

Online Appendix for Narajabad and Watson, ‘The dynamics of innovation and horizontal differentiation’,
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Appendix

In part one of the appendix we discuss a deterministic model of dynamic location choices in duopoly (with fixed product qualities). The solution to this model provides valuable insights into cases of multiple equilibria in the full stochastic model. We discuss these cases in detail in part two.

A.1 Deterministic model of dynamic location choices

The stage game is the same as in section 2. Here we assume that firms’ qualities are exogenously fixed, and thus that their investment decisions only affect their locations. Furthermore in order to solve the model analytically we assume here that location transitions are deterministic, not stochastic.

Let the quality differential be $\delta(t) = \delta$ for all t . Since qualities are fixed, define the industry’s current state $\omega(t)$ to be equal to the location state $s(t) \in S \equiv \{0, 1\}$. A firm i ’s Markov strategy comprises a state-conditional choice of whether or not to switch locations: a function $h_i : S \rightarrow \{0, 1\}$, $i = 1, 2$, where $h_i(\omega) = 0$ means ‘stay’ and $h_i(\omega) = 1$ means ‘switch’. By restricting attention to pure strategies, we rule out randomizations between switching and not switching.

The sequence of moves within each period is as follows:

- a. Firms observe the current state $\omega(t)$ and simultaneously decide whether or not to switch.
- b. The switching decisions are implemented and any costs of these decisions are incurred.
- c. Conditional on their new locations firms play the Hotelling stage game and receive their profits.

- d. Play moves on to the next period, with state $\omega(t+1)$ determined by the location choices implemented in (b).

A firm incurs a fixed cost of $T > 0$ each time that it switches location.

Where convenient we will use ω to denote the current state and ω' to denote next-period states. Given a rival's strategy h_j , the Bellman equation representing firm i 's optimization problem in this game is:

$$V_i(\omega) = \max_{h_i(\omega) \in \{0,1\}} (-Th_i(\omega) + \pi_i(\omega') + \beta V_i(\omega')) , \quad (7)$$

where, with \oplus representing binary addition, $\omega' \equiv \omega \oplus h_i(\omega) \oplus h_j(\omega)$. An equilibrium may be represented as a pair of value functions $(V_1(\cdot), V_2(\cdot))$ and a pair of policy functions $(h_1(\cdot), h_2(\cdot))$ such that for $i = 1, 2$, $V_i(\cdot)$ solves (7) given $h_j(\cdot)$, $j \neq i$, and such that

$$h_i(\omega) \in \arg \max_{h_i \in \{0,1\}} \{-Th_i + \pi_i(\omega') + \beta V_i(\omega')\} , \quad (8)$$

for all ω , given $h_j(\cdot)$, where $\omega' \equiv \omega \oplus h_i \oplus h_j(\omega)$.

For any given parameter values it is possible to solve for a pure-strategy equilibrium of this game analytically (if one exists). Suppose for example that $\delta \geq 3\alpha$. Then firm 1 earns zero profits no matter what the state, and any MPE must have $h_1(0) = h_1(1) = 0$, $V_1(0) = V_1(1) = 0$. Any other strategy would have firm 1 incurring some switching costs without any benefit. Firm 2's best response then depends on the parameters $(\delta, \alpha, T, \beta)$: if $\alpha < T(1 - \beta)$ we have $h_2(0) = h_2(1) = 0$, $V_2(0) = \delta/(1 - \beta)$, and $V_2(1) = (\delta - \alpha)/(1 - \beta)$ in equilibrium, while if $\alpha > T(1 - \beta)$ we have $h_2(0) = 0$, $h_2(1) = 1$, $V_2(0) = \delta/(1 - \beta)$, and $V_2(1) = -T + (\delta/(1 - \beta))$. The intuition for these best responses is straightforward: given that firm 1 is not going anywhere, firm 2 finds it optimal to switch in the 'apart' state ($\omega = 1$) if and only if the NPV of the extra stage-game profit, $\alpha/(1 - \beta)$, exceeds the one-off switching cost T . (Recall that with $\delta \geq 3\alpha$ firm 2 earns a stage-game profit of δ when $\omega = 0$ and profit of $\delta - \alpha$ when $\omega = 1$.)

Deriving the pure-strategy equilibria of this game for all parameter values is a laborious task. For brevity we omit the details and summarize the results of this effort in figure 10, which characterizes the type of equilibrium outcome according to the location in R_+^2 of the point $(\delta/\alpha, T/\alpha)$. (Note that the figure restricts attention to $\delta \geq 0$: by symmetry a mirror-image picture would apply in the case of $\delta < 0$.) The figure divides this space up into four areas. Three

of these represent different types of ‘steady states’, while the fourth represents the case of no equilibrium in pure strategies.

Take firstly the case of switching costs T that are high relative to α . Area I in the figure indicates that if T exceeds $\alpha/(1 - \beta)$ then there will be no movement at all in equilibrium, regardless of the difference in product qualities, i.e., in that area we have $h_1(0) = h_2(0) = h_1(1) = h_2(1) = 0$. Clearly the equilibrium steady-state location then just depends on the industry’s initial state $\omega(0)$. The same type of outcome also obtains for moderately high T , as long as the difference in product qualities is not too high. When δ is larger and T is at these moderate levels we move into area III, where the only equilibrium steady state is $\omega(t) = 0$: the firms end up together. In this region, for given values of T and α , the higher value of δ makes it worthwhile for firm 2 to switch locations to join firm 1 when the state is $\omega = 1$. As long as T/α is not too low, the low-quality producer, firm 1, will not respond to this move by itself switching locations (i.e., it will not ‘run away’).

If $\delta \geq 3\alpha$ then firm 1 will in fact stay put no matter what the values of T and α , since in those cases it earns zero profit in all states. But if $\delta < 3\alpha$ then firm 1 can make some profit in the apart states. Whether it finds it optimal to run away then depends on T and α – it will set $h_1(0) = 1$ if switching costs T are small enough, or if the consumer transport cost α is high enough (which would give firm 1 more local monopoly power). These cases are represented in the figure by areas II and IV.

In area IV there is no equilibrium in pure strategies. Firm 1 always finds it optimal to switch when $\omega = 0$ and firm 2 is not switching. But in this area firm 2 *ceteris paribus* prefers to be at the same location as firm 1. (Note that $\delta > 3(2 - \sqrt{3})\alpha$ in this region.) Furthermore since T is not too high firm 2 is willing to pay the cost of ‘following’ firm 1, i.e., if $\omega = 0$ and firm 1 is switching then firm 2 also wants to switch, in which case it would be better for firm 1 to stay put and we have a contradiction. Although there is no pure-strategy MPE in this region there would be a mixed-strategy MPE where both firms randomize between switching and not switching.

Finally in area II we have the case of low switching costs (or high α) and a low quality differential. Except where it overlaps area III, the equilibrium steady-state outcome in this area is always $\omega = 1$: the firms stay apart. The reasoning is simple: for obvious reasons the low-quality firm always prefers to be apart, and since δ is relatively low this is also true of the high-quality producer. Furthermore since switching costs are relatively low one or other of the firms will always be willing to shift away when $\omega = 0$.

A twist in this case is that this steady-state outcome can be arrived at via multiple equilibria. Two obvious such equilibria have: (i) $(h_1(0) = 0, h_1(1) = 0, h_2(0) = 1, h_2(1) = 0)$, and (ii) $(h_1(0) = 1, h_1(1) = 0, h_2(0) = 0, h_2(1) = 0)$. That is, an equilibrium involves only one firm moving away, but either firm may be the mover. There are also two other Pareto-dominated equilibria, in which firms needlessly swap places when they are already apart; these have: (iii) $(h_1(0) = 0, h_1(1) = 1, h_2(0) = 1, h_2(1) = 1)$, and $(h_1(0) = 1, h_1(1) = 1, h_2(0) = 0, h_2(1) = 1)$. In both cases firms could improve their payoffs by agreeing to stay put when they are already apart, but given that my rival fails to do this, it will be optimal for me to respond by also switching, in order to stay away from him.²⁴

Area II is the only region of multiple equilibria – outside this area the MPE is unique if it exists. In the region where areas II and III overlap we have not just multiple equilibria, but also multiple steady states (from a given initial state). In particular the equilibrium steady state could be either $\omega = 0$ or $\omega = 1$. The latter arises with the strategies $(h_1(0) = 1, h_1(1) = 0, h_2(0) = 0, h_2(1) = 0)$: firm 1 moves away when the firms are together, and (since δ is not high enough relative to T/α) firm 2 does not find it optimal to follow him. The former arises with the strategies $(h_1(0) = 0, h_1(1) = 0, h_2(0) = 0, h_2(1) = 1)$. Here firm 1 no longer finds it optimal to move away because his time apart would only last one period before his rival moved to join him. And since firm 1 is not moving when the firms are together it is optimal for firm 2 to switch when they are apart.

A.2 Multiple equilibria in the stochastic model

In the previous section we dwelt in some detail on the equilibrium outcomes in a deterministic model for two reasons. First, it is analytically tractable and we can therefore be certain of which types of equilibria occur where. Second, as is clear from comparison of figures 10 and 1, the intuition behind this case carries over to the model with stochastic transitions, where we see a similar pattern of equilibrium outcomes.

Each of the areas I–IV in figure 1 has an equivalent motivation to the corresponding area in figure 10. Furthermore, each of the different types of multiplicity in figure 10 crops up again in the model of figure 1. First, there

²⁴Note that multiple equilibria of this type do not exist throughout area II. Close to the origin all four equilibria exist. But in part of the area marked IIa (in particular, the part where δ is highest), the unique equilibrium has $(h_1(0) = 1, h_1(1) = 0, h_2(0) = 0, h_2(1) = 0)$.

are cases of ‘multiple steady states’, where two areas overlap, representing multiple types of equilibrium outcome for given parameters. In figure 10 there is just one such region, the overlap between areas II and III. A corresponding region is found in figure 1 around the parameter values $\gamma = 0.5$, $k = 2$, and there are also new areas of overlap not observed in figure 10. These new possibilities arise from the interactions between investments in quality and relocation, which interactions are not present in figure 10 because of fixed qualities. When $k = 0.8$ and $\gamma = 2$, for example, there is an equilibrium with no switching at all and an equilibrium where the firms will always end up together.²⁵

By analogy with figure 10 we should also expect to observe some cases of multiple equilibria of the same type, particularly in area II. In that region both firms want to locate apart. There are two ways of achieving this: either the quality leader can move away, or the laggard can move away. Both behaviors may be consistent with equilibrium (but not with the same equilibrium). Such cases of multiple equilibria are in fact observed for parameters in this area of figure 1 (although these multiples are not indicated on the figure for brevity). Since these equilibria have the same ergodic set (i.e., firms end up apart), this type of multiplicity is of no consequence for the analysis of long-run outcomes.

Also present in area II of figure 1 are the ‘pathological’ cases described in the previous section, where both firms prefer to be separate in all quality states but keep needlessly switching locations. That is, given that my rival is switching with high probability in state $s = 1$, it is optimal for me to do the same, even though we would both benefit from agreeing not to move at all in that state. Such behavior can be consistent with equilibrium in the stochastic model if the cost of switching γ is very low. We eliminate such cases of multiplicity by assuming that Pareto-dominated equilibria are not played (Assumption (b) in section 4) which here implies that the needless-switching outcome is rejected in favor of an equilibrium where firms cease all switching investment after reaching the state $s = 1$.

As noted in the text, the stochastic model with endogenous qualities also throws up the possibility of coordination failures in quality investment. When firms are in the equal-quality state $\delta = 0$ their quality investments are strategic complements (increasing best responses, all else equal), which gives

²⁵There is also an overlap between areas II and IV, which is not shown in figure 1 because all the ‘extra’ equilibria in this overlap are eliminated by Pareto-dominance.

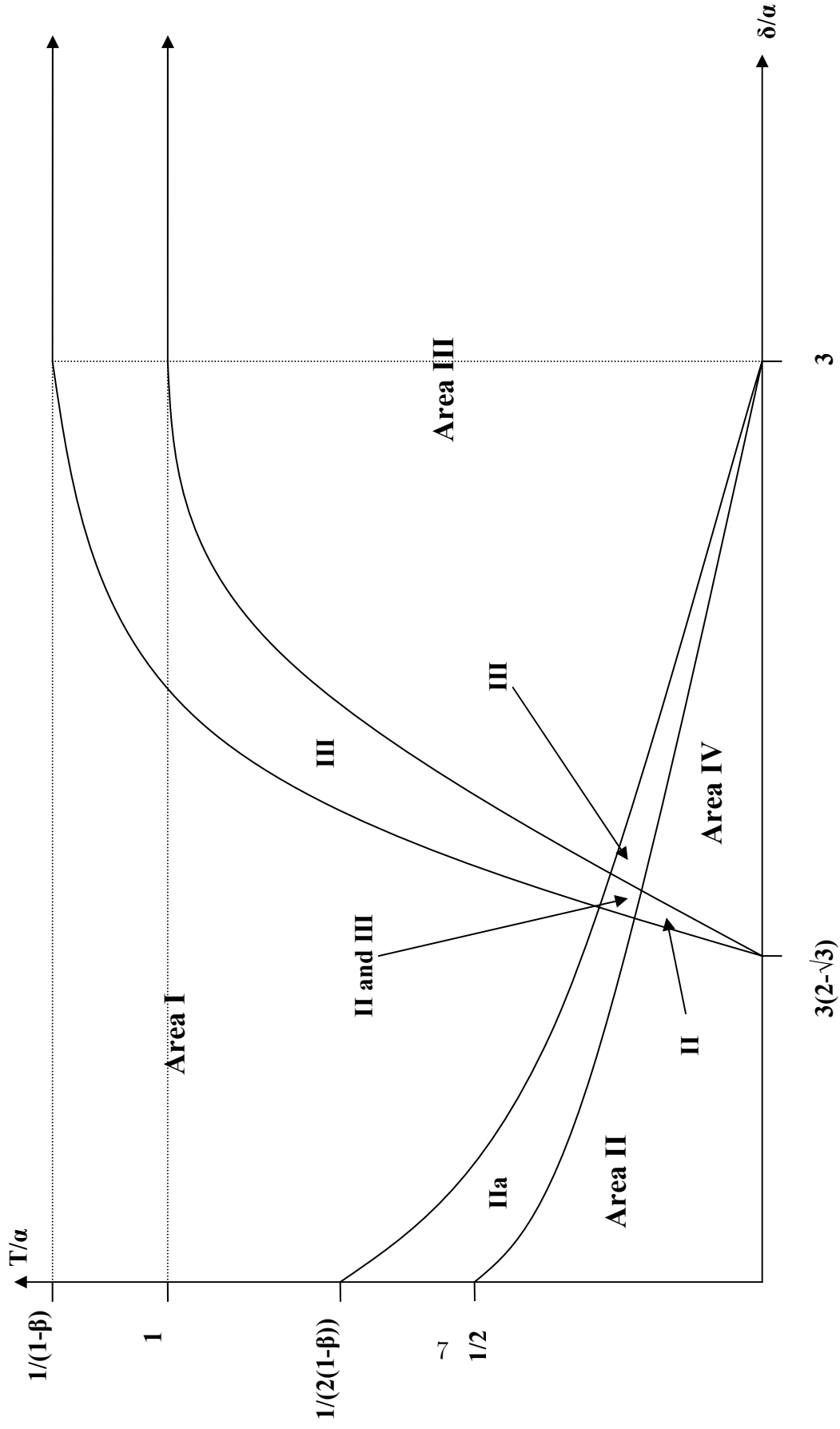
rise to the possibility of multiple symmetric equilibria. We eliminate such cases of multiplicity by assuming that the leader's quality advantage k is not too low (Assumption (a) in section 4).

In summary, we find cases of multiple equilibria for one of four reasons:

- i. near a boundary between regions representing two different classes of ergodic sets;
- ii. in area II, because the firms want to separate, and either the leader or the laggard may be the one to shift away;
- iii. in area II, if the switching cost γ is very low, because the firms want to separate, but they cannot refrain from needlessly switching when already apart;
- iv. when k is low, because of failures in the coordination of investments by firms of equal quality.

Recalling our assumptions in section 4, cases (iv) are eliminated with assumption (a), cases (iii) are eliminated with assumption (b), and cases (ii) are of no consequence for the long-run outcome. As discussed in the text, cases (i) can be reduced to just a single small area of overlap using assumptions (a) and (b).

Figure 10: Equilibrium outcomes in the model with fixed δ and deterministic location transitions



Area I: outcome is $s=0$ or 1 (no switching in equilibrium)
Area II: outcome is $s=1$ (firms end up separate)
Area III: outcome is $s=0$ (firms end up collocated)
Area IV: no equilibrium