6 Pyconuclear Reactions

• Heretofore, our discussion of nuclear effects in WDs has focused on structural influences, namely providing adjustments to dwarf radii from the ideal Chandrasekhar case. Here we discuss exothermic nuclear reactions as an intrinsic phenomenon to white dwarfs — they are not just dormant post-MS states, they generate heat, and emit radiation in the stellar interior that we eventually see in the UV after transport to the surface layers.

• In thermonuclear reactions, the thermal energy of the gas overcomes nuclear Coulomb repulsions via quantum tunneling to seed nuclear transitions that are predominantly exothermic. This is true for main sequence stars, and for primordial nucleosynthesis.

S & T, Sec. 3.7

• In condensed, white dwarf interiors, the zero-point Fermi energy at $T \approx 0$ can also induce nuclear reactions; these are termed **pyconuclear reactions**.

∗ derived from the Greek word **pyknos**, which means dense.

∗ an example is the familiar hydrogen fusion reaction $p + p \to D + e^+ + \nu$.

• Here we sketch an approximate treatment, based on the work of Salpeter & van Horn (1969). The essence is that the *electron* Fermi energy is not zero, and so provides an opportunity for tunneling:

∗ *nuclei can circumvent the Coulomb barrier and fuse* via quantum tunneling, if the *non-degenerate nuclei* can tap some of the e^- kinetic energy in a thermal equilibrium at ion temperature T_i .

∗ If we set $3kT_i/2 \sim p_{\rm F}^2/(2m_e)$, for non-relativistic degenerate electrons, we arrive at $T_i \sim 9.1 \times 10^8$ K. This is too high for WD interiors. The electrons are not quite perfectly degenerate, and so a small fraction of their degeneracy energy appears as thermal energy that can be accessed by ions.

[*Handout: Paper of Salpeter & van Horn (1969)*]

`https://ui.adsabs.harvard.edu/abs/1969ApJ...155..183S/abstract`

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6.1 Ion Penetration of the Coulomb Barrier

• First we consider an isolated positive nucleus colliding with another free one, before treating lattice modifications to tunneling rates.

Plot: Coulomb Barrier and Nuclear Potential

The tunneling probability transition rate takes the general form

$$W \;=\; v\,|\psi_{\rm inc}|^2\,\sigma(E) \quad , \quad \sigma(E) \;=\; \frac{\mathcal{T}\,\mathcal{S}(E)}{E} \quad . \tag{22}$$

Here v is the incident ion speed, and $\psi_{\rm inc}$ is its *incident* wavefunction.

• As a general consequence of Heisenberg's uncertainty principal, we expect the cross section $\sigma(E)$ to scale roughly as the square of the **de Broglie wavelength** λ, which for non-relativistic ions ($E \ll m_i c^2$) gives

$$\sigma \;\propto\; \pi\lambda^2 \;\propto\; \frac{1}{E} \quad , \quad \lambda \;=\; \frac{2\pi\hbar}{p} \quad , \quad E \;=\; \frac{1}{2}\mu v^2 \quad . \tag{23}$$

Here, $\mu \approx A m_p/2$ is the reduced nuclear mass. We expect $S(E)$ to be a slowly-varying function of energy.

• The remaining factor, $\mathcal{T}$, is known as the **Gamow penetration factor**, and represents the probability of tunneling through the Coulomb barrier. We now estimate $\mathcal{T}$ first *for an isolated nucleus.* It comes from solutions of the Schrödinger equation outside nuclei. Specifically we focus on the radial Schrödinger equation

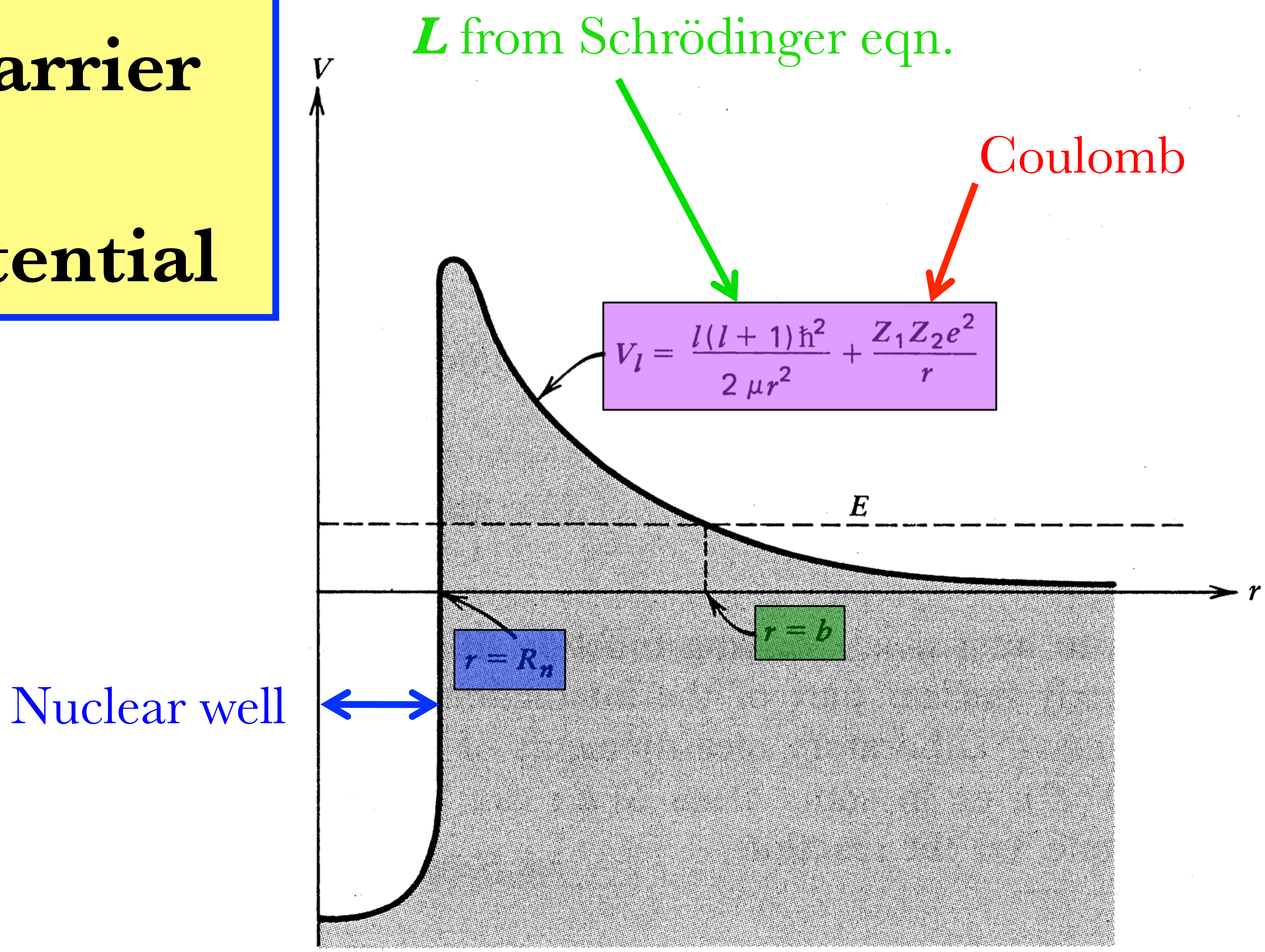
$$\frac{d^2\chi_l}{dr^2} + \frac{2\mu}{\hbar^2}\,[E - V_l(r)\,]\,\chi_l \;=\; 0 \quad , \quad \psi_l(r,\,\theta,\,\phi) \;=\; \frac{\chi_l(r)}{r}\,Y_{lm}(\theta,\,\phi) \quad . \tag{24}$$

The effective potential $V_l(r)$ includes an angular momentum portion, and a Coulomb repulsion term:

$$V_l(r) \;=\; \frac{l(l+1)\hbar^2}{2\mu r^2} + V_c(r) \quad , \quad V_c(r) \;=\; \frac{Z_1 Z_2 e^2}{r} \quad . \tag{25}$$

The angular momentum portion comes from the $(\theta,\,\phi)$ angular part via the separation of variables of the 3D ∇^2 operator.

Coulomb Barrier and Nuclear Potential



- The effective potential V(r) governing the radial motion of one nucleus relative to another (Fig. 3.4 of Shapiro & Teukolsky). The short range attractive nuclear force dominates at $r<R_n$. At larger distances, the potential is dominated by the repulsive Coulomb force.
- The classical turning point for an orbit is at $r = b = Z_1Z_2e^2/E$ for a CM energy E.

• Detailed knowledge of the nuclear potential is of secondary importance. The 1D **WKB approximation** for the solution to radial Schrödinger equation yields (Merzbacher 1970, *Quantum Mechanics*) a transition probability

$$\mathcal{T} \;\equiv\; \frac{|\chi_{\rm trans}|^2\, v_{\rm trans}}{|\chi_{\rm inc}|^2\, v_{\rm inc}} \;=\; \frac{4}{(2\theta+1/2\theta)^2} \tag{26}$$

where

$$\theta \;=\; \exp\left(\int_a^b k(r)\, dr\right) \quad , \quad k(r) \;=\; \sqrt{\frac{2\mu}{\hbar^2}\left|\Big[E - V_l(r)\Big]\right|} \tag{27}$$

In this casting, $k(r) = 2\pi/\lambda(r)$ is the effective wavenumber that relates to the wavelength scale $\lambda(r)$ for the barrier. Since the barrier spatial scale $b = Z_1Z_2e^2/E$ generally far exceeds $1/k$, we have $\theta \gg 1$, so that

$$\mathcal{T} \;\approx\; \frac{1}{\theta^2} \;=\; \exp\left(-2\int_a^b k(r)\, dr\right) \tag{28}$$

is small. Assuming that $a = R_n \ll b$, we can evaluate the integral:

$$\int_a^b k(r)\, dr \;\approx\; \frac{(2\mu E)^{1/2}}{\hbar}\int_0^b dr\sqrt{\frac{b}{r}-1} \;=\; \pi\eta \quad , \tag{29}$$

In forming this, we presume (correctly) that the "impact parameter" b far exceeds the r value at which the angular momentum contribution is significant relative to the Coulomb one. Then the Gamow penetration factor takes the form[1]

$$\boxed{\mathcal{T} \;\approx\; \exp(-2\pi\eta) \quad , \quad \eta \;=\; \frac{Z_1Z_2e^2}{\hbar}\sqrt{\frac{\mu}{2E}} \quad .} \tag{30}$$

While this yields the dominant behavior, the WKB approximation needs to be improved upon. The exact solution of the Schrödinger equation in a Coulomb potential introduces an extra logarithmic factor in the exponential that sets

$$\mathcal{T} \;=\; 2\pi\eta\, \exp(-2\pi\eta) \quad . \tag{31}$$

This resembles a Sommerfeld-Elwert factor. The pedagogy to this point gives the general structure of the pyconuclear reaction rate.

[1]G. Gamow, *Zeitschrift für Physik*, **51**, 204 (1928); the dominant term in the rate of many nuclear interactions at low bombarding energies.

6.2 Pyconuclear Reactions in a Crystal Ionic Lattice

• We turn now to reactions in a crystal lattice. This is an important upgrade *because the mean potential energy differs due to the proximity of neighboring nuclei.* In dense stars like white dwarfs, this modification is important.

• This change is significant since the calculation has an exponential dependence for the tunneling probability. The rate takes the same general form

$$W \;=\; v\,|\psi_{\rm inc}|^2\,\sigma(E) \quad , \quad \sigma(E) \;=\; \frac{\mathcal{T}\,\mathcal{S}(E)}{E} \quad . \tag{32}$$

The lattice potential is approximated using the dual Coulomb potential for nuclei separated by a fixed distance $2R_0$:

$$V(x) \;=\; \frac{Z^2e^2}{R_0 - x} + \frac{Z^2e^2}{R_0 + x} - \frac{2Z^2e^2}{R_0} \quad , \quad |x| \;<\; R_0 - R_n \quad . \tag{33}$$

The third term (constant) is introduced to set the internuclear midpoint to zero potential; one has the latitude to do this. The nuclear potential only contributes when $|x| > R_0 - R_n$. Note that we have now set $Z_1 = Z_2 = Z$.

Plot: Periodic Lattice Potential for Pyconuclear Reactions

• For a large Coulomb barrier, in the absence of tunneling, an ion will oscillate in the shallow inter-nuclear well in a region $x \ll R_0$. Performing a Taylor series expansion of Eq. (33) about $x = 0$, we can obtain

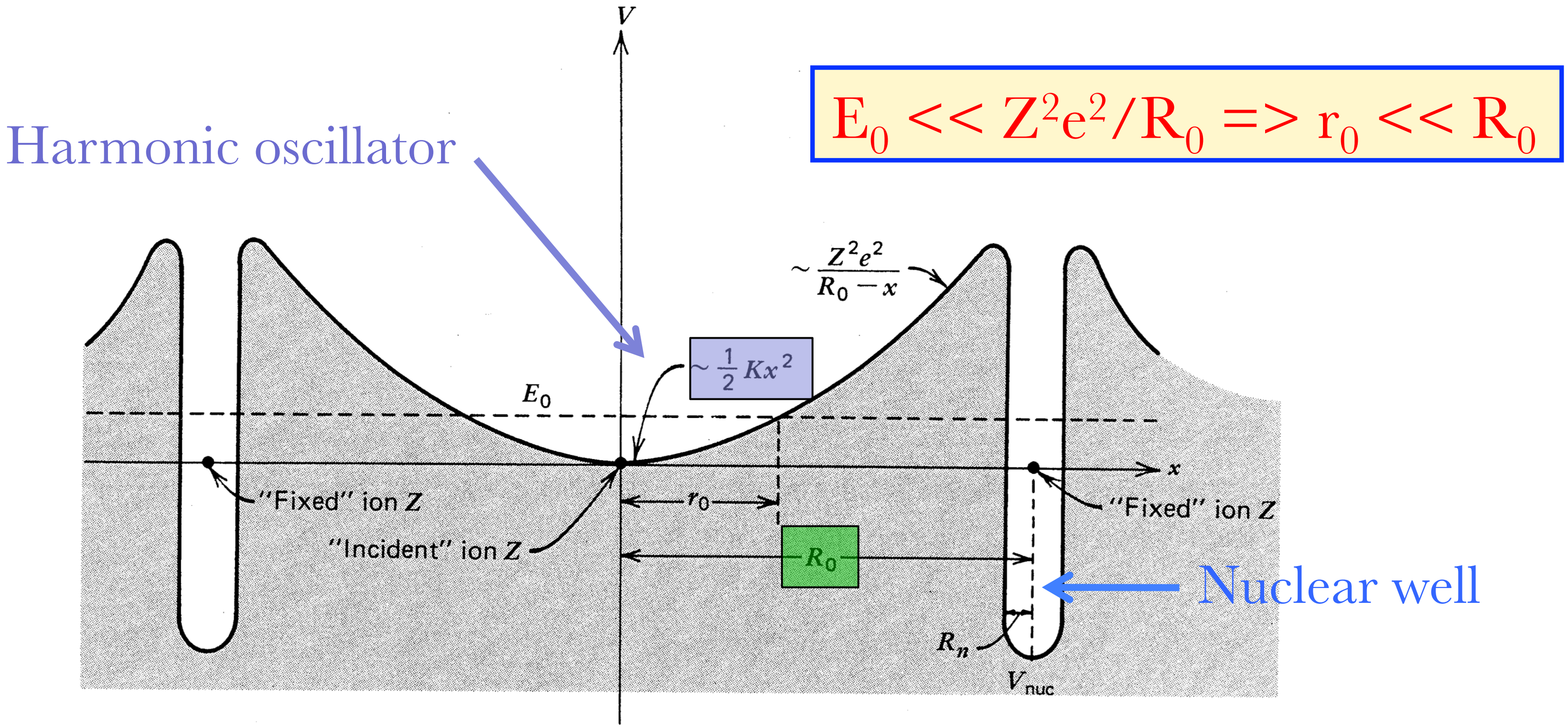
$$V(x) \;\approx\; \frac{K}{2}\,x^2 \quad , \quad K \;=\; \frac{4Z^2e^2}{R_0^3} \quad , \tag{34}$$

where K is the the **Fall constant** K . The classical equation of motion then represents a simple harmonic oscillator of angular frequency $\omega_0 = \sqrt{K/Am_p}$ that establishes the **zero-point energy** E_0 for a quantum interpretation:

$$\mu\,\frac{d^2x}{dt^2} \;=\; \omega_0^2 x^2 \quad \Rightarrow \quad E_0 \;\equiv\; \frac{\hbar\omega_0}{2} \;=\; \frac{Ze\hbar}{\sqrt{\mu R_0^3}} \quad . \tag{35}$$

Here $\mu \approx Am_p/2$ is the **nuclear reduced mass**.

Periodic Lattice Potential: Pyconuclear Reactions



- The potential governing the motion of an "incident" nucleus relative to its two neighbouring nuclei in a 1D lattice (Fig. 3.5 of Shapiro & Teukolsky).
- The nuclei are separated by distance R_0, and the turning point scale is $r_0 << R_0$.
- Zero-point fluctuations of energy E_0 in the harmonic potential near the "incident" ion lead to Coulomb barrier penetration and pyconuclear reactions.

$*$ It is trivially shown that for $n_A = 3/(4\pi R_0^3)$ as the number density of the nuclei, that $\omega_0 = (6\pi Z^2 e^2 n_A/\mu)^{1/2}$ is roughly the **plasma frequency**.

• The scale of the oscillation (classical turning point) arises for an ion of energy E_0 :

$$E_0 \;=\; V(r_0) \;=\; \frac{2Z^2e^2}{R_0^3}\,r_0^2 \quad \Rightarrow \quad r_0 \;=\; \left(\frac{\hbar}{2Ze}\right)^{1/2} \left(\frac{R_0^3}{\mu}\right)^{1/4} \quad . \tag{36}$$

This then sets the scale for ground state wave function ψ_{SHO} for the simple harmonic oscillator (SHO), which in 3D is

$$|\psi_{\text{SHO}}|^2 \;=\; \frac{e^{-(r/r_0)^2}}{(\pi r_0)^{3/2}} \quad , \tag{37}$$

and we specialize to the $x \equiv r \ll r_0$ limit to specify the incident wave function $\psi_{\text{inc}} \approx (\pi r_0)^{-3/2}$. This provides a first element of quantum input.

$*$ Observe that the maximum speed v of an ion in such an oscillation satisfies $E_0 = Am_p v^2/2$, giving $Am_p v = \sqrt{2Am_pE_0}$. Therefore, the de Broglie wavelength of the ion is $\hbar/(Am_pv) = \hbar(4\mu E_0)^{-1/2}$ for reduced mass $\mu \approx Am_p/2$. Substituting in the expression for E_0 ,

$$\frac{\hbar}{Am_pv} \;\approx\; \left(\frac{\hbar}{4Ze}\right)^{1/2} \left(\frac{R_0^3}{\mu}\right)^{1/4} \;=\; \frac{r_0}{\sqrt{2}} \quad . \tag{38}$$

Accordingly, we infer that *the scale* r_0 *of the oscillation is essentially the de Broglie wavelength of the ion.*

• Next, the transmission coefficient $\mathcal{T}$ is computed, applying the WKB approximation for an incident ion with ground state energy E_0 . Again,

$$\mathcal{T} \;=\; \exp\left\{-2\int_a^b k(x)\,dx\right\} \quad , \quad k(r) \;=\; \sqrt{\frac{2\mu}{\hbar^2}\left|\Big[E_0 - V(x)\Big]\right|} \quad , \tag{39}$$

i.e., $k(x)$ ($r \equiv x$) is the same wavenumber form as before, but with the upgraded lattice potential $V(x)$ deployed. This complicates the integration. We specialize to $R_n \ll R_0$, define $u = x/R_0$ as the integration variable, and set $\beta = r_0/R_0$. Since $V(\pm r_0) = E_0$, it follows that the integrand has zeros at $u = \pm\beta$, with

$$k(x) \;\propto\; \sqrt{\frac{u^2-\beta^2}{1-u^2}} \quad . \tag{40}$$

Then

$$\mathcal{T} \;\approx\; \exp\left\{ -\frac{2}{\beta^2\sqrt{1-\beta^2}} \int_\beta^1 du\,\sqrt{\frac{u^2-\beta^2}{1-u^2}} \right\} \;\approx\; \frac{1}{\beta}\,\exp\left\{ -\frac{2}{\beta^2} \right\} \quad , \tag{41}$$

for

$$\beta \;=\; \frac{r_0}{R_0} \;=\; \left(\frac{\hbar}{2Ze}\right)^{1/2} (\mu R_0)^{-1/4} \;\ll\; 1 \quad . \tag{42}$$

The integral in the exponential can be expressed in terms of elliptic functions, and the approximate form is obtained for $\beta \ll 1$.

$*$ Note that for $\beta \sim 1$, the ion de Broglie wavelength must be comparable to the lattice scale, for which nuclear scales must be sampled, and we have conditions for a neutron star.

• Collecting pieces, the pyconuclear reaction rate per nucleus for a lattice is

$$W \;=\; \frac{8}{\sqrt{2\pi^3}}\,\mathcal{S}(E)\,\frac{(Z^2e^2\mu)^{3/4}}{(\hbar^2 R_0)^{5/4}}\,\exp\left\{ -\frac{4Ze}{\hbar}\,\sqrt{\mu R_0} \right\} \quad , \tag{43}$$

a result that depends on both A and ρ. Including anisotropy and electron screening effects yields the same exponential, but a coefficient out the front that is enhanced by two orders of magnitude (Salpeter & van Horn 1969).

• Presuming charge neutrality, the internuclear spacing R_0 couples to the electron number density n_e, and therefore to the Fermi momentum:

$$n_e \;\equiv\; \frac{p_{\rm F}^3}{3\pi^2\hbar^3} \;=\; Zn_i \;=\; \frac{3Z}{4\pi R_0^3} \quad , \quad R_0 \;=\; \left(\frac{9\pi Z}{4}\right)^{1/3} \frac{\hbar}{p_{\rm F}} \quad . \tag{44}$$

Introducing the dimensionless parameter of Salpeter and van Horn (1969)

$$\Lambda \;=\; \frac{\hbar^2}{2\mu Z^2 e^2}\left(\frac{n_i}{2}\right)^{1/3} \;=\; \frac{\hbar^2}{2\mu Z^2 e^2}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{R_0} \quad , \tag{45}$$

for ion number density n_i, the argument of the exponential in the Gamow penetration factor can be re-expressed as (for $\mu \to Am_p/2$)

$$\frac{4Ze}{\hbar}\sqrt{\mu R_0} \;=\; \frac{\epsilon}{\sqrt{\Lambda}} \quad \text{for} \quad \epsilon \;=\; 2\sqrt{2}\left(\frac{\pi}{3}\right)^{1/6} \;\approx\; 2.85 \quad . \tag{46}$$

If we multiply Eq. (43) by the ion number density $n_i \approx \rho/(Am_p)$, and eliminate R_0 in favor of Λ using Eq. (45), the **total pyconuclear reaction rate** becomes

$$R_{\text{pyco}} \;=\; \Gamma\,\rho A Z^4\, \overline{S}\, \Lambda^{5/4} \exp\left\{-\frac{\epsilon}{\sqrt{\Lambda}}\right\} \;\; \text{cm}^{-3}\,\text{sec}^{-1} \quad , \tag{47}$$

where $\overline{S} \to S(E)$ is expressed in units of MeV barns (1 barn $= 10^{-24}$ cm^2), and comes from the nuclear physics of the particular interaction. The numerical coefficient out the front is

$$\Gamma \;=\; 1.1\times 10^{44} \quad . \tag{48}$$

Observe that Salpeter and van Horn (1969) introduce modifications for *more realistic lattice potentials and screening forms*, that amount to increasing Γ to around 4×10^{46}, modifying $\overline{S} \to \sqrt{\Lambda}\,S(E)$ and setting $\epsilon \sim 2.6$.

• To forge a connection to real rates, we express the principal parameter Λ in terms of the mass density:

$$\Lambda \;=\; \frac{1}{AZ^2}\,\frac{\lambda\!\!\!^{-}}{\alpha_{\text{f}}}\,\frac{m_e}{m_p}\left(\frac{\rho}{2Am_p}\right)^{1/3} \;\approx\; \frac{1}{A^{4/3}Z^2}\left(\frac{\rho}{1.39\times 10^{11}\ \text{g cm}^{-3}}\right)^{1/3} \quad . \tag{49}$$

Clearly this is small for white dwarf conditions well below neutron drip, so the exponential in the pyconuclear reaction rate has a large negative argument.

∗ To highlight this fact, one can evaluate Eq. (49) for a typical mean mass density $\rho \to \overline{\rho} \sim 1\,\text{g cm}^{-3}$ for a C/O white dwarf; this gives $1/\sqrt{\Lambda} \sim 226$ for carbon, and 366 for oxygen.

∗ If, instead, one uses the central density $\rho_c \sim 6\times 10^6\,\text{g cm}^{-3}$ for a $\Gamma = 5/3$ polytrope, then these estimates drop by around 25%.