howitt (1997).

Long-run innovation is defined as the average rate of improvement in the frontier technology according to the equilibrium invariant distribution over states d_i. Since quality leaders invest nothing in equilibrium, the frontier technology actually only changes in the neck-and-neck state, advancing by one step when either firm registers a quality improvement. Conditional on being in this state, the equilibrium hazard rate of an advance in the frontier technology is then $2w_0$. The long-run probability of being in the neck-and-neck state under the equilibrium invariant distribution is $\Pr(d_i = 0) = w_{-1}/(w_{-1} + 2w_0)$, and thus the equilibrium long-run innovation rate (LRI) is

$$LRI = 2w_0 \Pr(d_i = 0) = \left[\frac{1}{w_{-1}} + \frac{1}{2w_0} \right]^{-1} \; , \tag{5}$$

which is of course monotonically increasing in w_{-1} and w_0. (Note that investment by laggards does not directly advance the frontier technology. Instead it affects LRI by sending the industry back to the neck-and-neck state sooner.)

Recall from section 2 that the leader's incremental flow profits, $K_H \equiv \pi^1_{+1} - \pi^1_0$, are declining in α at all α. It follows from (3) that w_0 is declining in α everywhere. That is, neck-and-neck firms in equilibrium always invest less as taste heterogeneity increases, because more heterogeneity increases firms' local market power. This reduces the potential gains from quality improvement because it reduces consumers' propensity to switch purchases to a firm with a new and improved technology.

The effect of greater taste heterogeneity on investment by laggards is more complicated. To analyze this effect we assume that firms are very patient. That is, we set $r = 0$ in (3) and (4); then by continuity the ensuing results will also hold for small $r > 0$. Under this restriction the direction of the effect of α on w_{-1} depends on the level of α. Recall from section 2 that when firms are horizontally separated, they share the market if and only if $\alpha > 1/3$ (i.e., iff $\alpha > k/3$, since $k = 1$ here). Otherwise the leader gets all the sales. It can be seen from (3) and (4) that if $r = 0$ and $\alpha < 1/3$ then we have $dw_{-1}/d\alpha < 0$. On the other hand if $r = 0$ and $\alpha > 1/3$ then we have $dw_{-1}/d\alpha > 0$. Quality investment by laggards overall has a U shape in α, with a minimum at $\alpha = 1/3$.

It follows that if firms are patient and taste heterogeneity is low ($\alpha < 1/3$) then laggards and neck-and-neck firms both invest less as heterogeneity increases, and LRI consequently must fall. If $\alpha > 1/3$ then w_0 and w_{-1} move in opposite directions as α increases (w_0 falling, w_{-1} rising) and then it is not clear just from examination of (5) which effect will dominate. After some additional work it turns out that the effect on laggards' investment dominates, leading to an overall U-shaped pattern of long-run innovation in α, formalized as follows:

Proposition 1 *If firms are sufficiently patient (i.e., if r is small enough):*

a. $dLRI/d\alpha < 0$ *for* $\alpha \in [0, 1/3)$

b. $dLRI/d\alpha > 0$ *for* $\alpha > 1/3$

c. LRI is minimized at $\alpha = 1/3$ *and maximized at* $\alpha = 0$.

For illustration figure 1 shows LRI as a function of α in the case of $\phi = 1$. A relevant point of comparison here is the analysis of Aghion et al. (2001). Those authors studied innovation in an infinite-horizon continuous-time duopoly in which the demand function is that of a representative consumer with CES preferences. By varying the elasticity of substitution in demand they study the effects on long-run innovation of more or less competition, in the form of more or less product substitutability. They find that as products become more substitutable, i.e., as the elasticity rises, the innovation rate either increases throughout, or first increases and then decreases, exhibiting an inverse-U shape. This shape is attributed to the interaction of two countervailing influences. According to the first, 'escape-competition' effect, neck-and-neck firms innovate more as products become more substitutable, because that increases the returns to quality leadership (in terms of incremental flow profits). Against this must be measured a second, 'composition' effect, which recognizes that innovation is most intense when firms are neck-and-neck, but that if such firms innovate more then the industry leaves this state quicker, spending relatively more time over the long run in states of asymmetric quality, where innovative effort is low. This second effect may sometimes dominate the escape-competition effect as products become more substitutable, leading to the aforementioned inverse-U relationship. In related empirical work Aghion, Bloom, Blundell, Griffith and Howitt (2005) document an inverse-U relationship between profit margins and patent indicators, which under certain assumptions could be interpreted as supporting these theoretical predictions.

The key difference in the present analysis is that demand is derived from a heterogeneous-agent Hotelling model, in which product substitutability is represented by the (inverse of the) transport-cost parameter α. It is clear from Proposition 1 that this amendment changes the predicted relationship between substitutability and innovation. In particular figure 1 shows that the patterns reported in Aghion et al. (2001) are exactly those which are *not* predicted by our model. We find that if firms are patient enough then the relationship is always strictly quasiconvex with interior minimum (i.e., U-shaped), never inverse-U shaped or monotone increasing in substitutability (i.e., never inverse-U shaped or monotone decreasing in α).

The reason for this difference originates in the behavior of incremental flow profits around the point $\alpha = k/3$. Consider reductions in α from a relatively high level in figure 1 (e.g., from $\alpha = 2.5$). As noted above, as α falls we initially have equilibrium neck-and-neck investment (w_0) increasing (since $dw_0/d\alpha < 0$ everywhere), and equilibrium investment by the laggard (w_{-1}) decreasing. As a result as α falls the industry spends more and more time in the asymmetric-quality state, where innovation is in fact zero. This leads to an initial fall in long-run innovation – that is, as α falls the 'composition effect' noted above dominates first.

When α reaches $k/3$ there is a sharp change in incentives. From that point on all sales go to the quality leader, and a laggard's competitive ability is essentially exhausted, in the sense that his equilibrium price is fixed at zero. Further falls in α produce no price response from this rival. They simply reduce the mismatch between consumer tastes and the characteristics of the leader's good, which allows that firm to start raising price, so as to extract more surplus from the inframarginal consumers. This enhances the incentives for investment by neck-and-neck firms. It also raises the laggard's investment incentives, which are now largely driven by the prospect of eventually becoming the quality leader. A laggard who becomes a leader realizes an overall gain in flow profits of $K_H + K_L \equiv LS^1$, the leader's surplus defined in section 2. As noted there, this quantity is constant in α for $\alpha > k/3$, but decreasing in α (i.e., increasing in substitutability) for $\alpha < k/3$. The upswing in LRI as α falls below $k/3$ then reflects an escape-competition effect, with both laggards and neck-and-neck firms investing more in response to the greater returns to quality leadership. Overall, as product substitutability increases from a low level, we see the competition effect dominating early and the escape-competition effect dominating later, which is the opposite order to that observed in the model of Aghion et al. (2001).[7]

In terms of cumulative effects, part (c) of Proposition 1 indicates that the escape-competition effect is predominant. That is (if the discount rate is low enough), long-run innovation is maximized in industries that produce homogenous products. This is reminiscent of a result in Aghion and Howitt (1997), who found that the innovation rate for patient firms is higher under Bertrand product-market competition than under Cournot, unless the fixed size of increments in the technology is large.

4 Competition in product locations

Turn now to a variation on the model in the previous section, in which firms' product locations are endogenous, but their qualities evolve according to an exogenous stochastic process. Time is continuous, the horizon is infinite, and once again two single-product firms face a unit mass of consumers uniformly distributed on $[0, 1]$. For simplicity we still restrict firms to locating at the endpoints of this interval, but at any point in time they may now invest in moving to the opposite endpoint, either to duplicate, or differentiate their product from, the rival's product. We will show that for some parameter values the equilibrium of this game sees continual movement between product locations, with high-quality firms attempting to suppress competition from rivals by duplicating their product, and low-quality rivals attempting

[7] Effects in the case of impatient firms (i.e., when r is not close to zero) are not immediately clear. We know that $dw_0/d\alpha < 0$ still holds, but we do not know the general shape of $dw_{-1}/d\alpha$ in that case.

to expand sales by moving into their own product niche.

As previously $d_i \in \Delta \equiv \{-k, 0, +k\}$ denotes firm i's quality state relative to its rival, and $s \in S \equiv \{0, 1\}$ represents the industry's location state, i.e., whether firms are co-located or separate. Location changes are the outcomes of independent Poisson processes with hazard rates controlled separately by each firm. Let y^s, z^s, x^s denote firm i's hazard rate of changing product locations (from one endpoint to the other) when the industry's current location state is $s \in S$, and when the current quality state is $d_i = -k, 0, +k$, respectively. Thus z^0, for example, is i's hazard rate of changing locations (or 'switching') when the firms are co-located and have equal qualities. (For simplicity we suppress the subscript i on y, z, x.)

The costs of inducing a given hazard rate of switching are assumed to depend on the location state s. That is, the costs of moving away from a rival are allowed to differ from the costs of moving toward a rival. This might for example reflect patent policies, changes in which could move the costs of duplicating a rival's product and the costs of creating a new product in opposite directions. Specifically, let the flow cost of implementing a given hazard rate of switching w be $g(w) \equiv \gamma^0 w^2/2$ in state $s = 0$, and $f(w) \equiv \gamma^1 w^2/2$ in state $s = 1$, where γ^0 and γ^1 are strictly positive and not necessarily equal.

While changes in product locations (i.e., in horizontal differentiation) are controlled by firms, changes in product qualities are here assumed to evolve exogenously. Let either firm's quality q_i advance by one step according to a Poisson process with fixed hazard rate $\tau \geq 0$. These processes are assumed to be independent; thus at any time t the quality differential $d_i(t)$ advances by one step with hazard rate τ (if $d_i = -k$ or 0), and declines by one step with hazard rate τ (if $d_i = 0$ or $+k$), reflecting respectively an increment in firm i's quality and an increment in firm j's quality. In the special case of $\tau = 0$, firms' qualities are permanently fixed (or, alternatively, always advance in lockstep so that the quality differential never changes).

A firm's Markov strategy is now a choice of a vector $(y^0, y^1, z^0, z^1, x^0, x^1) \in R^6_+$, representing a switching hazard conditional on each of the six possible states $(s, d) \in S \times \Delta$. Once again we look for a symmetric equilibrium in such strategies, where symmetry means that i's action in state (s, d) is the same as j's action in state $(s, -d)$. Borrowing notation from Aghion and Howitt (1997), fix the rival firm j's strategy at $(\overline{y}^0, \overline{y}^1, \overline{z}^0, \overline{z}^1, \overline{x}^0, \overline{x}^1)$, and consider firm i's best response. Let V^s_d denote firm i's continuation value in location states $s = 0, 1$ when it has quality advantage d. Define the differences in these values as $\Delta V^{10}_{-1} \equiv V^1_{-1} - V^0_{-1}$, $\Delta V^{10}_0 \equiv V^1_0 - V^0_0$, $\Delta V^{01}_{+1} \equiv V^0_{+1} - V^1_{+1}$.

Then i's best response will be the policy function generated by the solution to the following Bellman equation:

$$\begin{aligned}
rV^0_{+1} &= \pi^0_{+1} + \max_{x^0\geq 0}[-x^0\Delta V^{01}_{+1} - g(x^0)] - \overline{y}^0\Delta V^{01}_{+1} + \tau(V^0_0 - V^0_{+1}) & (6)\\
rV^1_{+1} &= \pi^1_{+1} + \max_{x^1\geq 0}[x^1\Delta V^{01}_{+1} - f(x^1)] + \overline{y}^1\Delta V^{01}_{+1} + \tau(V^1_0 - V^1_{+1}) & (7)\\
rV^0_0 &= \pi^0_0 + \max_{z^0\geq 0}[z^0\Delta V^{10}_0 - g(z^0)] & \\
&\quad + \overline{z}^0\Delta V^{10}_0 + \tau(V^0_{+1} - V^0_0) - \tau(V^0_0 - V^0_{-1}) & (8)\\
rV^1_0 &= \pi^1_0 + \max_{z^1\geq 0}[-z^1\Delta V^{10}_0 - f(z^1)] & \\
&\quad - \overline{z}^1\Delta V^{10}_0 + \tau(V^1_{+1} - V^1_0) - \tau(V^1_0 - V^1_{-1}) & (9)\\
rV^0_{-1} &= \pi^0_{-1} + \max_{y^0\geq 0}[y^0\Delta V^{10}_{-1} - g(y^0)] + \overline{x}^0\Delta V^{10}_{-1} + \tau(V^0_0 - V^0_{-1}) & (10)\\
rV^1_{-1} &= \pi^1_{-1} + \max_{y^1\geq 0}[-y^1\Delta V^{10}_{-1} - f(y^1)] - \overline{x}^1\Delta V^{10}_{-1} + \tau(V^1_0 - V^1_{-1})\ . & (11)
\end{aligned}$$

Equation (7), for example, represents i's continuation value when it is the quality leader in state $s = 1$. The first two terms on the RHS represent the sum of current flow profits π^1_{+1}, plus the maximized expected gain in value from moving to the rival's location (thereby changing the location state to $s = 0$), net of the flow switching costs. The next two terms show the expected gain in value if the rival were to make an equivalent move (joining i's location), plus the expected loss in value if the rival realizes a random quality improvement (thereby changing the quality state to $d = 0$). Similar interpretations apply to the other equations. Note that if the current quality state is $d = 0$ (equations (8) and (9)) then either an increment or a decrement in d is possible, depending on who realizes the random quality improvement, with hazard rate τ in each case.

Given a strategy for the rival, it can be seen that the value function solving (6)-(11) is unique, and that it generates a unique policy function as i's best response. If the state is $(s = 1, d = +1)$, for example, then (7) implies that this best response must satisfy

$$x^1 = \max(0, \Delta V^{01}_{+1}/\gamma^1)\ .$$

It is easy to see that any best response only prescribes switching in one direction, given the quality state. That is, at any optimum we must have $x^1 > 0 \Rightarrow x^0 = 0$, $z^1 > 0 \Rightarrow z^0 = 0$, $y^1 > 0 \Rightarrow y^0 = 0$.

We continue to normalize the quality step k to one, since increases in k can equivalently be modelled as reductions in the switching-cost parameters γ^0 and γ^1. If $\alpha \geq k/(3(2-\sqrt{3})) \approx 1.244$, i.e., if taste heterogeneity is high, then the game has a trivial unique symmetric equilibrium in which firms try and separate from each other in all quality states, because separation always yields the highest flow profits. In this case the industry in the long run ends up in a state of permanent horizontal separation, regardless of its starting point.[8]

Consider then the case of taste heterogeneity that is not too high, i.e., $\alpha < 1.244$. In terms of the flow profit increments defined in section 2, we then have $a^{01} > 0$, $b^{10} > 0$, $c^{10} \geq 0$. That is, ignoring dynamic considerations, quality leaders now want to be co-located, while neck-and-neck firms and laggards still want to be separate. (Note that if $\alpha < 1/3$ then $c^{10} = 0$ because laggards get zero sales regardless of the location state.) If $\tau = 0$ then these short-run incentives carry over to the long run, because there is no prospect of reversals in the quality state. It can be shown in that case that i's best response to any rival's strategy always induces $\Delta V_{+1}^{01} > 0$, $\Delta V_0^{10} > 0$, $\Delta V_{-1}^{10} \geq 0$, which in turn implies $x^0 = 0$, $z^1 = 0$, $y^1 = 0$.

Using continuity in τ of the value function solving (6)-(11), it can be shown that these inequalities also hold for a range of small τ near zero:

Lemma 1 *For all $\alpha < 1.244$ and for all $r, \gamma^0, \gamma^1 > 0$ there is a $\overline{\tau}(\alpha) > 0$ that:*

- *a. depends only on $r, \alpha, \gamma^0, \gamma^1$, and*
- *b. is such that for all $\tau \in [0, \overline{\tau}]$, a best response to any strategy of the rival always induces $\Delta V_{+1}^{01} \geq 0$, $\Delta V_0^{10} \geq 0$, $\Delta V_{-1}^{10} \geq 0$.*

A proof of this lemma is available on request. The key point is that there is a set of small τ of non-zero measure over which we may restrict each firm's strategy space to be the set of triples (y^0, z^0, x^1). Furthermore the relevant non-empty set of τ can be defined independently of the rival's strategy. Accordingly we henceforth assume that $\alpha < 1.244$ and that τ is small. To write this concisely define $\overline{\tau}(\alpha) = 0$ at $\alpha = 1.244$ and $\overline{\tau}(\alpha) < 0$ at all $\alpha > 1.244$. Then $[0, \overline{\tau}(\alpha)]$ is non-empty iff $\alpha \leq 1.244$.

[8] In contrast with the discrete-time model in Narajabad and Watson (2008), there is no possibility here of the industry falling into a Pareto-dominated equilibrium where firms keep needlessly swapping locations even though they prefer to remain separated in all states.

If $\alpha < 1.244$ then a firm's continuation value in any state can never exceed π^0_{+1}/r, the present value of an uninterrupted stream of the maximum possible flow profits. The continuation value can also never fall below zero, since a firm can always guarantee itself a zero payoff by doing nothing. Therefore the value differences ΔV^{01}_{+1}, etc., are all bounded above, and hence in any state a firm i's optimal switching hazard rate is contained in the compact set $[0, \pi^0_{+1}/(r \min(\gamma^0, \gamma^1))]$, for all strategies of the rival. It is then straightforward to apply Lemma 1, the Theorem of the Maximum, and Brouwer's Fixed-Point Theorem to (6)-(11) to show that if $\tau \in [0, \overline{\tau}(\alpha)]$ then there exists a symmetric MPE that induces $\Delta V^{01}_{+1} \geq 0$, $\Delta V^{10}_{0} \geq 0$, $\Delta V^{10}_{-1} \geq 0$ for both firms, i.e., that induces $x^0 = 0$, $z^1 = 0$, $y^1 = 0$.

Taking differences in (6)-(11), and applying Lemma 1 and symmetry, we then arrive at the following necessary conditions for a symmetric MPE (y^*, z^*, x^*) when $\tau \in [0, \overline{\tau}(\alpha)]$:

$$
\begin{aligned}
(r + y^* + \tau)\Delta V^{01}_{+1} &= a^{01} - [x^* \Delta V^{01}_{+1} - f(x^*)] - \tau \Delta V^{10}_{0} && (12)\\
(r + z^* + 2\tau)\Delta V^{10}_{0} &= b^{10} - [z^* \Delta V^{10}_{0} - g(z^*)] - \tau \Delta V^{01}_{+1} + \tau \Delta V^{10}_{-1} && (13)\\
(r + x^* + \tau)\Delta V^{10}_{-1} &= c^{10} - [y^* \Delta V^{10}_{-1} - g(y^*)] + \tau \Delta V^{10}_{0}\ , && (14)
\end{aligned}
$$

where $y^* \equiv \Delta V^{10}_{-1}/\gamma^0$, $z^* \equiv \Delta V^{10}_{0}/\gamma^0$, $x^* \equiv \Delta V^{01}_{+1}/\gamma^1$, and where the ΔV's are derived from the solution to (6)-(11) given that the rival uses $y^0 = y^*$, $z^0 = z^*$, $x^1 = x^*$. The following result uses these necessary conditions to show that a sufficient condition for uniqueness if τ is small is that the directional switching costs not be too different from each other.

Proposition 2 *If $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ and if $\tau \in [0, \overline{\tau}(\alpha)]$ then there is a unique symmetric MPE in which $\Delta V^{01}_{+1} \geq 0$, $\Delta V^{10}_{0} \geq 0$, $\Delta V^{10}_{-1} \geq 0$ for both firms.*

It may be seen that if $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ and $\tau \in (0, \overline{\tau}(\alpha))$ then the three inequalities are all strict, implying that $y^* > 0$, $z^* > 0$, $x^* > 0$ at the symmetric MPE. In turn this implies that in equilibrium continual product relocations persist in the long run.[9]

[9] This is also true if $\tau = 0$ and $\alpha \in (1/3, 1.244)$. If $\tau = 0$ and $\alpha \leq 1/3$ then $x^* > 0$ and $z^* > 0$ but $y^* = 0$. Laggards invest nothing because they can never earn positive profits, leaders invest in moving to the rival's location, and neck-and-neck firms invest in moving away. The long-run location outcome then depends on the initial quality state: permanent co-location if firms have asymmetric qualities, and permanent separation if they have equal qualities.

Proposition 2 allows us to derive some comparative statics of the equilibrium switching hazards with respect to changes in α, γ^0, $\gamma^1$and τ. The following results are in each case derived by totally differentiating (12)-(14); details are available on request. Consider first the effect of changes in the switching-cost parameters γ^0 and γ^1.

Proposition 3 [Effects of γ^0, γ^1] *Suppose that* $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ *and* $\tau \in [0, \overline{\tau}(\alpha))$.

a. *(i) If* $\tau = 0$ *and* $\alpha > 1/3$, *then* $dx^*/d\gamma^0 > 0$, $dz^*/d\gamma^0 < 0$, $dy^*/d\gamma^0 < 0$.

(ii) If $\tau = 0$ *and* $\alpha \leq 1/3$, *then* $dx^*/d\gamma^0 = 0$, $dz^*/d\gamma^0 < 0$, $dy^*/d\gamma^0 = 0$.

b. *(i) If* $\tau = 0$ *or* $\gamma^1 < 1.5\gamma^0$, *then* $dz^*/d\gamma^1 < 0$.

(ii) If $\tau = 0$ *then* $dz^*/d\gamma^1 = 0$.

(iii) If $\tau = 0$ *and* $\alpha > 1/3$, *then* $dy^*/d\gamma^1 > 0$.

(iv) If $\tau = 0$ *and* $\alpha \leq 1/3$, *then* $dy^*/d\gamma^1 = 0$.

Under the conditions of Proposition 2 the leader is trying to reduce horizontal differentiation while neck-and-neck firms and the laggard are trying to increase it. It is then intuitive that changes in γ^0 and γ^1 should move the switching investments in opposite directions. The above proposition confirms this intuition, at least for the case of $\tau = 0$. For example, if $\tau = 0$ and the cost γ^0 of moving away from one's rival increases (and if γ^0 and γ^1 are not too different), then z^* and y^* fall and x^* increases. Note that the effect on x^* is an indirect response – the leader is more inclined to 'chase' his rivals if it is harder for them to escape once he has succeeded in duplicating their products.

If $\alpha \leq 1/3$ and $\tau = 0$ then a laggard makes zero profits in all states. It is then trivially true that $dy^*/d\gamma^0 = dy^*/d\gamma^1 = 0$, and also that $dx^*/d\gamma^0 = 0$, but $dx^*/d\gamma^1$ is still non-zero because the leader still responds to his own cost of switching. Similarly if $\tau = 0$ then $dz^*/d\gamma^0 < 0$ but $dz^*/d\gamma^1 = 0$, because with no changes in the quality differential the only incentive for neck-and-neck firms is toward permanent horizontal separation.

By continuity, results in Proposition 3 for $\tau = 0$ should also hold for τ near zero. But we do not know whether they hold at all $\tau \in [0, \overline{\tau}(\alpha))$ (except for $dx^*/d\gamma^1$ in (b)(i), if $\gamma^1 < 1.5\gamma^0$). When there is a non-zero probability of a quality reversal, a laggard's attitude to a higher value of γ^1 depends not only on the response of his immediate rival, the current leader, but also on the possibility that the laggard himself will soon be in that position. In such cases the comparative statics are not immediately obvious.

Next consider the effects of changes in α, the degree of taste heterogeneity. Here the intuition derives from the effects of α on the profit differences a^{01}, b^{10}, c^{10} defined in section 2. Recall that these are the profit differences between location states at fixed quality differentials. We have $db^{10}/d\alpha \geq 0$ and $dc^{10}/d\alpha \geq 0$ everywhere, while $da^{01}/d\alpha > 0$ iff $\alpha < 1/3$. In static terms these derivatives imply that neck-and-neck firms and laggards have more incentive to differentiate horizontally when consumer tastes are more heterogeneous. For the leader the incentive to reduce horizontal differentiation is increasing in α if and only if taste heterogeneity is not too high, otherwise it is decreasing in α.

Proposition 4 [Effects of α] *Suppose that* $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ *and* $\tau \in [0, \overline{\tau}(\alpha))$.

a. If $\alpha > 1/3$, *then* $dx^*/d\alpha < 0$, $dz^*/d\alpha > 0$, $dy^*/d\alpha > 0$.

b. If $\alpha < 1/3$, *then* $dx^*/d\alpha > 0$.

c. If $\alpha < 1/3$ *and* $\tau = 0$, *then* $dz^*/d\alpha > 0$, $dy^*/d\alpha = 0$.

The missing cases here are the signs of $dz^*/d\alpha$, $dy^*/d\alpha$ when $\alpha < 1/3$ and $\tau \neq 0$. As noted above, the open question in these cases is whether z^* and y^* respond mainly to changes in their short-run incentives, or to the new incentives that would arise in the event of a quality reversal.

For changes in τ we have the following:

Proposition 5 [Effects of τ] *If* $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ *and* $\tau = 0$, *then* $dx^*/d\tau < 0$.

This result captures the intuition that if we increase τ from zero, then the leader's incentive to chase its rivals can only decrease, since there is now some probability of losing his quality advantage, in which case he would prefer maximal horizontal differentiation. One would expect that an opposite intuition would hold for the laggard's switching investment y^* (and perhaps also for z^*), but to date we have not been able to determine the circumstances in which this is true.

5 Welfare effects in the game of product locations

To analyze the welfare effects of parameter changes we first determine the industry's equilibrium invariant distribution(s) over states (s, d). Let μ denote the equilibrium long-run probability of firms being co-located. In the case of $\tau \neq 0$ it is straightforward to show that $\mu = x^*/(x^* + z^* + y^*)$; thus μ is increasing in x^* and decreasing in z^* and y^*, which is quite intuitive.[10] However we do not have many comparative statics for this case of $\tau \neq 0$, so we will assume $\tau = 0$.

If $\tau = 0$ the only non-degenerate case arises when $1/3 < \alpha < 1.244$, $d_i \neq 0$. (Any other case yields equilibrium ergodic sets that are singletons, because the quality differential is fixed and firms in the long run end up either permanently separated or permanently co-located.) In this case we must have $\mu y^* = (1 - \mu)x^*$, i.e., $\mu = x^*/(x^* + y^*)$, which is naturally increasing in x^* and decreasing in y^*. Effects of parameter changes on μ are then clear if the comparative statics of x^* and y^* can both be signed and if they run in opposite directions. For example an increase in taste heterogeneity α leads to firms spending less time co-located over the long run (Proposition 4(a)).

With fixed quality differentials it can be shown that instantaneous consumer welfare is higher with separate than with co-located firms if and only if $\alpha < [3(\sqrt{12} - 3)]^{-1} \approx 0.718$.[11] That is, if α is high then consumers prefer firms to be co-located so as to limit their local market power. Hence, all else equal, an increase in μ raises long-run average consumer welfare iff $\alpha > 0.718$. The following result is then an immediate corollary of Proposition 3:

Proposition 6 *Suppose that* $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$, *that* $1/3 < \alpha < 1.244$, *and that* $\tau = 0$ *with* $d_i \neq 0$.

- *If* $\alpha < 0.718$ *then equilibrium long-run average consumer welfare is decreasing in* γ^0 *and increasing in* γ^1.
- *If* $\alpha > 0.718$ *then equilibrium long-run average consumer welfare is increasing in* γ^0 *and decreasing in* γ^1.

[10] It is easy to see that if $\tau > 0$ the long-run proportion of time spent in the $d_i = 0$ quality state is 1/3. In long-run equilibrium we must then have (no. of flows out of state $s = 0$) $= \mu((1/3)2z^* + (2/3)y^*) = (1 - \mu)(2/3)x^* =$ (no. of flows out of state $s = 1$), implying $\mu = x^*/(x^* + z^* + y^*)$.

[11] The same threshold holds for both linear and quadratic transport costs.

For example think of γ^0 and γ^1 as being influenced by patent policy, and consider an increase in patent protection that raises γ^1 and/or reduces γ^0. Under the conditions of Proposition 6 this raises consumer welfare if taste heterogeneity is low, and reduces it if heterogeneity is high. In either case it causes firms to spend more time at separate locations, and otherwise has no direct effect on instantaneous consumer welfare (since the γ's are not demand parameters.)

A notable consequence of Proposition 6 is that the relationship between long-run consumer welfare and the costs of horizontal differentiation is rather sensitive to the degree of taste heterogeneity in the market. This suggests that the impact on consumer welfare of a policy-induced change in these costs will ultimately be an empirical question. Following the work of Berry, Levinsohn and Pakes (1995) and earlier authors there is now a well-established set of tools for measuring such taste heterogeneity in any given empirical context.

6 Conclusion

Our companion paper (Narajabad and Watson 2008) studies a discrete-time version of this duopoly, in which firms may invest in quality improvement and horizontal differentiation simultaneously. These modifications render the model analytically intractable, so we use numerical methods to study the symmetric MPE's over reasonable ranges of parameters. Much of the intuition from the present model carries over to that framework with a few modifications. For example under certain conditions we again get a U-shaped relationship between innovation and substitutability, as seen in section 3. However in some cases the relationship can be monotonically increasing instead – the explanation lies in the behavior of equilibrium investment by the quality leader, which may be non-zero in a discrete-time model. That paper also discusses some empirical implications of the analysis, which would be an interesting area for further study.

Appendix

A.1 Proof of Proposition 1

(a) As noted in the text, we have $dw_0/d\alpha < 0$, all $\alpha \geq 0$, all $r > 0$. Consider then $dw_{-1}/d\alpha$. If $r = 0$ then (3) implies $w_0^2 = 2K_H/\phi$. Also if $\alpha < 1/3$ then $K_H = k - 1.5\alpha$ and $K_L = \alpha/2$ (see section 2). Substituting in (4) with $r = 0$, $k = 1$, we get

$$\begin{aligned} w_{-1} &= \sqrt{(2/\phi)}\left[\sqrt{2-2.5\alpha} - \sqrt{1-1.5\alpha}\right] \\ \Rightarrow \frac{dw_{-1}}{d\alpha} &= \frac{1}{4}\sqrt{\frac{2}{\phi}}\left[\frac{3}{\sqrt{1-1.5\alpha}} - \frac{5}{\sqrt{2-2.5\alpha}}\right] . \end{aligned}$$

Thus $\text{sgn}(dw_{-1}/d\alpha) = \text{sgn}(3\sqrt{2-2.5\alpha} - 5\sqrt{1-1.5\alpha})$, which is strictly negative if $\alpha < 1/3$. The result then immediately follows from (5).

(b) If $\alpha > 1/3$ then $K_H = \frac{1}{6}[2 + \frac{1}{3\alpha}]$ and $K_L = \frac{1}{6}[2 - \frac{1}{3\alpha}]$. Using $r = 0$, $k = 1$ and $w_0^2 = 2K_H/\phi$ in (4), we then arrive at

$$\frac{dw_{-1}}{d\alpha} = \frac{1}{4}\frac{1}{9\alpha^2}\sqrt{\frac{2}{\phi}}\left[\left(\frac{1}{3}+\frac{1}{18\alpha}\right)^{-0.5} - \left(1+\frac{1}{18\alpha}\right)^{-0.5}\right] > 0 \quad (15)$$

$$\frac{dw_0}{d\alpha} = \frac{-1}{18\alpha^2\sqrt{2K_H\phi}} = \frac{-1}{18\alpha^2\sqrt{\phi[2+(1/3\alpha)]/3}} < 0 . \quad (16)$$

Since $LRI = [(w_{-1})^{-1} + (2w_0)^{-1}]^{-1}$, consider the sign of $\frac{d}{d\alpha}[(w_{-1})^{-1} + (2w_0)^{-1}]$. From (15) we get

$$\frac{d}{d\alpha}\left(\frac{1}{w_{-1}}\right) = -\frac{\sqrt{\phi}/36\alpha^2}{\sqrt{2}[m_0\sqrt{m_1} - m_1\sqrt{m_0}]} , \quad (17)$$

where

$$m_0 \equiv 1 + \frac{1}{18\alpha} , \; m_1 \equiv \frac{1}{3} + \frac{1}{18\alpha} .$$

And from (16) we get

$$\frac{d}{d\alpha}\left(\frac{1}{2w_0}\right) = \frac{\sqrt{\phi}/36\alpha^2}{[2m_1]^{3/2}} . \quad (18)$$

From (17) and (18) it may be seen that

$$\text{sgn}\left\{\frac{d}{d\alpha}\left[(w_{-1})^{-1} + (2w_0)^{-1}\right]\right\} = \text{sgn}\left\{m_0\sqrt{m_1} - m_1\sqrt{m_0} - 2m_1\sqrt{m_1}\right\} ,$$

which is strictly negative, implying that if $r = 0$ and $\alpha > 1/3$ then $dLRI/d\alpha > 0$.

(c) Parts (a) and (b) indicate that if $r = 0$ then LRI is minimized at $\alpha = 1/3$ (given continuity of LRI in α), and that LRI is maximized either at $\alpha = 0$ or at α that are 'large' (which we may interpret as $\alpha \to +\infty$ for present purposes). Consider then LRI at $r = 0$ and $\alpha = 0$, where $K_H = 1$ and $K_L = 0$. We have $w_0 = \sqrt{2/\phi}$ and $w_{-1} = (2 - \sqrt{2})/\sqrt{\phi}$. On the other hand if $\alpha \to +\infty$ with $r = 0$ then $w_0 \to \sqrt{1/(3\phi)}$, $w_{-1} \to (1 - (1/\sqrt{3}))/\sqrt{\phi}$, which values are in each case less than the values at $\alpha = 0$. Hence if $r = 0$ then $LRI(\alpha = 0) > \lim_{\alpha \to +\infty} LRI$.

A.2 Proof of Proposition 2

A preliminary lemma is useful. Add (12) and (14) and substitute in for $\Delta V^{01}_{+1} \equiv \gamma^1 x^*$, $\Delta V^{10}_{-1} \equiv \gamma^0 y^*$, $f(x^*)$ and $g(y^*)$. It can then be seen that a necessary condition for a symmetric equilibrium is $F(x^*, y^*) = 0$, where

$$\begin{aligned} F(x,y) \equiv\ & \gamma^1(r+\tau)x + \gamma^0(r+\tau)y + xy(\gamma^1 + \gamma^0) \\ & - \left[a^{01} + c^{10} - 0.5(\gamma^1 x + \gamma^0 y^2)\right] . \end{aligned} \tag{19}$$

Lemma 2 *Suppose that (x', y'), (x'', y'') satisfy $F(x', y') = 0$, $F(x'', y'') = 0$, $x', x'' \geq 0$, $y', y'' \geq 0$.*

a. $x' = x'' \Leftrightarrow y' = y''$

b. If $x'' > x'$ then $y'' < y'$.

c. For all $x \in [x', x'']$ there exists a unique y such that $F(x, y) = 0$.

d. Define a function y(x) such that $F(x, y(x)) = 0$ at all $x \in [x', x'']$. Then dy/dx exists, with $dy/dx < 0$ at every $x \in [x', x'']$.

Proof: It may be seen that the partial derivatives F_x and F_y are both strictly positive, e.g.,

$$F_x = \gamma^1(r + \tau) + (\gamma^1 + \gamma^0)y + \gamma^1 x > 0 \ .$$

(a) Suppose, e.g., that $x'' > x'$ but $y' = y''$. But then we must have $F(x'', y'') > 0$, since $F_x > 0$, $F(x', y') = 0$, which is a contradiction. Other cases yield a contradiction in similar fashion.

(b) If $x'' > x'$ and $y'' \geq y'$ then $F(x'', y'') > 0$, since $F_x > 0$, $F_y > 0$, which is a contradiction.

(c) We have $F(x', y') = 0$. Fix y at y'. And consider raising x from x'. Since $F_x > 0$, we have $F(t, y') > 0 \; \forall \; t \in (x', x'']$. We also have $F(x'', y'') = 0$, and we know from (b) that $y'' < y'$. Fix y at y''. And consider reducing x from x''. Since $F_x > 0$ at any $y \geq 0$, we have $F(t, y'') < 0 \; \forall \; t \in [x', x'')$. Furthermore $F(x, y)$ is differentiable and strictly increasing in y at any $x \in [x', x'']$. It follows that $\forall \; t \in [x', x''] \; \exists$ a unique y such that $F(t, y) = 0$.

(d) Since $F_x, F_y > 0$, the Implicit Function Theorem can be applied to show that dy/dx exists at any $x \in [x', x'']$ and $dy/dx < 0$. ■

Suppose then that there are two symmetric equilibria (y', z', x') and (y'', z'', x''). It can be seen that we cannot have $z' \neq z''$, $x' = x''$, $y' = y''$. (Take the difference (9) minus (8) to get an expression for firm i's best response to the rival's $\overline{z}$ at fixed values of ΔV^{01}_{+1}, ΔV^{10}_{-1}. This best response can only have one fixed point.) In view of Lemma 2 (a) this implies that we must have $x' \neq x''$ and $y' \neq y''$. Suppose w.l.o.g. that $x'' > x'$.

Plug (y', z', x') into (13), using $\Delta V^{01}_{+1} = \gamma^1 x'$, $\Delta V^{10}_{0} = \gamma^0 z'$, $\Delta V^{10}_{-1} = \gamma^0 y'$. Repeat for (y'', z'', x''), take differences and re-arrange to get

$$(z'' - z') = \frac{1}{\gamma^0} \left[\frac{\tau}{(r + 2\tau) + 1.5(z' + z'')} \right] [\gamma^0 (y'' - y') - \gamma^1 (x'' - x')] \; . \quad (20)$$

Similarly plug (y', z', x') into (12), repeat for (y'', z'', x''), take differences and re-arrange to get:

$$\begin{aligned} \tau\gamma^0 (z' - z'') &= \gamma^1 (r + \tau)(x'' - x') - 0.5\gamma^1 ((x')^2 - (x'')^2) \\ &\quad + \gamma^1 (x'' y'' - x' y') \; . \end{aligned} \quad (21)$$

Do the same for (14) to get

$$\begin{aligned} \tau\gamma^0 (z' - z'') &= -\gamma^0 (r + \tau)(y'' - y') + 0.5\gamma^0 ((y')^2 - (y'')^2) \\ &\quad - \gamma^0 (x'' y'' - x' y') \; . \end{aligned} \quad (22)$$

Add (21) and (22):

$$2\tau\gamma^0 (z' - z'') = (r + \tau)(\gamma^1 (x'' - x') - \gamma^0 (y'' - y')) + A \; , \quad (23)$$

where

$$A \equiv 0.5[\gamma^1((x'')^2 - (x')^2) - \gamma^0((y'')^2 - (y')^2)] + (\gamma^1 - \gamma^0)(x''y'' - x'y') \ .$$

Suppose that $\tau = 0$. Then, since $x'' > x'$ and $y'' < y'$, (23) implies $A < 0$. That is, $B(x'', y'') < B(x', y')$, where

$$B(x, y) \equiv 0.5[\gamma^1 x^2 - \gamma^0 y^2] + (\gamma^1 - \gamma^0)xy \ . \tag{24}$$

Or suppose $\tau > 0$. Write (23) with $z'' - z'$ on the LHS and equate with (20) to get

$$\begin{aligned} &\frac{\tau}{(r + 2\tau) + 1.5(z' + z'')}[\gamma^1(x'' - x') - \gamma^0(y'' - y')] = \\ &\quad \frac{1}{2\tau}(r + \tau)(\gamma^1(x'' - x') - \gamma^0(y'' - y')) + \frac{A}{2\tau} \ . \end{aligned} \tag{25}$$

The first term on the RHS of (25) is strictly greater than the LHS. Therefore we again require $A < 0$, i.e., $B(x'', y'') < B(x', y')$.

Recall that we must have $F(x', y') = F(x'', y'') = 0$, where F is defined in (19). Consider then the behavior of $B(x, y(x))$ for $x \in [x', x'']$, where $y(x)$ is such that $F(x, y(x)) = 0$. From Lemma 2, $y(x)$ exists and is differentiable in x. We have:

$$\frac{dB}{dx} = \left[\gamma^1 x - \gamma^0 y \frac{dy}{dx}\right] + (\gamma^1 - \gamma^0)\left(y + x\frac{dy}{dx}\right) \ . \tag{26}$$

From the Implicit Function Theorem in (19) we get

$$\frac{dy}{dx} = -\left\{\frac{\gamma^1(r + \tau) + y(\gamma^1 + \gamma^0) + \gamma^1 x}{\gamma^0(r + \tau) + x(\gamma^1 + \gamma^0) + \gamma^0 y}\right\} \ . \tag{27}$$

Substitute (27) into (26), express as a single ratio, and collect terms to see that

$$\begin{aligned} \mathrm{sgn}(dB/dx) \ &= \ \mathrm{sgn}\{2\gamma^1\gamma^0(x^2 + xy + y^2) \\ &\quad +(r + \tau)[\gamma^0(2\gamma^1 - \gamma^0)y + \gamma^1(2\gamma^0 - \gamma^1)x]\} \ . \end{aligned}$$

Therefore if $\gamma^1 \in (0.5\gamma^0, 2\gamma^0)$ then $dB/dx \geq 0$, which implies $B(x'', y'') \geq B(x', y')$, which is a contradiction.

References

Aghion, Philippe, and Peter Howitt (1997), 'Competition and growth with step-by-step innovation: An example', *European Economic Review*, 41, 771-82.

Aghion, Philippe, Christopher Harris, Peter Howitt, and John Vickers (2001), 'Competition, imitation and growth with step-by-step innovation', *Review of Economic Studies*, 68, 467-92.

Aghion, Philippe, Nick Bloom, Richard Blundell, Rachel Griffith and Peter Howitt (2005), 'Competition and innovation: An inverted-U relationship', *Quarterly Journal of Economics*, 120, 2, 701-28.

Bernard, Andrew B., Stephen J. Redding, and Peter K. Schott (2006), 'Multiproduct firms and product switching', NBER w.p. no. 12293.

Berry, Steven, James Levinsohn and Ariel Pakes (1995), 'Automobile prices in market equilibrium', *Econometrica*, 63, 4, 841-90.

Gilbert, Richard (2006), 'Looking for Mr. Schumpeter: Where are we in the competition-innovation debate?', *NBER Innovation Policy and the Economy*, 6, 1, 159-215.

Maskin, Eric, and Jean Tirole (1988a), 'A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs', *Econometrica*, 56, 3, 549-69.

———————— (1988b), 'A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles', *Econometrica*, 56, 3, 571-99.

Narajabad, Borghan, and Randal Watson (2008), 'The dynamics of innovation and horizontal differentiation', mimeo., Rice University Economics.

Van Cayseele, P.J.G. (1998), 'Market structure and innovation: A survey of the last twenty years', *De Economist*, 146, 3, 391-417.

Figure 1: Equilibrium long-run innovation as a function of taste heterogeneity