

**STABILIZED FINITE ELEMENT METHODS
FOR INCOMPRESSIBLE FLOWS WITH EMPHASIS ON
MOVING BOUNDARIES AND INTERFACES**

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Abstract

Two finite element formulations for incompressible fluid dynamics are presented. The space-time velocity-pressure formulation is used for flow problems involving moving boundaries and interfaces. The formulation is based on a time-discontinuous Galerkin method, supplemented with least-squares-type stabilization terms. The discussion centers on the application of the method to various problems with deforming domains. In particular, the formulation is employed to simulate three-dimensional sloshing in a storage tank subjected to external vibrations.

The stress-velocity-pressure formulation is developed, essentially for viscoelastic flows. Treatment of stress as a separate unknown allows for inclusion of complex constitutive relations. Least-squares-type stabilization terms provide, once more, robustness to otherwise potentially unstable formulation. The method is first tested on Newtonian fluid flows past a circular cylinder in two dimensions. Then, by using simple viscoelastic constitutive model, the formulation is applied to a standard test problem.

The strategies for the solution of large systems of equations arising from the finite element discretization of the above formulations are also discussed. Particular emphasis is placed on iterative methods and the implementations on massively parallel supercomputers, paving the way for solving very large-scale practical problems, including those in three dimensions.

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List of Figures

3.1	Numerical methods applicable to problems involving deforming domains: author's decision tree (see text for explanation of terms). . . .	11
3.2	Space-time discretization: concept of a space-time slab.	14
3.3	Space-time discretization: deformed element.	16
3.4	Mesh moving options at the free surface: prescribed direction (left), normal direction (center) and local velocity direction (right).	22
3.5	Mesh moving options: movement with no remeshing.	23
3.6	Mesh moving options: movement with discrete remeshing.	24
3.7	Mesh moving options: movement with continuous remeshing.	26
3.8	Fountain flow: domain description.	26
3.9	Fountain flow: finite element mesh at $t = 0.0, 1.0$ (top row), $t = 2.0, 3.0$ (middle row) and $t = 4.0, 5.0$ (bottom row).	28
3.10	Fountain flow: finite element mesh at $t = 6.0$	29
3.11	Fountain flow: pressure field at $t = 0.0, 1.0$ (top row), $t = 2.0, 3.0$ (middle row) and $t = 4.0, 5.0$ (bottom row).	30
3.12	Fountain flow: pressure field at $t = 6.0$	31
3.13	Parallel implementation: element-level (left) and equation-level (right) data storage modes.	37
4.1	Interpolation sets: bi-linear stress sub-elements, bi-quadratic velocity element and bi-linear pressure element of Marchal and Crochet. . . .	44

4.2	Interpolation sets: discontinuous quadratic stress element, bi-quadratic velocity element and discontinuous linear pressure element of Fortin and Fortin.	44
4.3	Interpolation sets: equal-order bi-linear stress, velocity and pressure element admissible by the stabilized formulation.	45
4.4	Strain stabilization: streamwise velocity for the “stick-slip” problem.	46
5.1	Typical finite element matrix: location of non-zero entries.	50
5.2	Typical finite element matrix: skyline profile.	51
5.3	Finite element matrix for an unstructured grid: skyline profile before reordering.	52
5.4	Finite element matrix for an unstructured grid: skyline profile after reordering.	53
6.1	Horizontally oscillating tank: domain description.	62
6.2	Horizontally excited tank: time history of the wave height at the $x = 0$ wall.	65
6.3	Horizontally excited tank: time history of the wave height at the $x = W$ wall.	65
6.4	Horizontally excited tank: finite element mesh at $t = 314.77, 316.06$ (top row), $t = 317.34, 318.63$ (middle row), and $t = 319.91, 321.20$ (bottom row).	66
6.5	Horizontally excited tank: velocity field at $t = 314.77, 316.06$ (top row), $t = 317.34, 318.63$ (middle row), and $t = 319.91, 321.20$ (bottom row).	67
6.6	Vertically excited tank: time history of the wave height at the $(x, y) = (0, 0)$ corner.	72
6.7	Vertically excited tank: time history of the wave height at the $(x, y) = (W, 0)$ corner.	72
6.8	Vertically excited tank: time history of the wave height at the $(x, y) = (0, H)$ corner.	73

6.9	Vertically excited tank: time history of the wave height at the $(x, y) = (W, H)$ corner.	73
6.10	Vertically excited tank: time history of the wave height at point A1.	74
6.11	Vertically excited tank: time history of the wave height at point A2.	74
6.12	Vertically excited tank: free surface view and isolines at $t = 13.74, 14.20,$ and 14.67 (from top to bottom).	75
6.13	Vertically excited tank: free surface view and isolines at $t = 15.13, 15.60,$ and 16.06 (from top to bottom).	76
6.14	Vertically excited tank: free surface view and isolines at $t = 16.52, 16.99,$ and 17.45 (from top to bottom).	77
6.15	Vertically excited tank: free surface view and isolines at $t = 71.29, 71.76,$ and 72.22 (from top to bottom).	78
6.16	Vertically excited tank: free surface view and isolines at $t = 72.69, 73.15,$ and 73.61 (from top to bottom).	79
6.17	Vertically excited tank: free surface view and isolines at $t = 74.08, 74.54,$ and 75.00 (from top to bottom).	80
6.18	Flow past a circular cylinder: domain description.	82
6.19	Flow past a circular cylinder: finite element mesh.	82
6.20	Flow past a circular cylinder: extended finite element mesh.	83
6.21	Flow past a circular cylinder at Reynolds number 1,000: vorticity field at $t = 0, 25, 50, 75, 100, 125, 150$ and 175	85
6.22	Flow past a circular cylinder at Reynolds number 1,000: vorticity field at $t = 221.7, 223.6, 225.9,$ and 227.7	86
6.23	Flow past a circular cylinder at Reynolds number 1,000: time history of the drag coefficient.	87
6.24	Flow past a circular cylinder at Reynolds number 1,000: time history of the lift coefficient.	87
6.25	Flow past a circular cylinder at Reynolds number 2,000: vorticity field at $t = 10, 20, 30, 40, 50, 60$ (left column), $70, 80, 90, 100, 110,$ and 120 (right column).	89

6.26	Flow past a circular cylinder at Reynolds number 2,000: vorticity field at $t = 130, 140, 150, 160, 170, 180$ (left column), $190, 200, 210, 220, 230,$ and 240 (right column).	90
6.27	Flow past a circular cylinder at Reynolds number 2,000: time history of the drag coefficient.	91
6.28	Flow past a circular cylinder at Reynolds number 2,000: time history of the lift coefficient.	91
6.29	Flow past a circular cylinder at Reynolds number 4,000: vorticity field at $t = 10, 20, 30, 40, 50, 60$ (left column), $70, 80, 90, 100, 110,$ and 120 (right column).	93
6.30	Flow past a circular cylinder at Reynolds number 4,000: vorticity field at $t = 130, 140, 150$ (left column), $160, 170$ and 180 (right column).	94
6.31	Flow past a circular cylinder at Reynolds number 4,000: time history of the drag coefficient.	95
6.32	Flow past a circular cylinder at Reynolds number 4,000: time history of the lift coefficient.	96
6.33	Flow past a circular cylinder at Reynolds number 10,000: vorticity field at $t = 50, 55, 60,$ and 65	98
6.34	Flow past a circular cylinder at Reynolds number 10,000: time history of the drag coefficient.	99
6.35	Flow past a circular cylinder at Reynolds number 10,000: time history of the lift coefficient.	99
6.36	Contraction of a viscoelastic fluid: domain description.	100
6.37	Contraction of a viscoelastic fluid: finite element mesh.	101
6.38	Contraction of a viscoelastic fluid: $De = 0.0$ case.	103
6.39	Contraction of a viscoelastic fluid: $De = 0.8$ case.	104
6.40	Contraction of a viscoelastic fluid: $De = 1.6$ case.	105
6.41	Contraction of a viscoelastic fluid: $De = 3.2$ case.	106
6.42	Contraction of a viscoelastic fluid: \mathbf{T}_{1xx} profiles along the $y = 3.0$ line for (from top to bottom) $De = 0.0, De = 0.8, De = 1.6$ and $De = 3.2$	107

List of Tables

6.1 Horizontally excited tank: parameters. 63
6.2 Vertically excited tank: parameters. 69

List of Boxes

5.1	GMRES algorithm: control flow	54
5.2	GMRES algorithm: solution of the reduced system	55
5.3	GMRES algorithm: modified control flow	57

Contents

Abstract	ii
Acknowledgments	iii
List of Figures	iv
List of Tables	v
List of Boxes	vi
1 Introduction	1
1.1 Overview	3
2 Problem Statement	5
2.1 Equations of Motion for Incompressible Fluid Flow	5
2.2 Constitutive Relations	7
3 Space-Time Velocity-Pressure Formulation	10
3.1 Background	10
3.2 Variational Formulation	14
3.3 Stabilization Details	17
3.3.1 Parameter Design	19
3.3.2 Low Order Elements	21
3.4 Moving Boundary Treatment	22

3.4.1	Mesh Moving Options	23
3.4.2	Surface Tension	32
3.5	Matrix Form	32
3.6	Parallel Implementation	36
4	Stress-Velocity-Pressure Formulation	39
4.1	Background	39
4.2	Variational Formulation	41
4.3	Stabilization Details	43
4.3.1	Parameter Design	47
5	Solution Methods	48
5.1	Direct Solution Techniques	48
5.2	Iterative Solution Techniques	52
5.2.1	GMRES Algorithm	53
5.2.2	Preconditioning	58
5.3	Parallel Implementation	59
6	Numerical Examples	61
6.1	Sloshing in a Rectangular Tank	61
6.1.1	Horizontal excitation	61
6.1.2	Vertical excitation	68
6.2	Flows Past a Circular Cylinder	81
6.2.1	Reynolds number 1,000	84
6.2.2	Reynolds number 2,000	88
6.2.3	Reynolds number 4,000	92
6.2.4	Reynolds number 10,000	95
6.3	Contraction of a Viscoelastic Fluid	100
7	Conclusions	108
7.1	Future Research Directions	110

