3 The Influence of Magnetic Fields

Here the analysis assesses the role of magnetic fields in modifying the white dwarf stability conditions. To isolate the magnetic influences, we neglect rotational contributions, by setting $T = 0$. The virial theorem (derived in detail in Sec. 7.1 of the book) then relates the gravitational potential energy W, the internal (pressure) energy Π and the magnetic energy $\mathcal{M}$ via **S & T, Sec. 7.2**

$$W + 3\Pi + \mathcal{M} \ = \ 0 \quad \Rightarrow \quad W + 3M \left\langle \frac{P}{\rho} \right\rangle + \frac{4\pi}{3} R^3 \left\langle \frac{B^2}{8\pi} \right\rangle \ = \ 0 \ . \tag{34}$$

We would like to ascertain how mass and radius change in response to different magnitudes of the magnetic field. Presuming **high electric conductivity**, then **magnetic flux** $\Phi_{\rm M} \sim \langle B \rangle R^2$ is conserved when the stellar radius changes. Since dimensional analysis dictates that $W \propto GM^2/R$, and $P \propto (M/R^3)^{\Gamma-1}$, we can write down the general scaling of the specialized virial equation with M and R:

$$0 \ = \ -\alpha_g \frac{GM^2}{R} + \alpha_i \frac{\mathcal{K} M^\Gamma}{R^{3(\Gamma-1)}} + \alpha_m \frac{\Phi_{\rm M}^2}{R} \quad . \tag{35}$$

Algebraically, the magnetic field effectively serves as a correction to the gravitational binding energy. Solving for R when $\Gamma \neq 4/3$ gives

$$R \ = \ \left\{ \frac{\alpha_i \mathcal{K} M^\Gamma}{\alpha_g G M^2 - \alpha_m \Phi_{\rm M}^2} \right\}^{1/(3\Gamma-4)} \quad . \tag{36}$$

Expressing this in terms of $\delta_{\rm B} = \mathcal{M}/|W|$, the **magnetic perturbation parameter**, as this is usually small, if R_0 represents the WD radius for $B = 0$, then

$$\frac{R}{R_0} \ \approx \ 1 + \frac{\delta_{\rm B}}{3\Gamma - 4} \quad , \quad \delta_{\rm B} \ \equiv \ \frac{\mathcal{M}}{|W|} \ = \ \frac{\alpha_m}{\alpha_g} \frac{\Phi_{\rm M}^2}{GM^2} \quad . \tag{37}$$

Thus *the magnetic field oblates the star slightly* (a few percent) because of the added push provided by its pressure.

For the relativistic case, $\Gamma \approx 4/3$, and Eq. (35) cannot be solved for R. Instead, it solves for mass and yields the modification to the Chandrasekhar mass limit, which is $M_{\rm CH} = (\alpha_i \mathcal{K}/\alpha_g G)^{3/2}$. Then, to leading order in $\delta_{\rm B}$

$$M \ \approx \ M_{\rm CH} \Big(1 + \frac{3}{2} \, \delta_{\rm B} \Big) \quad . \tag{38}$$

In this domain, the radius and central density can vary widely. The polytropic virial theorem is, for $B \to 0$

$$W + U \;=\; E \;=\; -\frac{3-n}{3}\,|W| \;\propto\; \frac{GM^2}{R} \quad , \tag{39}$$

from which it follows that small energy perturbations induced by finite B satisfy

$$\Delta(\log_e R) \;\equiv\; \frac{\Delta R}{R} \;=\; -\frac{\Delta E}{E} \;=\; \frac{3}{3-n}\,\frac{\mathcal{M}}{|W|} \;=\; \frac{3\delta_{\rm B}}{3-n} \quad . \tag{40}$$

In differential form this is a logarithmic derivative, so that

$$R \;=\; R_0\,\exp\biggl\{\frac{3\delta_{\rm B}}{3-n}\biggr\} \quad , \tag{41}$$

and the increase in white dwarf radius is exponential in the magnetic field parameter for a white dwarf near the Chandrasekhar mass limit.

Plot: Radii of Magnetic White Dwarfs

- Thus, in the extreme case of $\delta_{\rm B} \sim 0.1$, corresponding to $B \sim 10^{12}\,$G near the center of the star, the change in core radius is $\sim 40\,\%$, which will *impact pyconuclear reaction rates*. Using flux freezing arguments, $\Phi_{\rm M} \sim const$, during core collapse of a main sequence star to a WD, this demands $B \sim 10^8\,$G in the core of the progenitor star, a high value indeed!

What is $\delta_{\rm B}$ for a WD? Using flux freezing arguments, $\Phi_{\rm M} \sim const$, during core collapse of a main sequence star to a WD, the field is amplified by 4 orders of magnitude. For interior fields of $B \sim 10^3\,$G this yields $B \sim 10^7\,$G, a value that is borne out by observations. Thus,

$$\delta_{\rm B} \;=\; \frac{\mathcal{M}}{|W|} \;\sim\; 10^{-8}\left(\frac{B}{10^8\text{G}}\right)^2\left(\frac{R}{10^{-2}R_{\odot}}\right)^4\left(\frac{M}{M_{\odot}}\right)^{-2} \;\sim\; 10^{-10} - 10^{-6} \quad . \tag{42}$$

Thus $\delta_{\rm B}$ is small overall.

Plot: White dwarf magnetic fields

- Conclusion: the energy equilibrium is only weakly influenced by **B**: there is little influence on permitted masses, but 5-20% modification to radii.

Radii of Magnetic White Dwarfs

- From Ostriker & Hartwick (1968, ApJ **153**, 797).

J=IΩ > 0 provides centripetal support

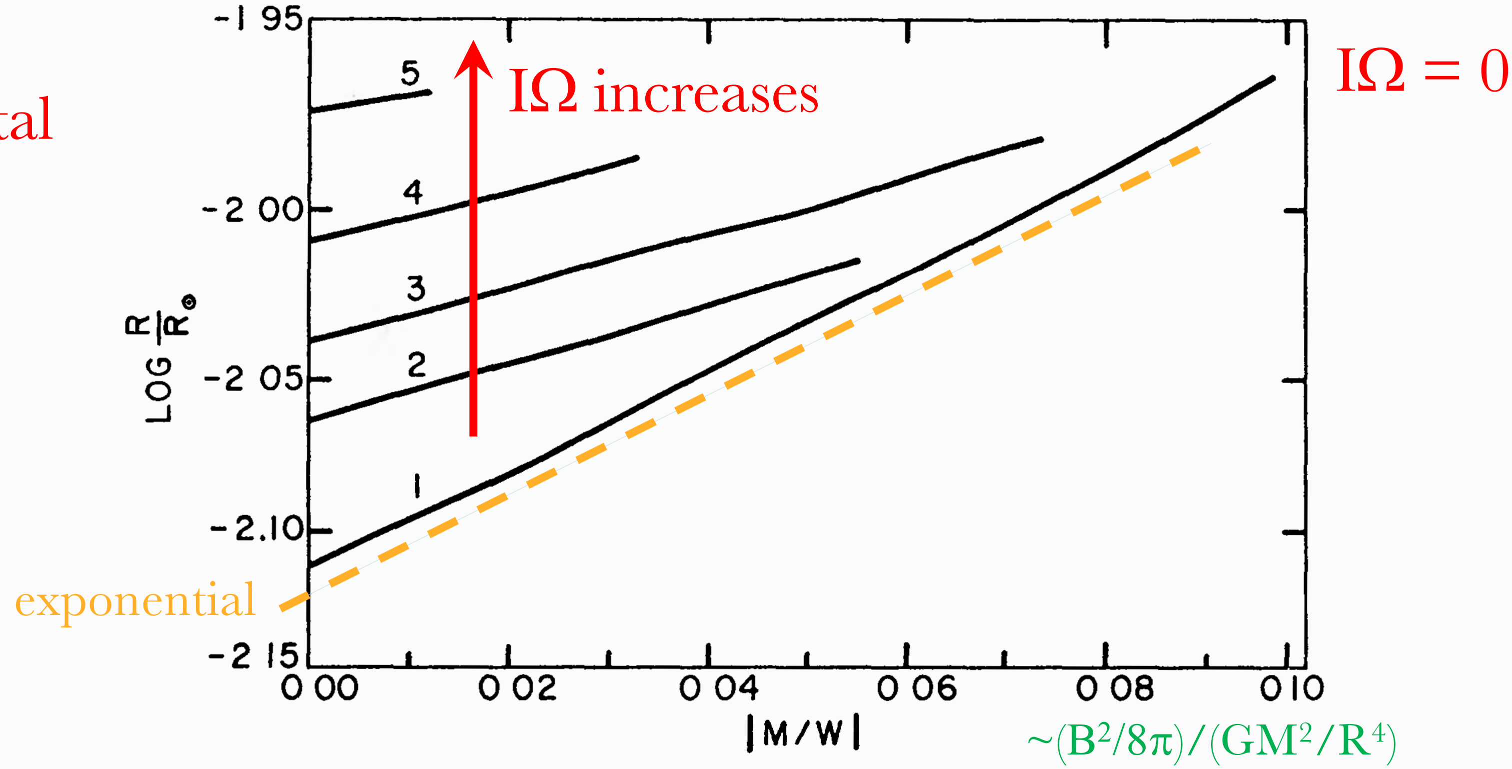
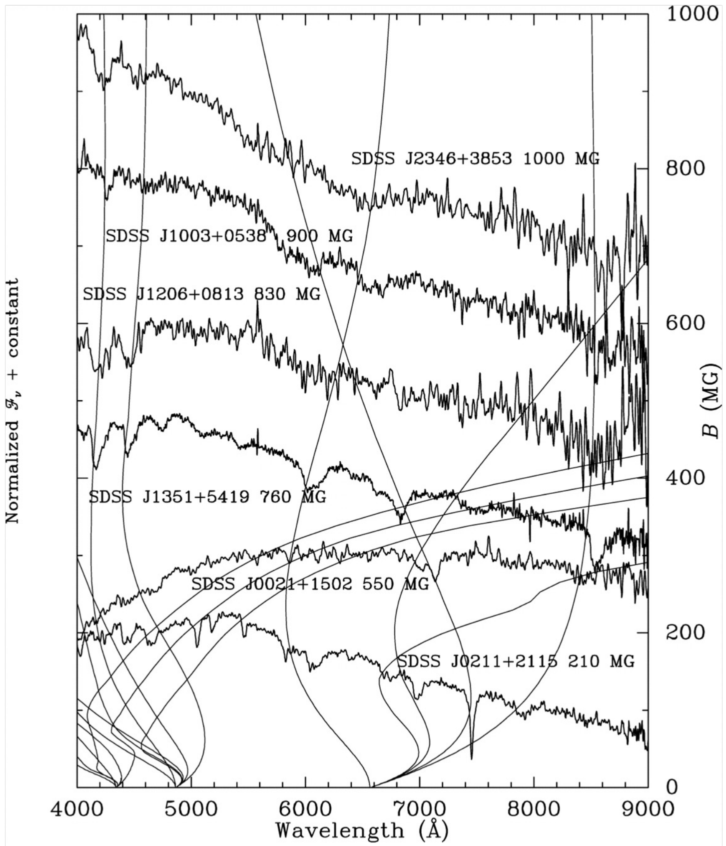
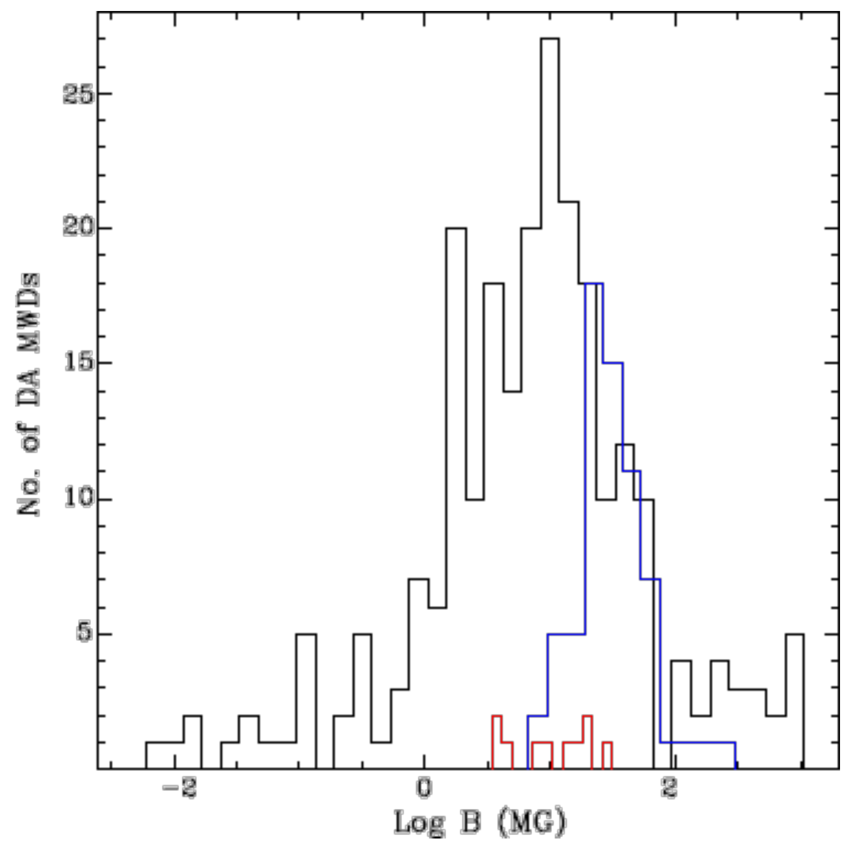


FIG. 2.—Radius of a white dwarf as a function of magnetic energy and angular momentum. For a 1 05 $M_\odot$ star (representing Sirius B), we plot the radius R versus the ratio of magnetic to gravitational energies $|\mathfrak{M}/W|$; R is the larger of the equatorial and polar radii. The curves labeled 1, 2, 3, 4, and 5 represent uniformly rotating sequences having angular momenta equal to (0,1,2,3,4) $\times$ 1.92 $\times 10^{49}$ g cm^2 sec^{-1}, respectively.

White Dwarf Magnetic Fields





- *Left panel*: Optical spectra for six WDs with Hα and Hβ lines split by the Zeeman effect, which is used to measure B. Curves are wavelength variations of split lines as a function of field strength (right axis). Fig. 1 from Vanlandingham et al. (2005, AJ **130**, 734).
- *Right panel*: Magnetic field distribution of ~600 magnetic white dwarfs in the Sloan Digital Sky Survey (SDSS). Black histogram is for isolated WDs, and blue is for polars. Fig. 8 from Ferrario, de Martino & Gaensicke (2015, SSRv **191**, 111).

4 Rotating White Dwarfs

Just as magnetic fields can add buoyancy to white dwarfs, so also are there profound influences of rotation on their structure. To isolate the rotational effects, here $B = 0$ is set throughout.

4.1 Rotating, Spherical Stars

The first stage of the rotational analysis proceeds in similar fashion to the magnetic field study. The rotational kinetic energy scales as

S & T, Sec. 7.4

$$T \; = \; \frac{1}{2}\int \rho \mathbf{v}_{\mathrm{rot}}^2 \, d^3x \; \sim \; M\Omega^2 R^2 \; \sim \; \frac{J^2}{MR^2} \quad , \quad J \; = \; I\Omega \; \sim \; MR^2\Omega \quad . \tag{43}$$

Here $\mathbf{J}$ is the angular momentum of the star, which we employ as a system parameter since it is a conserved quantity during collapse and stellar evolution, modulo mass loss in the explosion. Thus,

$$0 \; = \; -\alpha_g \, \frac{GM^2}{R} + \alpha_i \, \frac{\mathcal{K}M^\Gamma}{R^{3(\Gamma-1)}} + 2\alpha_r \, \frac{J^2}{MR^2} \quad . \tag{44}$$

The angular momentum accordingly serves as an "expulsive contribution," oblating stars and *possibly inducing their break up.* Solving for R when $\Gamma \neq 4/3$ gives

$$R \; = \; \left\{ \frac{\alpha_i \mathcal{K} M^\Gamma}{\alpha_g G M^2 (1 - 2T/|W|)} \right\}^{1/(3\Gamma-4)} \quad . \tag{45}$$

Implicitly, the J contribution complicates the solution for R, so that a numerical determination is required. Our main interest concerns the mass correction in the $n = 3$ domain. This is quickly found:

$$M \; \approx \; M_{\mathrm{CH}} \left(1 + \frac{3T}{|W|} \right) \quad . \tag{46}$$

In practice, the rotational correction is at most a few percent for a spherical white dwarf, as will soon become apparent. Asphericity is another matter!

• Consider a uniformly-rotating polytrope that is spherically symmetric. The surface **escape velocity** limits the rotational angular velocity via

$$\Omega^2 R^2 \;\leq\; \frac{GM}{R} \quad\Rightarrow\quad T \;=\; \frac{1}{2}\, I\Omega^2 \;\lesssim\; \frac{GM^2}{2R} \quad . \tag{47}$$

For polytropes, the gravitational potential energy satisfies

$$|W| \;=\; \frac{3}{5-n}\,\frac{GM^2}{R} \quad . \tag{48}$$

For an incompressible fluid of uniform density (i.e., an $n = 0$ polytrope), the moment of inertia is $I = 2MR^2/5$, so that we obtain the bound

$$\frac{T}{|W|} \;\leq\; \frac{1}{5}\,\frac{GM^2}{R} \bigg/ \frac{3}{5}\,\frac{GM^2}{R} \;=\; \frac{1}{3} \quad . \tag{49}$$

On the surface, this suggests large rotational corrections to white dwarf structure. However, their interior fluids are compressible.

Now consider an $n = 3$ polytrope, highly compressible and close to the Chandrasekhar limit. The density concentration near the center increases the total gravitational potential energy, so that

$$|W| \;=\; \frac{3}{2}\,\frac{GM^2}{R} \quad . \tag{50}$$

In contrast, the density concentration reduces the moment of inertia, and thus T. The computation of the value of T is more involved. Using the Lane-Emden solutions, the moment of inertia satisfies

$$\frac{I}{MR^2} \;=\; \frac{2}{3MR^2}\int_0^M r^2\, dm \;=\; \frac{2}{3\xi_1^4|\theta'(\xi_1)|}\int_0^{\xi_1} \theta^n\, \xi^4\, d\xi \;\xrightarrow{\text{n=3}}\; 0.07535 \; . \tag{51}$$

The numerical value of the integral then yields the result

$$\frac{T}{|W|} \;\leq\; \frac{I}{3MR^2} \;\approx\; 0.025 \quad . \tag{52}$$

This a much smaller number than the incompressible fluid result.

• Mass concentration in the polytropic case reduces the moment of inertia and hence increases rotational energy for fixed angular momentum.

4.2 Maclaurin Spheroids

• In the uniform rotation, spherical star case, the white dwarf mass limit is changed only by around 1 percent. Yet, we note that differential rotation and aphericity of a star are potentially profound influences; we now consider the latter by introducing spheroids, neglecting discussion of Jacobi ellipsoids.

∗ Such spheroids are explored extensively by Binney & Tremaine, and important for (i) Earth oblateness, (ii) gas condensation in protostellar clouds, (iii) galactic dynamics, and (iv) rotating white dwarfs and neutron stars.

Uniform density rotators are now considered. They are stable solutions of self-gravitating systems. Such homogenous ellipsoid systems are termed **Maclaurin spheroids**, which are flattened, axisymmetric figures that are **oblate**, or squashed like an alien spacecraft! For a spheroid with circular cross section in the (x, y)-plane of radius a,

$$\frac{x^2}{a^2} + \frac{z^2}{a^2(1-e^2)} \; = \; 1 \quad , \quad x \; = \; a \sin\theta \quad , \quad z \; = \; a\sqrt{1-e^2}\,\cos\theta \quad . \tag{53}$$

The gravitational potential Maclaurin spheroids satisfies Poisson's equation

$$\nabla^2 \Phi \; = \; 4\pi G\rho \quad \Rightarrow \quad \Phi \; = \; -G\rho \int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \quad , \tag{54}$$

where the Green's function solution is provided for ρ being uniform throughout the star. Integrating the Cartesian form of Poisson's equation yields

$$\Phi \; = \; -\pi G\rho \Big[A a^2 - A_x x^2 - A_y y^2 - A_z z^2 \Big] \quad , \quad A_x + A_y + A_z \; = \; 2 \quad . \tag{55}$$

The constraint on the A_i coefficients is imposed by Poisson's equation. Solving Eq. (54), perhaps via the Green's function approach, the coefficients are

$$\begin{aligned} A_x \; = \; A_y \;\; &= \;\; 1 - \frac{1}{e^2} + \frac{\sqrt{1-e^2}}{e^3}\,\arcsin e \\ A_z \;\; &= \;\; \frac{2}{e^2} - \frac{2\sqrt{1-e^2}}{e^3}\,\arcsin e \\ A \;\; &= \;\; \frac{2\sqrt{1-e^2}}{e}\,\arcsin e \quad . \end{aligned} \tag{56}$$

In the limit $e \to 0$, $A_x \to 2/3$, $A_z \to 2/3$ and $A \to 2$.

An expedient path to determining the coefficients is to generate the Bessel function solution in cylindrical coordinates for a disk density source, and then integrate over the elliptical profile.

Plot: Maclaurin Spheroids and their potentials

• The next task is to couple the geometry of the uniform density spheroid to the hydrostatic nature of the spinning star. The spin vector $\mathbf{\Omega}$ is coincident with the z-axis, and this rotation is presumed to result in an **equatorial bulge** due to the expulsive effect of angular momentum.[2] For rotational velocity $\mathbf{v}_{\rm rot} = \mathbf{\Omega} \times \mathbf{r}$, the Euler equation of motion for this system is

$$-\Omega^2\bigl(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}\bigr) \;\equiv\; \mathbf{\Omega} \times \bigl(\mathbf{\Omega} \times \mathbf{r}\bigr) \;=\; \frac{d\mathbf{v}_{\rm rot}}{dt} \;=\; -\frac{\nabla P}{\rho} - \nabla\Phi \quad . \tag{57}$$

The centripetal acceleration is obvious. Forming the axial (z) and azimuthal (x) components (the y component is obtained via $x \to y$ correspondence) yields

$$-\frac{1}{\rho}\,\frac{\partial P}{\partial z} - \frac{\partial \Phi}{\partial z} \;=\; 0 \quad \text{and} \quad -\frac{1}{\rho}\,\frac{\partial P}{\partial x} - \frac{\partial \Phi}{\partial x} \;=\; -\Omega^2 x \quad . \tag{58}$$

Since Φ is a quadratic function of the (x, y, z) coordinates that has zero derivative at the origin (no cusp), then so also is the pressure function:

$$P \;=\; P_c\Bigl(1 - p_x\bigl[x^2 + y^2\bigr] - p_z z^2\Bigr) \quad . \tag{59}$$

Inserting this into the force equations yields

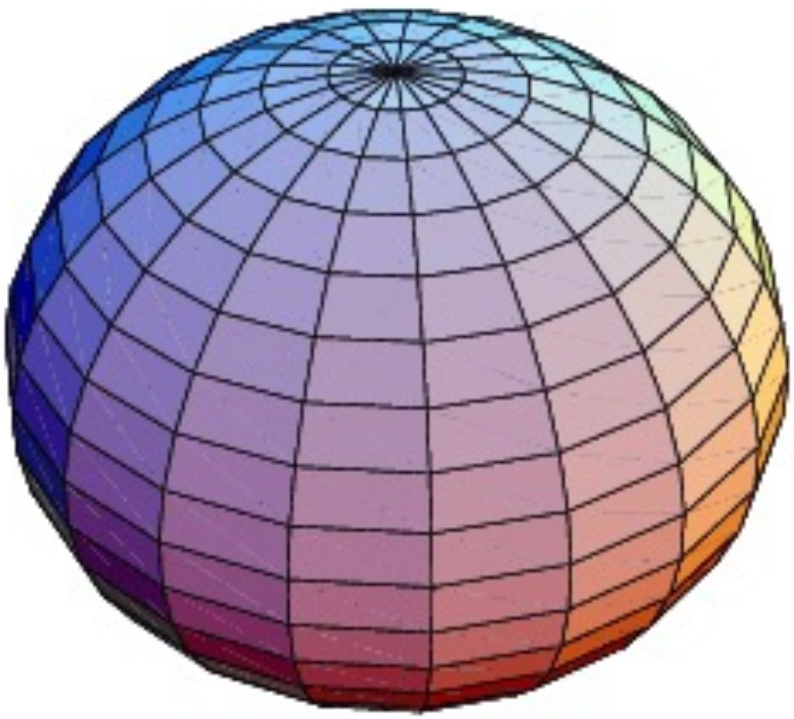
$$\frac{P_c p_z}{\rho} \;=\; \pi G\rho\, A_z \quad \text{and} \quad \frac{P_c p_x}{\rho} \;=\; \pi G\rho\, A_x - \frac{1}{2}\,\Omega^2 \quad , \tag{60}$$

a system of two equations in four unknowns, P_c, p_x, p_z and Ω. To complete the path to solution, we note that the star can only be stable if the pressure P goes to zero at the surface. Then, $p_x = 1/a^2$ and $p_z = 1/[a^2(1-e^2)]$ are geometrically determined! It then follows that

$$P_c \;=\; \pi G\rho^2 a^2\bigl(1 - e^2\bigr)\, A_z \tag{61}$$

[2]Much like Kepler's Laws, a self-gravitating mass distribution responds to rotation so that Ω couples to oblateness or eccentricity: $\Omega = \Omega(e)$.

Maclaurin Spheroids and their Gravitational Potentials



- Maclaurin spheroid (from mathworld.wolfram.com), and the potential for such an oblate spheroid.

An oblate **Maclaurin spheroid** has an elliptical cross section:

$$\frac{x^2}{a^2} + \frac{z^2}{a^2(1-e^2)} = 1 \quad .$$

Its gravitational potential satisfies Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho \quad \Rightarrow \quad \Phi = -G\rho \int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} \quad .$$

and is of the form

$$\Phi = -\pi G\rho \left[Aa^2 - A_x x^2 - A_y y^2 - A_z z^2 \right] \quad .$$

The solution is

$$A_x = A_y = 1 - \frac{1}{e^2} + \frac{\sqrt{1-e^2}}{e^3} \arcsin e$$

$$A_z = \frac{2}{e^2} - \frac{2\sqrt{1-e^2}}{e^3} \arcsin e$$

$$A = \frac{2\sqrt{1-e^2}}{e} \arcsin e \quad .$$

In the limit $e \to 0$, $A_x, A_z \to 2/3$ and $A \to 2$.

defines the central pressure, and the *angular rotation speed is also coupled to the eccentricity*:

$$\Omega^2 \;=\; 2\pi G\rho \left[3 - \frac{3}{e^2} + \left(3 - 2e^2\right) \frac{\sqrt{1-e^2}}{e^3} \arcsin e \right] \quad . \tag{62}$$

Observe that $P \equiv 2\pi/\Omega \sim 1/\sqrt{G\rho}$ is the natural timescale for rotation of a self-gravitating system. In the limit $e \ll 1$, $\Omega^2 \approx 8\pi G\rho e^2/15$ and the rotation speed must drop to zero to accommodate zero oblatenss.

• This connection is a consequence of hydrostatic equilibrium for the rotation of a uniform density star. Introducing density stratification will change the coupling, *as will differential rotation.*