Inflation with Random Consumption and Production Timing

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Abstract

When people hold money, the delay between production and consumption has to be compensated to attain optimality. However, when the time of consumption is random, raising inflation will increase the risk of holding money. In particular, in addition to the well-studied distortionary effect of inflation, which reduces agents' incentives to produce, higher risk would induce agents to produce more in order to reduce the risk associated with the uncertainty about the real value of their money holdings when they consume. I show that increasing inflation from zero, using the appropriate monetary policy tool, always increases aggregate output, which is consistent with the empirical evidence of positive correlation between output level and long run inflation for low inflation countries. Moreover, I show that if agents are sufficiently risk averse, increasing inflation from zero via lump-sum monetary transfers, not only increases aggregate output, but is also welfare improving.

Keywords: Micro foundation for money, Inflation, Monetary Policy.

JEL Classification: E42, E52.

1 Introduction

People rarely enjoy the fruits of their labor immediately. There is usually a gap between the time of production and the time of consumption. Current literature has mostly focused on environments with a constant length of time between production and consumption. This paper studies an environment with a random length of time between production and consumption. In addition to the usual tax-like distortionary effect, inflation increases the risk associated with uncertainty about the level of consumption. The *risk effect* induces workers to supply more labor to insure their consumption level. I show that using the appropriate monetary tool, the *risk effect* can overcome the *distortionary effect*, generating a positive correlation between inflation and aggregate output in inflation levels close enough to zero.

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I provide a framework with anonymous ex-ante identical infinitely lived agents who receive privately observable production opportunities and consumption desires for a divisible non-storable good. Similar to Kocherlakota (1998), money works as a record keeping device and provides an incentive for incurring a production cost when a production opportunity is received, in exchange for a promise of consumption when a consumption desire arrives. Money is the only asset available in the environment. Therefore, the inflation cost associated with the resources spent for economizing on cash holdings by converting non-interest-bearing money to other interest-bearing assets is not studied in this paper. However, the environment is rich enough to study the implications of changing the real rate of return of money for labor supply. (See Lucas(2000) for an extensive review of inflation costs.)

As in Lagos and Wright (2005), the micro foundation is explicitly modeled and no restriction is imposed on how much cash agents can hold. However, unlike their paper, the distribution of (real) money holdings is not degenerate. Nonetheless, the framework delivers analytical results.

In this environment, agents use money to facilitate intertemporal transactions between when they can produce but do not desire to consume and when they desire to consume but cannot produce. This can be interpreted as a precautionary motive for holding money, following Bewley (1983). However, unlike his paper, in which the fiscal burden of financing interest payments on money makes the Friedman rule suboptimal, the fiscal burden plays no role in my analysis.

Similar to the search models of money (See Shi (1995) and Trejos and Wright (1995)) agents receive consumption and production opportunities at random times. However, unlike bilateral search models, at any point in time there is a Walrasian central market where consumers and producers meet. Therefore, the price is set in a competitive market rather than by a bargaining process. (See Aruoba, Rocheteau and Waller (2007) for an analysis of the importance of the bargaining protocols for the welfare cost of inflation in the bilateral search models.) As studied by Rocheteau and Wright (2005) the monetary equilibrium and effects of policy are distinctively different under search and competitive market structures. In their environment each period is divided into two sub-periods, day and night, and money balances are readjusted during day for night use, hence there is no randomness in the length of time between two consecutive money balance adjustments. This paper, however, preserves the random timing of consumption and production, which is common to many search models of money, while abstracting from the intricacies of the bargaining process by imposing competitive market structure.

The most interesting result of the environment is that inflation could have a positive correlation with aggregate output. Although the regular distortionary effect of inflation is in play, since the time of consumption is random, an increase in inflation increases the rate at which real balances decrease. This results in higher risk associated with the uncertainty about the amount of consumption that agents are expecting in exchange for their labor effort. Facing a higher risk, agents produce more in order to insure themselves. This second channel works opposite to the well-studied first channel. In particular, I show that if agents are risk averse enough, then the *risk effect* can overcome the *distortionary effect* and increasing inflation via lump-sum transfers of money promote more production. Despite increasing the risk of the payoff received from labor supply, inflation via lump-sum transfer reduces the ex-ante risk due to its redistributive nature, and therefore improves the ex-ante welfare.

I do not explicitly answer the question of what is the optimal level of inflation in this paper. However, the question is quantitatively answerable in this environment. The main question of this paper is whether it is optimal to increase the inflation level from zero.

The model also allows one to study the role of different monetary policy tools. In particular, I show that when agents are not sufficiently risk averse, increasing the inflation rate from zero via lump-sum transfers would not increase the aggregate output level, whereas raising inflation by injecting money via a proportional subsidization of production (or consumption) increases the aggregate output level. This contrasts with the cash-in-advance framework, in which injecting money via proportional subsidies is neutral and does not affect the output level.

Empirical studies have been suggestive of a positive correlation between long-term inflation and output level for low inflation countries. (See Bullard and Keating (1995) and Rapach (2003).) Although some of the micro foundation models of money are capable of generating a positive effect of inflation on output (See Lagos and Rocheteau (2005)), to my knowledge none of them has shown possibility of a positive correlation of aggregate output and welfare with the inflation for low inflation economies.[1] Moreover, I show that, using the appropriate monetary tool, it is always possible to increase output by raising inflation from zero.

Section 2 describes the environment and characterizes the competitive equilibrium with constant money supply. Section 3 studies the equilibrium with positive money supply growth, when money injection is conducted by lump-sum transfers. Section 4 characterizes the equilibrium with positive inflation, when monetary policy is conducted by proportional subsidization of production. Section 4 concludes.

2 Model

Consider a continuous time economy with a unit measure of infinitely lived ex-ante identical anonymous agents who discount the future at rate r (that is, time t utility is discounted by e^{-rt} at time zero.) Agents receive production opportunities of a non-storable good according to a Poisson process with the arrival rate β_p. At the instant of production opportunity, agents can produce any amount $q \in \mathbb{R}_+$, by incurring disutility cost $-q$. Moreover, agents receive consumption desires for this non-storable good according to a Poisson process with the arrival rate β_c. At the instant of consumption desire, agents derive instantaneous utility $u(q)$ from consuming $q \in \mathbb{R}_+$ units. I assume, $u(0) = 0$, $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $1 < u'(0) < \infty$[2].

Since the probability of receiving a production opportunity and a consumption desire at the same time is zero, the value of autarky is zero. Moreover, since agents are anonymous they cannot arrange long-term contracts with each other. However, there exists a divisible storable good, money, which is supplied monopolistically by the monetary authority. At any time t, there is a Walrasian competitive market for goods and money to exchange each unit of good for p_t units of money.

[1] The positive correlation studied in these papers and mine is between aggregate output and long run expected inflation, and is not generated due to an unexpected shock to money growth rate and/or sticky prices.

[2] It is possible to assume $u'(0) = \infty$ and $u(0) < 0$, while agents have access to a costless home production function which produces $\underline{c}$ units of good, where $u(\underline{c}) = 0$ and $u'(\underline{c}) < \infty$. The results will be the same.

Before proceeding with the study of monetary equilibria, it is instructive to see what the social optimum allocation would be if the arrival of production and consumption opportunities were publicly observable. Since goods are not storable, all goods produced by the β_p flow of agents should be consumed by the β_c flow of agents who have consumption desires. Therefore, the solution of social planner's problem is given by any agent with a production opportunity producing q^* units and giving c^* units to agents with consumption desire, with $u'(c^*) = 1$ and $q^* = (\beta_c/\beta_p)c^*$.

In the next sections, I will study the monetary equilibrium when the money supply is constant, when it grows at a constant rate by a uniform lump-sum transfer of money (i.e. helicopter drops) and finally when money injection is conducted by a proportional subsidization of production.

3 Constant Money Supply

In this section, I will study a stationary monetary equilibrium when the aggregate supply of money, M, is constant. In particular, I form an equilibrium with a constant price level $p_t = p$. Let's denote the value of holding real money balance $q = \frac{m}{p}$ for an agent by $V(q)$. Moreover, let's denote the value of receiving production opportunity and consumption desire , while holding real money balance $q = \frac{m}{p}$, by $V_p(q)$ and $V_c(q)$. Given the price level after receiving production (consumption) opportunities, agents decide how much good to produce (consume) by solving the following problems:

$$V_p(q) = \max_{q' \geq q} -(q' - q) + V(q') \tag{1}$$

$$V_c(q) = \max_{q' \leq q} u(q - q') + V(q'). \tag{2}$$

Moreover, the Bellman equation for the value function implies:

$$V(q) = \frac{1}{r + \beta_p + \beta_c} \left\{ \beta_p V_p(q) + \beta_c V_c(q) \right\}. \tag{3}$$

A stationary competitive equilibrium consists of a price level p^*, decision rules $q_p'^*(\cdot)$ and $q_c'^*(\cdot)$ and a distribution of real money holdings $\mu(\cdot)$, such that, given p^*, decision rules $q_p'^*(\cdot)$ and $q_c'^*(\cdot)$ solve (1)-(3), and both money and goods markets clear at any point in time. That is:

$$M = p^* \cdot \int q \cdot d\mu(q) \tag{4}$$

$$\beta_p \cdot \int \left(q_p'^*(q) - q \right) \cdot d\mu(q) = \beta_c \cdot \int \left(q - q_c'^*(q) \right) \cdot d\mu(q). \tag{5}$$

Moreover, the real balance distribution should be invariant. That is for any real balance level q the flow of agents whose real balance changes from below q to above it should be equal to the flow of agents whose real balance changes from above q to below it. In short:

$$\int_0^q \beta_p I_{\{q_p'^*(\tilde{q}) > q\}} d\mu(\tilde{q}) = \int_q^\infty \beta_c I_{\{q_c'^*(\tilde{q}) \leq q\}} d\mu(\tilde{q}) \qquad \forall q \in [0, \infty) \tag{6}$$

where $I_{\{x\}} = 1$ if x is true, and zero otherwise.

It is important to realize that since the production cost function is linear, agents will not find it optimal to produce in two consecutive production opportunities without consuming between them. As we will show below the value function, defined in (3), is strictly concave. Hence the solution for an agent's problem when a production opportunity arrives is unique. When an agent comes out of a production opportunity and adjusts her real balance to solve (1), since the price level is constant, if she receives another production opportunity before realizing any consumption desire, her real balance would be the same level as when she left the previous production opportunity. Therefore, in the stationary equilibrium she would not find it optimal to adjust her real balance to another level. That is, there exists $\overline{q}_p \geq 0$ such that

$$q_p^{'*}(q) = \begin{cases} \overline{q}_p & q \leq \overline{q}_p \\ q & q > \overline{q}_p \end{cases} . \tag{7}$$

If the Inada condition holds and $u'(0) = \infty$ then $V'(0) = \infty$ and in the the presence of a consumption desire agents would not consume all of their real balances since it would violate (2). But if $u'(0) < \infty$ then when the real balance falls below a threshold an agent would consume all of her balance. In particular, we show that, if the following assumption holds, then on the equilibrium path agents would consume all of their real balances whenever they receive a consumption desire. That is, $\mu(\cdot)$ has only two mass points at $q = 0$ and $q = \overline{q}_p$.

Assumption 1: $\frac{r+\beta_c}{\beta_c} < u'(0) < (\frac{1}{\beta_c})^2 \cdot [r(r + \beta_p + \beta_c) + \beta_c(r + \beta_c)]$

If agents spend all of their real balance when $q \leq \overline{q}_p$ for consumption, then the solution for (1)-(3) would be given by:

$$\forall q \leq \overline{q}_p : \quad V_p(q) = -(\overline{q}_p - q) + V(\overline{q}) \tag{8}$$

$$\forall q \leq \overline{q} : \quad V_c(q) = u(q) + V(0) \tag{9}$$

$$\begin{aligned} \forall q \leq \overline{q} : \quad V(q) &= \frac{\beta_p}{r + \beta_p + \beta_c} V_p(q) + \frac{\beta_c}{r + \beta_p + \beta_c} V_c(q) \\ &= \frac{\beta_p}{r + \beta_p + \beta_c} \left[-(\overline{q}_p - q) + V(\overline{q}_p) \right] + \frac{\beta_c}{r + \beta_p + \beta_c} \left[u(q) + V(0) \right]. \end{aligned} \tag{10}$$

From (10) it follows that:

$$\forall q \leq \overline{q} : \quad V^{'}(q) = \frac{\beta_p}{r + \beta_p + \beta_c} + \frac{\beta_c}{r + \beta_p + \beta_c} \cdot u'(q). \tag{11}$$

Therefore, the solution for (1), given by $V'(\overline{q}_p) = 1$, implies

$$u'(\overline{q}_p) = \frac{r + \beta_c}{\beta_c}. \tag{12}$$

Notice that, following (12), the second inequality of *Assumption 1* guarantees $u'(\overline{q}_p) \geq V'(0)$, hence

$$q_c^{'*}(q) = 0 \ \ \forall q \leq \overline{q}_p. \tag{13}$$

Moreover, the first inequality of *Assumption 1* guarantees the existence of $\overline{q}_p > 0$. Finally, concavity of $V(\cdot)$ follows from (11) and concavity of $u(\cdot)$.

Following (6), in the equilibrium, we have $\mu(0) = \frac{\beta_p}{\beta_c+\beta_p}$ and $\mu(\overline{q}_p) = 1$. That is $\frac{\beta_p}{\beta_c+\beta_p}$ fraction of agents have the positive real balance $\overline{q}_p$, and the remaining $\frac{\beta_c}{\beta_c+\beta_p}$ of them have zero money holding. Agents with the real balance $\overline{q}_p$ will forgo all production opportunities without producing and consuming all of their money holdings when they receive a consumption desire. Agents with zero money holding cannot consume when they receive consumption desire, but they produce $\overline{q}_p$ units of goods when their production opportunities arrive. The average real money holding is $\frac{\beta_p}{\beta_c+\beta_p} \cdot \overline{q}_p$, and since the aggregate money supply is M, following (4), the price level will be $p = \frac{M}{\overline{q}_p} \cdot \frac{\beta_c+\beta_p}{\beta_p}$, where $\overline{q}_p$ is given by (12).

Notice that in this equilibrium a flow of $\frac{\beta_c\beta_p}{\beta_c+\beta_p}$ agents consume $\overline{q}_p$ units of goods. Comparing this allocation with the social optimal allocation under full information, there are losses in both extensive and intensive margins. First, instead of a flow of β_c agents, a lower flow of them, $\frac{\beta_c\beta_p}{\beta_c+\beta_p}$ consume at any moment. Second, the amount of consumption is also less than the optimal level. Recall that under full information $u'(c^*) = 1$, but in this monetary equilibrium we have $u'(\overline{q}_p) = (\beta_c+r)/\beta_c > 1$.

Finally, although *Assumption 1* is restrictive enough to guarantee that the invariant distribution has two mass points, the equilibrium will remain similar after relaxing this assumption. In particular, if $u'(0)$ is high enough but bounded away from infinity, then the invariant distribution will have a finite number of mass points, $\left\{q^*_{-n}\right\}_{n=0}^{N}$, were $q^*_{-N} = 0$ and $q^*_{-n-1} < q^*_{-n}$. An agent with the positive real balance q^*_{-n} will consume $q^*_{-n} - q^*_{-n-1}$ units in a consumption opportunity and adjust her real balance to q^*_{-n-1}, and produce $q^*_0 - q^*_{-n}$ units in a production opportunity and adjust her real balance to q^*_0. [3]

4 Positive Inflation with Lump-sum Transfers

In this section we will study the monetary equilibrium when the supply of money grows by a continuous uniform lump-sum transfer of money to all agents.[4] Let's denote the real value of the lump-sum rate of transfer per unit of time by τ and the growth rate of price level by $\pi = \frac{\dot{p}}{p}$. Although the value of receiving a production opportunity or a consumption desire will still be governed by (1) and (2), the Bellman equation for the continuation value is given by:

$$V(q) \quad = \quad \frac{1}{r+\beta_p+\beta_c}\left\{\beta_p V_p(q) + \beta_c V_c(q) + V'(q)\cdot(-\pi\cdot q+\tau)\right\}. \tag{14}$$

Notice that the rate of change in the real balance of an agent whose real money holding is q and receives neither a production nor consumption opportunity is given by $-\pi \cdot q + \tau$. She receives τ units of real lump-sum transfer of money and the inflation rate π reduces her real balance by $-\pi \cdot q$ units.

Notice that in the equilibrium, similar to the no-inflation case, whenever agents receive a production opportunity they adjust their real money holding to a fixed level, that is the solution for

[3] If $u'(0) = \infty$, then $\mu(\cdot)$ will have infinite countable positive mass points, that is $N = \infty$.

[4] I restrict my attention to positive inflation cases. Since the support of the equilibrium distribution of real balances includes zero, implementing a negative inflation rate via lump-sum transfer (tax) involves some ad hoc rules, which would make the analysis excessively complicated.

(1) is given by (7). Moreover, I will study an equilibrium in which agents spend all of their money holdings when they receive a consumption desire. That is, the solution for (2) is given by (13). As we will see below, *Assumption 1* guarantees the existence of such an equilibrium for a small enough inflation rate, $\pi \geq 0$.

If we denote the invariant distribution of real money holdings by $\mu(\cdot)$, (7) implies $\mu(\overline{q}_p) = 1$. Moreover, if (13) holds then (5) implies:

$$\int_0^{\overline{q}_p} q \cdot d\mu(q) \quad = \quad \frac{\beta_p}{\beta_p + \beta_c} \cdot \overline{q}_p. \tag{15}$$

Since the real lump-sum transfer is financed by seniorage revenue from taxing agents' real balances, that is $\tau = \int (\pi \cdot q) d\mu(q)$, (15) implies:

$$\tau \quad = \quad \pi \cdot \frac{\beta_p}{\beta_p + \beta_c} \cdot \overline{q}_p. \tag{16}$$

In order to determine the distribution of real balances, $\mu(\cdot)$, we characterize $\overline{q}_p$. If the solutions for (1) and (2) are given by (7) and (13), and $\dot{q} = -\pi \cdot q + \tau$, then the value function (14) can be rewritten as follows.

$$V(q) \quad = \quad \int_0^{\infty} e^{-(r+\beta_c+\beta_p)t} \left\{ \begin{array}{l} \beta_c[u(\overbrace{e^{-\pi t} \cdot q + (1 - e^{-\pi t}) \cdot \frac{\tau}{\pi}}^{q(t)}) + V(0)] \\ +\beta_p[-(\overline{q}_p - (\underbrace{e^{-\pi t} \cdot q + (1 - e^{-\pi t}) \cdot \frac{\tau}{\pi}}_{q(t)})) + V(\overline{q}_p)] \end{array} \right\} dt. \tag{17}$$

Notice that if $\dot{q} = -\pi \cdot q + \tau$, then $q(t) = e^{-\pi t} \cdot q(0) + (1 - e^{-\pi t}) \cdot \frac{\tau}{\pi}$. Therefore, the marginal value of real balance is given by:

$$V'(q) \quad = \quad \frac{\beta_p}{\beta_p + \beta_c + r + \pi} + \beta_c \cdot \int_0^{\infty} e^{-(\beta_p+\beta_c+r+\pi)t} \cdot u'(e^{-\pi t} \cdot q + (1 - e^{-\pi t}) \cdot \frac{\tau}{\pi}) dt. \tag{18}$$

Concavity of $V(\cdot)$ follows from concavity of $u(\cdot)$, hence (1) and (2) have unique solutions. In order to have (7) as the solution for (1), we should have $V'(\overline{q}_p) = 1$. Using (16) and (18) we can characterize $\overline{q}_p$ by the following expression.

$$\frac{\beta_c + r + \pi}{\beta_p + \beta_c + r + \pi} \quad = \quad \beta_c \int_0^{\infty} e^{-(\beta_p+\beta_c+r+\pi)t} \cdot u'(e^{-\pi t} \cdot \overline{q}_p + (1 - e^{-\pi t}) \cdot \frac{\beta_p}{\beta_c + \beta_p} \overline{q}_p) \, dt. \tag{19}$$

Notice that for a given value of β_p ,β_c, r and π, the left hand side of (19) is constant while the right hand side of it is decreasing in $\overline{q}_p$, since $u(\cdot)$ is concave. Therefore if $u'(0) > (\beta_c + r + \pi)/\beta_c$, there is a unique positive solution, $\overline{q}_p > 0$, for (19). Notice that if *Assumption 1* holds and $\pi > 0$ is small enough, then $u'(0) > (\beta_c + r + \pi)/\beta_c$ holds.

Moreover, in order to have (13) as the solution for (2), it is necessary and sufficient to have $u'(\overline{q}) \geq V'(0)$. Notice that if $\pi = 0$, then (19) is equivalent to (12), that is $u'(\overline{q}_p) = \frac{r+\beta_c}{\beta_c}$, and

$V'(0) = (\beta_p + \beta_c \cdot u'(0))/(\beta_p + \beta_c + r)$. In this case, as we studied in the previous section, *Assumption 1* is sufficient to have $u'(\overline{q}_p) = \frac{r+\beta_c}{\beta_c} > V'(0)$. If $\pi > 0$ is small enough, such that

$$V'(\overline{q}_p) - \frac{\beta_p + \beta_c \cdot u'(\overline{q}_p)}{\beta_p + \beta_c + r + \pi} \quad < \quad \frac{\beta_c}{(\beta_p + \beta_c + r + \pi)^2} \left\{ (\frac{1}{\beta_c})^2 \left[(\beta_c + r + \pi)(\beta_p + \beta_c + r + \pi) - \beta_c \beta_p \right] - u'(0) \right\} \tag{20}$$

then $u'(\overline{q}_p) \geq \frac{\beta_p + \beta_c \cdot u'(0)}{\beta_p + \beta_c + r + \pi} > V'(0)$. Notice that if *Assumption 1* holds then the right hand side of (20) is strictly positive, and as $\pi \to 0$, the left hand side of (20) goes to zero.

For an arbitrary $0 < q < \overline{q}_p$, there are $\mu(q)$ agents whose real money holding is less than or equal to q, and there are $1 - \mu(q)$ agents whose real money holding is greater than q. Consider the set of agents whose money holding is less than or equal to q. At any point in time, $\beta_p \cdot \mu(q)$ flow of these agents receive production opportunities and leave this set, and $\beta_c \cdot (1 - \mu(q))$ flow of agents in the complement set, receive consumption desire, adjust their real money holding to zero and enter the set of agents with money holding less than or equal to q. Moreover, the real money holding of agents with real balance q, changes at rate $-\pi \cdot q + \tau$, which is positive (negative) if $q < \frac{\tau}{\pi}$ ($q > \frac{\tau}{\pi}$). Therefore, at any point in time $(-\pi \cdot q + \tau) \cdot \mu'(q)$ flow of agents leave the set of agents with real balances less than or equal to q.[5] Since $\mu(\cdot)$ is the ergodic distribution, at any point the flow of agents who enter this set should be equal to the flow of agents who leave the set. Therefore, we can characterize $\mu(\cdot)$ by the following differential equation:

$$\beta_c \cdot (1 - \mu(q)) \quad = \quad \beta_p \cdot \mu(q) + (-\pi \cdot q + \tau) \cdot \mu'(q) \tag{21}$$

with the boundary conditions $\mu(0) = 0$ and $\mu(\overline{q}_p) = 1$. The unique solution for (21) with these boundary conditions is given by:

$$\mu(q) \quad = \quad \begin{cases} \frac{\beta_c}{\beta_c+\beta_p} - \frac{\beta_c}{\beta_c+\beta_p} \left[1 - q/(\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}) \right]^{(\frac{\beta_p+\beta_c}{\pi})} & if \quad q \leq \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_p} \\ \frac{\beta_c}{\beta_c+\beta_p} + \frac{\beta_p}{\beta_c+\beta_p} \left[q/(\frac{\beta_c \cdot \overline{q}_p}{\beta_p+\beta_c}) - \frac{\beta_p}{\beta_c} \right]^{(\frac{\beta_p+\beta_p}{\pi})} & if \quad q \geq \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c} \end{cases} . \tag{22}$$

The implied *probability density function* from (22) is given by:

$$\mu'(q) \quad = \quad \begin{cases} \left(\frac{\beta_c}{\pi} / (\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}) \right) \left[1 - q/(\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}) \right]^{(\frac{\beta_p+\beta_p}{\pi} - 1)} & if \quad q \leq \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c} \\ \left(\frac{\beta_p}{\pi} / (\frac{\beta_c \cdot \overline{q}_p}{\beta_p+\beta_c}) \right) \left[q/(\frac{\beta_c \cdot \overline{q}_p}{\beta_p+\beta_c}) - \frac{\beta_p}{\beta_c} \right]^{(\frac{\beta_p+\beta_c}{\pi} - 1)} & if \quad q \geq \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c} \end{cases} . \tag{23}$$

[5] Notice that when $q > \frac{\tau}{\pi}$ the net change in money holding of agents with real money balance q is negative, and hence agents with a real money balance just above q would enter the set of agents with real money balance less than or equal to q

Notice that the *pdf* of the invariant distribution is a decreasing function on $\left[0, \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}\right)$ and an increasing one on $\left(\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}, \overline{q}_p\right]$, with $\mu'(\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}) = 0$. The following observation follows from (7) and (13). After production, agents adjust their real money balance to $\overline{q}_p$. However, after that the net gain from inflation $-\pi \cdot q + \tau$ is negative for them. Therefore, the real balances of those who receive neither consumption nor production opportunities gradually decreases. After consumption, agents adjust their real money balance to zero. However, after that the net gain from inflation $-\pi \cdot q + \tau$ is positive for them. Therefore, the real balances of those who receive neither consumption nor production opportunities gradually increases. Notice that in order to have a real money balance equal to $\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}$ an agent should have received neither consumption nor production opportunities for an infinite time. Therefore, the probability density of $q = \frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}$ is zero.

Next we study the effect of inflation on the production level. In particular, I show that under certain assumptions, increasing inflation from zero will increase production; that is $\frac{d\overline{q}_p}{d\pi}|_{\pi=0} > 0$. Applying the implicit function theorem to (19) at $\pi = 0$, we have:

$$\begin{aligned}
\frac{d\overline{q}_p}{d\pi}|_{\pi=0} &= -\frac{\frac{\beta_p}{(\beta_p+\beta_c+r)^2} + \beta_c \int_0^\infty e^{-(\beta_p+\beta_c+r)t} \cdot t\left[u'(\overline{q}_p^*) + \frac{\beta_c \overline{q}_p^*}{\beta_c+\beta_p} u''(\overline{q}_p^*)\right] dt}{-\beta_c \int_0^\infty e^{-(\beta_p+\beta_c+r)t} u''(\overline{q}_p^*)} \\
&= \frac{-\frac{\beta_p}{\beta_c} - u'(\overline{q}_p^*) + \frac{\beta_c \overline{q}_p^*}{\beta_c+\beta_p}(-u''(\overline{q}_p^*))}{(\beta_p + \beta_c + r)(-u''(\overline{q}_p^*))} \qquad (24)
\end{aligned}$$

where $\overline{q}_p^*$ is the solution for (12), that is $u'(\overline{q}_p^*) = \frac{\beta_c+r}{\beta_c}$. Since the denominator of (24) is positive, the necessary and sufficient condition for $\frac{d\overline{q}_p}{d\pi}|_{\pi=0} > 0$ can be summarized as follows.

Proposition 1: $\frac{d\overline{q}_p}{d\pi}|_{\pi=0} > 0$, if and only if

$$-\frac{\overline{q}_p^* \cdot u''(\overline{q}_p^*)}{u'(\overline{q}_p^*)} > \frac{(\beta_p + \beta_c)(\beta_p + \beta_c + r)}{\beta_c(\beta_c + r)} \qquad (25)$$

where $u'(\overline{q}_p^*) = \frac{\beta_c+r}{\beta_c}$.

Notice that if $u(\cdot)$ is a *CRRA* utility function, that is $\frac{-c \cdot u''(c)}{u'(c)} = \sigma$, then (25) gives a lower bound for the coefficient of of relative risk aversion. In short, if agents are sufficiently risk averse then increasing inflation from zero would increase production. Total production equals total consumption, and at any point the flow of consumption is equal to β_c times the average real money holding, that is $\frac{\beta_p \beta_c}{\beta_p+\beta_c} \cdot \overline{q}_p$. Therefore, an increase in $\overline{q}_p$ increases the total production and consumption levels.

This result is contrary to the intuition behind the Friedman rule. According to the Friedman rule, when agents hold money for purchasing their consumption goods, inflation works as a tax on labor supply. Therefore increasing inflation distorts labor supply and reduces production. The same type of distortion is in play in the environment of this paper. However, there is a subtle difference between this environment and for example a cash-in-advance environment. In *c.i.a* environments the span of time between incurring the labor supply cost and receiving its benefit from consumption is constant. The time span is random in this environment. Therefore, unlike *c.i.a* models in which higher inflation only works as a higher tax rate and reduces the real pay-off of labor supply, in this environment higher inflation also increases the risk associated with the uncertainty about the amount of consumption. That is, since the time of consumption is random, high inflation has a larger negative effect on the consumption amount when a consumption desire arrives late than when it arrives early. Therefore, although higher inflation works as a distortionary tax, reducing agents incentive to produce, it also increases the risk associated with the consumption amount, inducing risk-averse agents to produce more.

It is important to notice that although inflation increases the risk for an agent who has produced, due to its redistributive nature, the ex-ante risk decreases. In particular, we can show that if $\overline{q}_p^0 < \overline{q}_p^\pi$, where $\overline{q}_p^0$ and $\overline{q}_p^\pi$ are respectively the optimal real balance after production when inflation rate is zero and $\pi > 0$, then $\int_0^q \mu_0(q')dq' \leq \int_0^q \mu_\pi(q')dq'$, $\forall q \geq 0$, where $\mu_0(\cdot)$ and $\mu_\pi(\cdot)$ are the equilibrium invariant distribution of real balances when the inflation rate is zero and when it is $\pi > 0$.[6]

The welfare increasing implication of a mild inflation, due to its redistributive role, is a well-known result. In this environment, however, under certain assumptions a mild inflation increases the aggregate production and consumption, generating a positive correlation between output level and inflation level when inflation rate is sufficiently close to zero.

5 Positive Inflation with Proportional Transfers

In this section we will study the monetary equilibrium when the supply of money grows by making transfers of money to producers, proportional to their production amount.[7] Before proceeding let's

[6] Recall that $\mu_0(\cdot)$ has two mass points of probability $\frac{\beta_c}{\beta_p+\beta_c}$ on $q = 0$ and probability $\frac{\beta_p}{\beta_p+\beta_c}$ on $q = \overline{q}_p^0$, therefore we have:

$$\int_0^q \mu_0(q')dq' = \begin{cases} \frac{\beta_c}{\beta_p+\beta_c} \cdot q & if\ q \leq \overline{q}_p^0 \\ \frac{\beta_c}{\beta_p+\beta_c} \cdot \overline{q}_p^0 + (q - \overline{q}_p^0) & if\ q \geq \overline{q}_p^0 \end{cases} . \tag{26}$$

Moreover, using (22) we have:

$$\int_0^q \mu_\pi(q')dq' = \begin{cases} \frac{\beta_c}{\beta_p+\beta_c} \cdot q - \frac{\beta_p\beta_p \cdot \overline{q}_p^\pi}{(\beta_p+\beta_c)^2} \cdot \frac{\pi}{\beta_p+\beta_p+\pi} \left\{ 1 - \left[1 - q/(\frac{\beta_p \cdot \overline{q}_p}{\beta_p+\beta_c}) \right]^{(\frac{\beta_p+\beta_c+\pi}{\pi})} \right\} & if\ q \leq \frac{\beta_p \cdot \overline{q}_p^\pi}{\beta_p+\beta_c} \\ \frac{\beta_c}{\beta_p+\beta_c} \cdot q - \frac{\beta_p\beta_p \cdot \overline{q}_p^\pi}{(\beta_p+\beta_c)^2} \cdot \frac{\pi}{\beta_p+\beta_p+\pi} \left\{ 1 - \left[q/(\frac{\beta_c \cdot \overline{q}_p}{\beta_p+\beta_c}) - \frac{\beta_p}{\beta_c} \right]^{(\frac{\beta_p+\beta_{p+\pi}}{\pi})} \right\} & if\ \frac{\beta_p \cdot \overline{q}_p^\pi}{\beta_p+\beta_c} \leq q \leq \overline{q}_p^\pi \\ \frac{\beta_c}{\beta_p+\beta_c} \cdot \overline{q}_p^\pi + (q - \overline{q}_p^\pi) & if\ q \geq \overline{q}_p^\pi \end{cases} \tag{27}$$

and $\int_0^q \mu_0(q')dq' \leq \int_0^q \mu_\pi(q')dq'$, $\forall q \geq 0$ follows from comparing (26) and (27).

[7] I again restrict my attention to positive inflation cases. As it becomes clearer below, deflation would increase the real balance of the agents who have produced but not consumed yet. Since their real balance can grow indefinitely, *Assumption 1* will not guarantee that agents would spend their entire real balance upon arrival of a consumption opportunity, which in turn makes the analysis excessively complicated.

recall that in a standard $c.i.a$ environment, injecting money by making proportional transfers to producers is neutral and has no real effect on the aggregate output level. In this environment, however, as I will show below a positive inflation, caused by a proportional transfer of money, can increase output.

Denote the subsidy rate by ρ, so that for price level p the monetary payoff of producing q units of goods is $(1+\rho)p \cdot q$. Also, denote the resulting inflation rate by $\pi = \frac{\dot{p}}{p}$. Although the value of receiving a consumption desire is still governed by (2), the value of receiving a production opportunity in the presence of the proportional transfer is given by:

$$V_p(q) \quad = \quad \max_{q' \geq q} - \frac{q' - q}{(1+\rho)} + V(q'), \tag{28}$$

since increasing the real money supply by one unit only requires $\frac{1}{(1+\rho)}$ units of production. Moreover, the Bellman equation for the continuation value is governed by:

$$V(q) \quad = \quad \frac{1}{r + \beta_p + \beta_c} \left\{ \beta_p V_p(q) + \beta_c V_c(q) + V'(q) \cdot (-\pi \cdot q) \right\}. \tag{29}$$

Notice that unlike (14), in which the net rate of change in real balances could be positive or negative, in this case the net change of the real money balance in the absence of production and consumption, given by $(-\pi \cdot q)$, is always negative.

Similar to the previous cases, the solution for the producer's problem, (28), is given by (7). However, in this case we have $V'(\overline{q}_p) = \frac{1}{(1+\rho)}$. Moreover, I will only study an equilibrium in which agents spend all of their real balance when they receive a consumption desire, that is the solution for the consumer's problem (2) is given by (13). Similar to before, (7) implies $\mu(\overline{q}_p) = 1$, where $\mu(\cdot)$ is the invariant distribution of real balances. Moreover, given (7) and (13) the goods market clearing condition is slightly different from (5) and is given by:

$$\beta_p \int_0^{\overline{q}_p} \frac{\overline{q}_p - q}{(1+\rho)} \cdot d\mu(q) \quad = \quad \beta_c \int_0^{\overline{q}_p} q \cdot d\mu(q) \tag{30}$$

which implies

$$\int q \cdot d\mu(q) \quad = \quad \frac{\beta_p}{\beta_p + \beta_c(1+\rho)} \cdot \overline{q}_p. \tag{31}$$

Since real proportional production transfers are financed by seniorage revenues from taxing agents' real balances, we have:

$$\beta_p \int_0^{\overline{q}_p} \rho \cdot \frac{\overline{q}_p - q}{(1+\rho)} \cdot d\mu(q) \quad = \quad \pi \int_0^{\overline{q}_p} q \cdot d\mu(q)$$

which along (30) implies:

$$\pi \quad = \quad \beta_c \cdot \rho. \tag{32}$$

Since the solutions for the producer's and consumer's problems are given by (7) and (13), the continuation value (29), can be rewritten as follows.

$$V(q) \quad = \quad \int_0^\infty e^{-(r+\beta_c+\beta_p)t}\left\{\beta_c[u(e^{-(\beta_c\cdot\rho)t}\cdot q)+V(0)]+\beta_p[-\frac{(\overline{q}_p-e^{-(\beta_c\cdot\rho)t}\cdot q)}{(1+\rho)}+V(\overline{q}_p)]\right\}dt \tag{33}$$

where $q(t)=e^{-(\beta_c\cdot\rho)t}\cdot q$, follows from $\dot{q}=-\pi\cdot q=-(\beta_c\cdot\rho)\cdot q$. From (33) the marginal value of real balances is given by:

$$V'(q) \quad = \quad \frac{\beta_p}{(1+\rho)(\beta_p+\beta_c(1+\rho)+r)}+\beta_c\cdot\int_0^\infty e^{-(r+\beta_c(1+\rho)+\beta_p)t}\cdot u'(e^{-(\beta_c\cdot\rho)t}\cdot q)dt. \tag{34}$$

Therefore, the solution for (28), characterized by $V'(\overline{q}_p)=\frac{1}{(1+\rho)}$, implies

$$\frac{\beta_c(1+\rho)+r}{\beta_p+\beta_c(1+\rho)+r} \quad = \quad \beta_c(1+\rho)\cdot\int_0^\infty e^{-(r+\beta_c(1+\rho)+\beta_p)t}\cdot u'(e^{-(\beta_c\cdot\rho)t}\cdot\overline{q}_p)dt. \tag{35}$$

If $u'(0)>(\beta_c(1+\rho)+r)/(\beta_c(1+\rho))$, the existence of a unique positive solution for (35) follows from concavity of $u(\cdot)$. Similar to the lump-sum transfer case, it can be shown that *Assumption 1* guarantees the optimality of (13) as the solution for (2). That is, for a small enough inflation rate, $\pi=(\beta_c\cdot\rho)$, we have $u'(\overline{q}_p)\geq V'(0)$, .

For an arbitrary $q\in(0,\overline{q}_p)$, at any point in time, a flow of $\beta_p\cdot\mu(q)$ agents with real money balance less than or equal to q receive a production opportunity and adjust their real balance to $\overline{q}_p>q$, and a flow of $\beta_c\cdot(1-\mu(q))$ agents with real money balance greater than q receive a consumption desire and adjust their real balance to zero which is less than q. Moreover, a flow of $(\pi\cdot q)\cdot\mu'(q)$ agents leave the set of agents with real balances greater than q and enter the set of agents with real balances less than or equal to q. Since $\mu(\cdot)$ is the ergodic distribution, the flow of agents who leave the set should be equal to the flow of agents who enter the set. Hence we can characterize $\mu(\cdot)$ by the following differential equation:

$$\beta_c\cdot(1-\mu(q))+(\pi\cdot q)\cdot\mu'(q) \quad = \quad \beta_p\cdot\mu(q) \tag{36}$$

with the boundary condition $\mu(\overline{q}_p)=1$. The unique solution for (36) is given by:

$$\mu(q) \quad = \quad \frac{\beta_c}{\beta_p+\beta_c}+\frac{\beta_p}{\beta_p+\beta_c}\cdot\left(\frac{q}{\overline{q}_p}\right)^{\frac{\beta_p+\beta_c}{\pi}} \qquad 0\leq q\leq\overline{q}_p. \tag{37}$$

Notice that the ergodic distribution has a mass point with probability $\frac{\beta_c}{\beta_p+\beta_c}$ on $q=0$, and the probability density function, $\mu'(q)=\frac{\beta_p}{\pi\cdot\overline{q}_p}\cdot\left(\frac{q}{\overline{q}_p}\right)^{(\frac{\beta_p+\beta_c}{\pi}-1)}$, is increasing on $(0,\overline{q}_p]$. Thus, agents who receive production opportunities adjust their real balances to $\overline{q}_p$. Then their real balances gradually decrease, unless they receive a consumption desire and adjust their real balances to zero or receive another production opportunity and readjust their real balances to $\overline{q}_p$.

Next we study the effect of transfer rate, ρ, on the level of aggregate production and consumption. In particular I show that increasing inflation, via increasing the proportional transfer rate from zero, will always increase $\overline{q}_p$ and under certain assumptions it also increases the aggregate production and consumption levels. Applying the implicit function theorem to (35) at $\pi = \beta_c \cdot \rho = 0$, we have:

$$\begin{aligned} \frac{d\overline{q}_p}{d\rho}|_{\rho=0} &= -\frac{\frac{\beta_c\beta_p}{(\beta_p+\beta_c+r)^2} - \beta_c \int_0^\infty e^{-(\beta_p+\beta_c+r)t} \cdot u'(\overline{q}_p^*)dt + \beta_c \int_0^\infty e^{-(\beta_p+\beta_c+r)t} \cdot (\beta_c t) \left[u'(\overline{q}_p^*) + \overline{q}_p \cdot u''(\overline{q}_p^*)\right] dt}{-\beta_c \int_0^\infty e^{-(\beta_p+\beta_c+r)t} \cdot u''(\overline{q}_p^*)dt} \\ &= \frac{-\beta_p + (\beta_p+r)u'(\overline{q}_p^*) + \beta_c(-\overline{q}_p^* \cdot u''(\overline{q}_p^*))}{(\beta_p+\beta_c+r)(-u''(\overline{q}_p^*))} > 0 \end{aligned} \tag{38}$$

where $\overline{q}_p^*$ is the solution for (12), that is $u'(\overline{q}_p^*) = \frac{\beta_c+r}{\beta_c}$, and the inequality follows from concavity of $u(\cdot)$ and (12). However, $\frac{d\overline{q}_p}{d\rho}|_{\rho=0} > 0$ does not translate to an increase in the aggregate production and consumption levels. Using (31) and (38), the change of the aggregate consumption and production levels, $\beta_c \cdot \int q \cdot d\mu(q)$, is given by:

$$\begin{aligned} \frac{d}{d\rho}\left(\beta_c \cdot \int q \cdot d\mu(q)\right)|_{\rho=0} &= \frac{d}{d\rho}\left(\frac{\beta_p\beta_c}{\beta_p+\beta_c(1+\rho)} \cdot \overline{q}_p\right)|_{\rho=0} \\ &= \frac{\beta_p\beta_c}{\beta_p+\beta_c}\left\{\frac{-\beta_c}{\beta_p+\beta_c} \cdot \overline{q}_p^* + \frac{d\overline{q}_p}{d\rho}|_{\rho=0}\right\} \\ &= \frac{\beta_p\beta_c}{\beta_p+\beta_c}\left\{\frac{-\beta_p + (\beta_p+r)u'(\overline{q}_p^*) - r \cdot \frac{\beta_c\overline{q}_p^*}{\beta_p+\beta_c}(-u''(\overline{q}_p^*))}{(\beta_p+\beta_c+r)(-u''(\overline{q}_p^*))}\right\}. \end{aligned} \tag{39}$$

Since the denominator of (39) is positive, using (12), the necessary and sufficient condition for $\frac{d}{d\rho}\left(\beta_c \cdot \int q \cdot d\mu(q)\right)|_{\rho=0} > 0$ can be summarized as follows.

Proposition 2: $\frac{d}{d\rho}\left(\beta_c \cdot \int q \cdot d\mu(q)\right)|_{\rho=0} > 0$, if and only if

$$-\frac{\overline{q}_p^* \cdot u''(\overline{q}_p^*)}{u'(\overline{q}_p^*)} < \frac{(\beta_p+\beta_c)(\beta_p+\beta_c+r)}{\beta_c(\beta_c+r)} \tag{40}$$

where $u'(\overline{q}_p^*) = \frac{\beta_c+r}{\beta_c}$.

If $u(\cdot)$ is a *CRRA* utility function, that is $\frac{-c \cdot u''(c)}{u'(c)} = \sigma$, then (40) gives an upper bound for the coefficient of relative risk aversion. In short, increasing inflation via proportional monetary transfers to producers (i.e. production subsidization) increases production, provided agents are not very risk averse. Notice that condition (40) is the exact opposite of (25). That is:

Proposition 3: When inflation is zero, production increases by raising inflation either via lump-sum transfers or proportional transfers, unless

$$-\frac{\overline{q}_p^* \cdot u''(\overline{q}_p^*)}{u'(\overline{q}_p^*)} = \frac{(\beta_p+\beta_c)(\beta_p+\beta_c+r)}{\beta_c(\beta_c+r)}, \tag{41}$$

where $u'(\overline{q}_p^*) = \frac{\beta_c + r}{\beta_c}$, in which case raising inflation by neither lump-sum nor proportional transfers does not affect the aggregate output level.

It is important to notice that although the proportional transfers could induce agents to produce more, the resulting inflation subjects them to higher risk associated with the amount of their consumption. Unlike the lump-sum transfer case in which the redistributive effect of inflation reduces the ex-ante risk, proportional transfers are made to producers alone, who are going to be better off than those who do not receive production opportunities. This would increase inequality instead of alleviating it. More specifically, if $\overline{q}_p^0$ and $\overline{q}_p^\rho$ denote the optimal real balance after production respectively at transfer rates zero and $\rho > 0$, and $\mu_0(\cdot)$ and $\mu_\rho(\cdot)$ denote the corresponding real balance distributions, then from (37) we have $\int_0^q \mu_\rho(q')dq' > \int_0^q \mu_0(q')dq' = \frac{\beta_c}{\beta_p + \beta_c} \cdot q$ for $0 \leq q < \overline{q}_p^0$. However, if the average real balance is higher with the transfer rate $\rho > 0$, that is $\frac{\beta_p \beta_c}{\beta_p + \beta_c(1+\rho)} \cdot \overline{q}_p^\rho > \frac{\beta_p \beta_c}{\beta_p + \beta_c} \cdot \overline{q}_p^0$, then $\int_0^{\overline{q}_p^\rho} \mu_\rho(q')dq' < \int_0^{\overline{q}_p^\rho} \mu_0(q')dq'$.[8] Therefore, $\mu_\rho(\cdot)$ and $\mu_0(\cdot)$ are neither first order nor second order stochastically ordered.

Although in this section we only studied an equilibrium with proportional transfers to producers (i.e. production subsidization) the exact same results hold in the case of proportional transfer to consumers (i.e. consumption subsidization.) It is important to notice that the proportional transfer model captures the essential features of an environment in which only those who participate in market activities are recipients of money injections.

6 Last Thoughts

In the environment studied in this paper, it is always possible to increase aggregate output by raising inflation from zero, either by injecting money via lump-sum transfers or via proportional subsidization of production (or consumption.) If agents are sufficiently risk averse then injecting money via lump-sum transfers not only increases aggregate output, as a result of the redistributive role of inflation, it also reduces the ex-ante risk associated with the amount of consumption. When agents are not too risk averse, an inflation resulting from a proportional subsidization of production (or consumption) increases aggregate output. However, the effect of such a policy on ex-ante risk associated with the amount of consumption is not clear.

The choice between the monetary policy tools, lump-sum transfer or proportional subsidization, depends in particular on the arrival rates of consumption and production opportunities. A high arrival rate of production opportunities makes the condition for optimality of increasing inflation via lump-sum transfers more difficult to satisfy. The intuition is simple: when production opportunities arrive very often, agents do not have to worry about losing the real value of their money holdings while waiting for arrival of a consumption opportunity. They can re-adjust their real balances to the optimal level whenever a production opportunity arrives. Hence, increasing inflation does not make the amount of their consumption prone to a lot of risk, and only reduces the marginal benefit of production.

On the other hand, a high arrival rate of consumption opportunities, makes increasing inflation via lump-sum transfers more likely to be optimal. However, the optimality will not happen unless

[8] Recall that if $\rho > 0$ then $\overline{q}_p^0 < \overline{q}_p^\rho$ therefore if $\int_0^{\overline{q}_p^\rho} \mu_\rho(q')dq' \geq \int_0^{\overline{q}_p^\rho} \mu_0(q')dq'$, then $\int_0^q \mu_\rho(q')dq' \geq \int_0^q \mu_0(q')dq'$ $\forall q \geq 0$, which contradicts $\int q \cdot d\mu_\rho(q) > \int q \cdot d\mu_0(q)$.

agents' coefficient of risk aversion is bounded above one. Notice that incurring production cost to increase the real balance upon arrival of a production opportunity has two benefits: higher consumption upon arrival of the next consumption opportunity, and lower production cost upon arrival of the next production opportunity. While inflation's effect on the first benefit is ambiguous and could be positive or negative depending on agents' relative risk aversion, its effect on the second benefit is always negative. Therefore, if agents are sufficiently risk averse and hence inflation has positive effect on the first benefit, higher arrival rate of consumption opportunities makes it more likely for the positive effect of inflation on the first benefit to over come its negative effect on the second benefit.

The interesting result of the *Proposition 3* is that when increasing inflation (from zero) via lump-sum transfer is not optimal, raising inflation via proportional subsidization of production (or consumption) increases the aggregate output. The welfare implication of such a policy, however, is a subject for further investigation.

The intuition provided in this paper has been the trade-off between the tax-like effect of inflation, and the risk effect caused by the interaction of inflation with random consumption timing. Alternatively, it might be possible to explain the mechanisms as the trade-off between the substitution and income effects of inflation. While inflation reduces the real rate of return on holding money and therefore the substitution effect reduces agents incentive to produce, it also reduces agents real balance at the time of consumption and therefore the income effect increases their incentive to produce.[9] The distinction between these alternatively explanations boils down to the interpretation of $\sigma = -u''(c) \cdot c/u'(c)$ as the coefficient of relative risk aversion or as the inverse of the elasticity of intertemporal substitution. This distinction has been well studied in the literature. (See Weil (1990) and Epstein & Zin (1989).) Differentiating between these two intuitions is beyond the scope of this paper.

Since households receive expenditure shocks (i.e. consumption desires) for which they need cash for purchasing certain goods and services, the assumption of this paper regarding random arrival of consumption desire could be easily motivated. However, given the steadiness of households' labor income, one might find it difficult to justify the assumption of this paper regarding random arrival of production opportunities. In the appendix I study an environment with random consumption timing, however, production spells deterministic production timing. Similar to the *Propositions 1* and *2*, the *Propositions 4* and *5* show that if agents are risk averse enough then increasing inflation from zero via lump-sum transfer of money increases the aggregate output, and if they are not too risk averse, then increasing inflation from zero via proportional subsidization of production increases the aggregate output. Therefore, it is possible to conclude that only the randomness of consumption timing is essential for the paper's results.

The environment of this paper lacks interest-bearing assets. Obviously if there is an interest-bearing asset which could be used for purchasing goods, then agents would abandon non-interest-bearing money and only hold interest-bearing assets in their portfolio. However, if money is the only universally acceptable asset for exchanging with goods, then agents would hold at least part of their

[9] In the *c.i.a.* models with degenerate distribution of real balances, lump-sum transfers return the entire lose of real balances back to the agents and therefore the income effect is absent. Proportional transfer has neither income nor substitution effect in these set ups.

portfolio in the form of money. In particular, let's assume the authenticity of an interest-bearing asset, called *bond*, is not easily verifiable, while the authenticity of money is universally recognized. Moreover, agents do not have permanent access to the bond market and the opportunity to access this market arrives following a poisson process. In this case, agents receive money in exchange for their production and always keep a fraction of their portfolio in the form of money in order to be able to consume when they receive a consumption desire. Of course, agents would utilize opportunities of access to the bond market to adjust their portfolio of money/bond. The rate of access to the bond market, which should be interpreted as the degree of liquidity (convertibility) of bonds, has important implications for how changing the inflation rate affects agents' incentives to work, and therefore the aggregate production level. This is a subject of further study, but my conjecture is that as long as agents hold part of their portfolio in the form of non-interest-bearing money, and inflation increases the risk associated with the purchasing power of their money holdings at the time of receiving consumption desire, there is a channel which induces agents to work more in order to secure themselves against risk when inflation increases. Whether this effect is large enough to overcome the distortionary effect of increasing inflation on labor supply should be studied.

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Appendix A Deterministic Production Time

Consider an environment similar to the one described in the paper with unit one measure of infinitely lived agents in a continuous time setup. Let's assume agents receive production opportunities deterministically every T period. That is at any point in time, a flow of agents with size $\frac{1}{T}$ receive production opportunities. Consumption opportunities, however, arrive according to a Poisson process with the arrival rate β_c. There is an inflation rate of π due to a lump-sum transfer of money with real value τ.

Let's $\mu(q)$ denotes the distribution of agents' real balance and $\mu_T(q)$ denotes the real balance distribution of agents who have produced T units of time before, right before they produce again.

The relation between inflation rate and the real value of lump-sum monetary transfer is given by:

$$\tau \quad = \quad \pi \cdot \int q d\mu(q). \tag{42}$$

Moreover from the goods market clearing condition we have:

$$\beta_c \cdot \int q d\mu(q) \quad = \quad \frac{1}{T} \cdot \int (\overline{q} - q) d\mu_T(q) \tag{43}$$

where $\overline{q}$ is the level to which agents adjust their real balances once they receive a production opportunity. Notice that the density of agents who produce is $\frac{1}{T}$ while a flow of them with size β_c, consumes.

Plugging (43) into (42), we can determine the relation between τ and π in the equilibrium by using $\mu_T(\cdot)$. μ_T has a mass of size $e^{-\beta_c \cdot T}$ at $q = e^{-\pi \cdot T} \cdot \overline{q} + (1 - e^{-\pi \cdot T}) \cdot \frac{\tau}{\pi}$. These are agents who did not receive consumption opportunity between their production opportunities, and their real balance evolved by losing its value at rate π while receiving a stream of real balance τ. Those who receive consumption opportunities will have real balances in $[0, (1 - e^{-\pi \cdot T}) \cdot \frac{\tau}{\pi})$ based on the time they have received consumption opportunities. From those who have produced T periods ago, $1 - e^{-\beta_c \cdot \tilde{t}}$ fraction of them have received a consumption opportunity in the last $\tilde{t}$ periods. The real balance of these agents will be less than or equal to $(1 - e^{-\pi \cdot \tilde{t}}) \cdot \frac{\tau}{\pi}$. Therefore, for $q \in [0, (1 - e^{-\pi \cdot T}) \cdot \frac{\tau}{\pi})$ we have

$$\mu_T(q) = 1 - (1 - \frac{q}{\tau/\pi})^{\frac{\beta_c}{\pi}} \tag{44}$$

which in turn implies the probability density funcion:

$$\mu_T'(q) = \frac{\beta_c}{\tau} \cdot (1 - \frac{q}{\tau/\pi})^{(\frac{\beta_c}{\pi} - 1)}. \tag{45}$$

Using (42), (43) and (45) we have:

$$\begin{aligned}
\frac{\tau}{\pi} &= \frac{1}{\beta_c \cdot T} \int (\overline{q} - q) d\mu_T(q) \\
&= \frac{1}{\beta_c \cdot T} \left\{ e^{-\beta_c T} \cdot (1 - e^{-\pi T})(\overline{q} - \frac{\tau}{\pi}) + \int_0^{(1-e^{-\pi T})\frac{\tau}{\pi}} (\overline{q} - q) \cdot \frac{\beta_c}{\tau} \cdot (1 - \frac{q}{\tau/\pi})^{(\frac{\beta_c}{\pi} - 1)} dq \right\} \\
&= \frac{1}{\beta_c \cdot T} \left\{ (1 - e^{-(\beta_c + \pi)T})\overline{q} + \frac{\tau}{\pi} \cdot \frac{1}{1 + \frac{\beta_c}{\pi}} (e^{-(\beta_c + \pi)T} - 1) \right\}
\end{aligned}$$

which in turn implies the following relation between the inflation rate and the real value of the lump-sum transfer.

$$\frac{\tau}{\pi} = \frac{1-e^{-(\beta_c+\pi)T}}{\beta_c \cdot T + \frac{1}{1+\frac{\beta_c}{\pi}} \cdot (1-e^{-(\beta_c+\pi)T})} \cdot \overline{q} \tag{46}$$

Let's $V_t(q)$ denotes the value of having a real balance of q, t periods after the last production opportunity. The only possible event until the next production opportunity is the arrival of a consumption desire, in which case a modified version of *Assumption 1* guarantees consumption of the entire real balance. Therefore,

$$\begin{aligned} V_t(q) &= \int_0^{T-t} e^{-(r+\beta_c)\tilde{t}} \cdot \beta_c \left\{ u(e^{-\pi\tilde{t}} \cdot q + (1-e^{-\pi\tilde{t}}) \cdot \frac{\tau}{\pi}) + V_{\tilde{t}}(0) \right\} d\tilde{t} \\ &\quad + e^{-r\cdot(T-t)} \cdot (1-e^{-\beta_c\cdot(T-t)}) \cdot \left\{ -\overline{q} + (e^{-\pi\cdot(T-t)} \cdot q + (1-e^{-\pi\cdot(T-t)}) \cdot \frac{\tau}{\pi}) + V_0(\overline{q}) \right\} \end{aligned} \tag{47}$$

Notice that since the real balance follows $\dot{q} = -\pi \cdot q + \tau$, the real balance of the agent evolves to $(e^{-\pi\tilde{t}} \cdot q + (1-e^{-\pi\tilde{t}}) \cdot \frac{\tau}{\pi})$ after $\tilde{t}$ periods. Since the cost of production is linear, $\overline{q}$ is determined by $V_0'(\overline{q}) = 1$. Taking derivative of (47) with respect to q at $t=0$ and substituting for $\frac{\tau}{\pi}$ from (46) the equilibrium value of $\overline{q}$ is characterized as follows.

$$1 = \int_0^T e^{-(r+\beta_c+\pi)t} \cdot \beta_c \cdot u'([e^{-\pi t} + \frac{(1-e^{-\pi t})(1-e^{-(\beta_c+\pi)T})}{\beta_c \cdot T + \frac{1}{1+\frac{\beta_c}{\pi}} \cdot (1-e^{-(\beta_c+\pi)T})}] \cdot \overline{q}_\pi)dt + e^{-(r+\pi)T}(1-e^{-\beta_c T}) \tag{48}$$

Note that if $\pi = 0$, then (48) implies

$$u'(\overline{q}_0) = \frac{r+\beta_c}{\beta_c} \cdot \frac{1-e^{-rT}(1-e^{-\beta_c T})}{1-e^{-(r+\beta_c)T}}. \tag{49}$$

Taking derivative from the implicit function (48) at zero inflation rate will yield the following.

$$\begin{aligned} \frac{d\overline{q}}{d\pi}|_{\pi=0} &= -\frac{\int_0^T e^{-(r+\beta_c)t}\beta_c \cdot (-t)[u'(\overline{q}_0) + u''(\overline{q}_0) \cdot \overline{q}_0(1-\frac{1-e^{-\beta_c T}}{\beta_c T})]dt + (-T)e^{-rT}(1-e^{-\beta_c T})}{\int_0^T e^{-(r+\beta_c)t}\beta_c \cdot u''(\overline{q}_0)dt} \\ &= -\frac{\frac{\beta_c}{r+\beta_c}(Te^{-(r+\beta_c)T} - \frac{1-e^{-(r+\beta_c)T}}{(r+\beta_c)})[u'(\overline{q}_0) + u''(\overline{q}_0) \cdot \overline{q}_0(1-\frac{1-e^{-\beta_c T}}{\beta_c T})] - Te^{-rT}(1-e^{-\beta_c T})}{\frac{\beta_c}{r+\beta_c}(1-e^{-(r+\beta_c)T})u''(\overline{q}_0)}. \end{aligned} \tag{50}$$

By (42), the output level, Y, which is same as the total consumption level, $\beta_c \cdot \int q d\mu(q)$, equals to $\beta_c \cdot \frac{\tau}{\pi}$. Then using (46) we have:

$$\frac{dY}{d\pi}|_{\pi=0} = \frac{1-e^{-\beta_c T}}{T} \cdot \frac{d\overline{q}}{d\pi}|_{\pi=0} + e^{-\beta_c T} \cdot \overline{q}_0 \tag{51}$$

By (50) and (51) we have the following proposition.

Proposition 4: Increasing inflation from zero via lump-sum transfer of money, increases the output (i.e. $\frac{dY}{d\pi}|_{\pi=0} > 0$) if and only if

$$-\frac{u''(\overline{q}_0)\cdot\overline{q}_0}{u'(\overline{q})} \geq \frac{\frac{1-e^{-\beta_c T}}{T}(\frac{1-e^{-(r+\beta_c)T}}{r+\beta_c} - Te^{-(r+\beta_c)T}) + e^{rT}(1-e^{-\beta_c T})^2 \cdot \frac{1-e^{-(r+\beta_c)T}}{1-e^{-rT}(1-e^{-\beta_c T})}}{\frac{1-e^{-\beta_c T}}{T}(\frac{1-e^{-(r+\beta_c)T}}{r+\beta_c} - Te^{-(r+\beta_c)T})(1-\frac{1-e^{-\beta_c T}}{\beta_c T}) + e^{-\beta_c T}(1-e^{-(r+\beta_c)T})} \tag{52}$$

where $u'(\overline{q}_0) = \frac{r+\beta_c}{\beta_c}\cdot\frac{1-e^{-rT}(1-e^{-\beta_c T})}{1-e^{-(r+\beta_c)T}}$.

Notice that the left hand side of (52) is the coefficient of risk aversion at $\overline{q}_0$ and the right hand side of it is positive. The positivity of the right hand side follows from $\frac{1}{1+(r+\beta_c)T} > e^{-(r+\beta_c)T}$, $\forall T > 0$. In short, provided agents are risk averse enough, increasing inflation from zero via lump-sum transfer of money increases the aggregate output.

Next we study the case of injecting money via proportional subsidization of production. Suppose the monetary authority subsidizes production at rate ρ. In particular, in order to adjust their real balance to $\overline{q}$ agents should only produce $\frac{1}{1+\rho}(\overline{q}-q)$, and the monetary authority will provide the remaining $\rho\cdot\frac{1}{1+\rho}(\overline{q}-q)$.

Suppose $\mu_t(\cdot)$ denotes the real balance distribution of agents who have had their last production opportunity, t periods ago, and $\mu(\cdot)$ denotes the real balance distribution of all agents. The relation between the inflation rate and the subsidization rate is given by:

$$\pi\cdot\int q d\mu(q) = \frac{1}{T}\cdot\int\frac{\rho}{1+\rho}(\overline{q}-q)d\mu_T(q). \tag{53}$$

The goods market clearing condition implies:

$$\beta_c\cdot\int q d\mu(q) = \frac{1}{T}\cdot\int\frac{1}{1+\rho}(\overline{q}-q)d\mu_T(q) \tag{54}$$

and by combining (53) and (54) we have $\rho = \frac{\pi}{\beta_c}$.

Since agents do not receive any monetary transfer when they do not produce, μ_T will have a mass point of size $(1-e^{-\beta_c T})$ at $q=0$. These are agents who have received a consumption opportunity during the past T periods, and have spent all of their real balances on consumption. (A modified version of *Assumption 1* guarantees that agents use all of their real balances upon arrival of a consumption desire.) μ_T will also have a mass point of size $e^{-\beta_c T}$ at $q = e^{-\pi T}\overline{q}$. These are agents who have not received any consumption opportunity during the past T periods, and their real balances have shrunken due to inflation. Using (54), the output level, Y, which is same as the consumption level $\beta_c\cdot\int q d\mu(q)$ is given by:

$$Y = \frac{1}{T}\cdot\frac{1}{1+\rho}\cdot\Big((1-e^{-\beta_c T}) + e^{-\beta_c T}(1-e^{-(\rho\beta_c)T})\Big)\cdot\overline{q}$$

which in turn implies:

$$\frac{dY}{d\rho}|_{\rho=0} = \frac{1-e^{-\beta_c T}}{T} \cdot \frac{d\overline{q}}{d\rho}|_{\rho=0} + (\beta_c e^{-\beta_c T} - \frac{1}{T}(1-e^{-\beta_c T})) \cdot \overline{q}_0. \tag{55}$$

Denoting the value of having real balance of q after t periods since the last production by $V_t(q)$, we have:

$$\begin{aligned} V_t(q) &= \int_0^{T-t} e^{-(r+\beta_c)\tilde{t}} \cdot \beta_c \left\{ u(e^{-\pi\tilde{t}} \cdot q) + V_{\tilde{t}}(0) \right\} d\tilde{t} \\ &\quad + e^{-r\cdot(T-t)} \cdot (1-e^{-\beta_c\cdot(T-t)}) \cdot \left\{ -\frac{\beta_c}{\beta_c+\pi}(\overline{q} - e^{-\pi\cdot(T-t)} \cdot q) + V_0(\overline{q}) \right\}. \end{aligned} \tag{56}$$

Since the linear production is subsidized at rate $\rho = \frac{\pi}{\beta_c}$, in the equilibrium we have $V_0^{'}(\overline{q}_\pi) = \frac{1}{1+\rho}$. Taking derivative of (56) with respect to q at $t=0$, the equilibrium $\overline{q}$ is determined by the following.

$$\frac{1}{1+\rho} = \int_0^T e^{-(r+\beta_c+\rho\beta_c)t} \cdot \beta_c \cdot u'(e^{-(\rho\beta_c)t} \cdot \overline{q}_\rho)dt + e^{-(r+\rho\beta_c)T}(1-e^{-\beta_c T}) \cdot \frac{1}{1+\rho}. \tag{57}$$

Note that similar to the lump-sum transfer case, if $\pi = 0$, then (57) implies (49). However, taking derivative from the implicit function (57) at zero inflation rate yields a different result:

$$\begin{aligned} \frac{d\overline{q}}{d\rho}|_{\rho=0} &= -\frac{\int_0^T e^{-(r+\beta_c)t}\beta_c \cdot (-\beta_c t)[u'(\overline{q}_0) + u''(\overline{q}_0) \cdot \overline{q}_0]dt - (\beta_c T+1)e^{-rT}(1-e^{-\beta_c T}) + 1}{\int_0^T e^{-(r+\beta_c)t}\beta_c \cdot u''(\overline{q}_0)dt} \\ &= -\frac{\frac{\beta_c^2}{r+\beta_c}(Te^{-(r+\beta_c)T} - \frac{1-e^{-(r+\beta_c)T}}{(r+\beta_c)})[u'(\overline{q}_0) + u''(\overline{q}_0) \cdot \overline{q}_0] - (\beta_c T+1)e^{-rT}(1-e^{-\beta_c T}) + 1}{\frac{\beta_c}{r+\beta_c}(1-e^{-(r+\beta_c)T})u''(\overline{q}_0)}. \end{aligned} \tag{58}$$

Plugging (58) into (55) we get the following proposition.

Proposition 5: Increasing inflation from zero via proportional subsidization, increases the output (i.e. $\frac{dY}{d\rho}|_{\rho=0} > 0$) if and only if

$$-\frac{u''(\overline{q}_0) \cdot \overline{q}_0}{u'(\overline{q})} \leq \frac{\frac{1-e^{-(r+\beta_c)T}}{T}(\frac{1-e^{-\beta_c T}}{\beta_c} - T\frac{e^{-rT}(1-e^{-\beta_c T})^2}{1-e^{-rT}(1-e^{-\beta_c T})}) - \frac{1-e^{-\beta_c T}}{T}(\frac{1-e^{-(r+\beta_c)T}}{r+\beta_c} - Te^{-(r+\beta_c)T})}{\frac{1-e^{-(r+\beta_c)T}}{T}(\frac{1-e^{-\beta_c T}}{\beta_c} - Te^{-\beta_c T}) - \frac{1-e^{-\beta_c T}}{T}(\frac{1-e^{-(r+\beta_c)T}}{r+\beta_c} - Te^{-(r+\beta_c)T})}. \tag{59}$$

where $u'(\overline{q}_0) = \frac{r+\beta_c}{\beta_c} \cdot \frac{1-e^{-rT}(1-e^{-\beta_c T})}{1-e^{-(r+\beta_c)T}}$.

Notice that the right hand side is greater than or equal to 1, iff $\frac{e^{-\beta_c T}}{1-e^{-\beta_c T}} \geq e^{-rT}$. In short, provided agents are not too risk averse, increasing inflation from zero via proportional subsidization of production increases the aggregate output.

Overall, the two propositions state that, even with the deterministic production timing, increasing inflation from zero, via lump-sum transfers of money or proportional subsidization, can increase the aggregate output, provided that the agents' coefficient of risk-aversion is high enough or low enough.