

## 2 Extrasolar Planets

A classic 21st Century application of our binary system understanding is in the study of **extrasolar planets**, also called **exoplanets**. This field has become possible with enhanced technologies on two fronts.

C & O,  
Sec. 7.4

[Reading Assignment: C & O: Section 7.4]

- The first technical advance was in the ability to measure Doppler shift velocities in spectral lines at the precision of better than 10 metres/sec. If one applies Kepler III to Jupiter, one arrives at a perihelion speed of  $v_J \sim 13 \text{ km/sec}$ , i.e. around  $2\pi a_J/P_J$ . The center of mass considerations then simply yield the corresponding value for the sun about Jupiter:

$$v_{\odot} \approx \frac{m_J}{M_{\odot}} v_J \sim 12.5 \text{ metres/sec} . \quad (13)$$

This sets the Doppler precision scale to infer the presence of Jupiters around nearby stars using state of the art spectrometers.

\* Sub-Jupiter objects (Neptunes) were beginning to be detected by this Doppler technique by 2003-2004. There are now over 1200 exoplanets discovered via this technique (2025).

**Plot:**  $\mu$  Arae Velocity Curves from La Silla and AAT - a Neptune

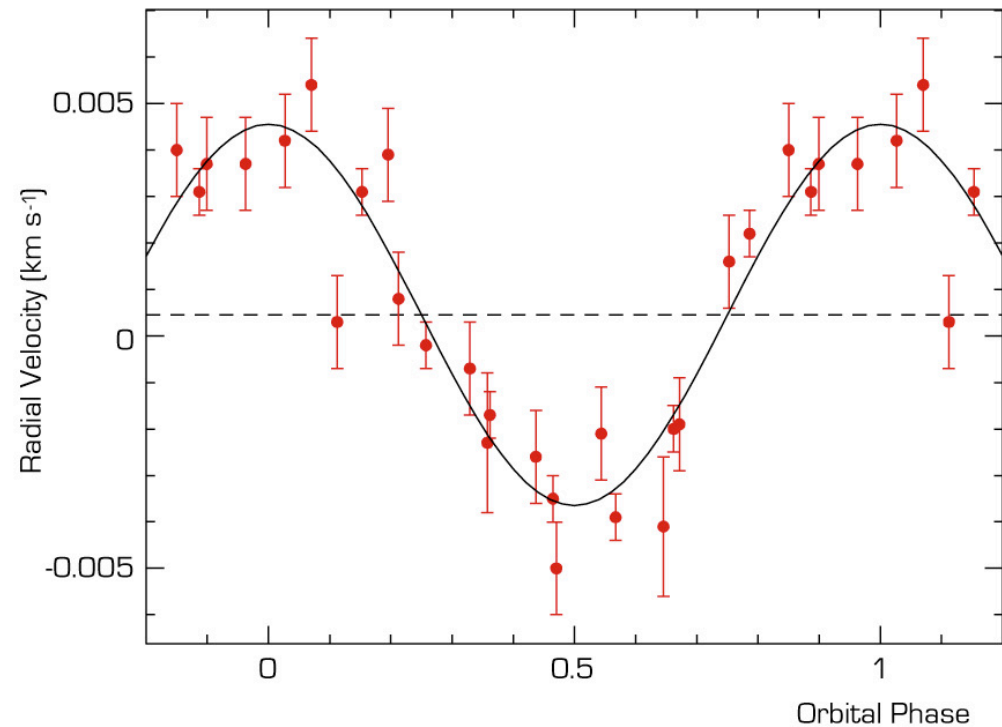
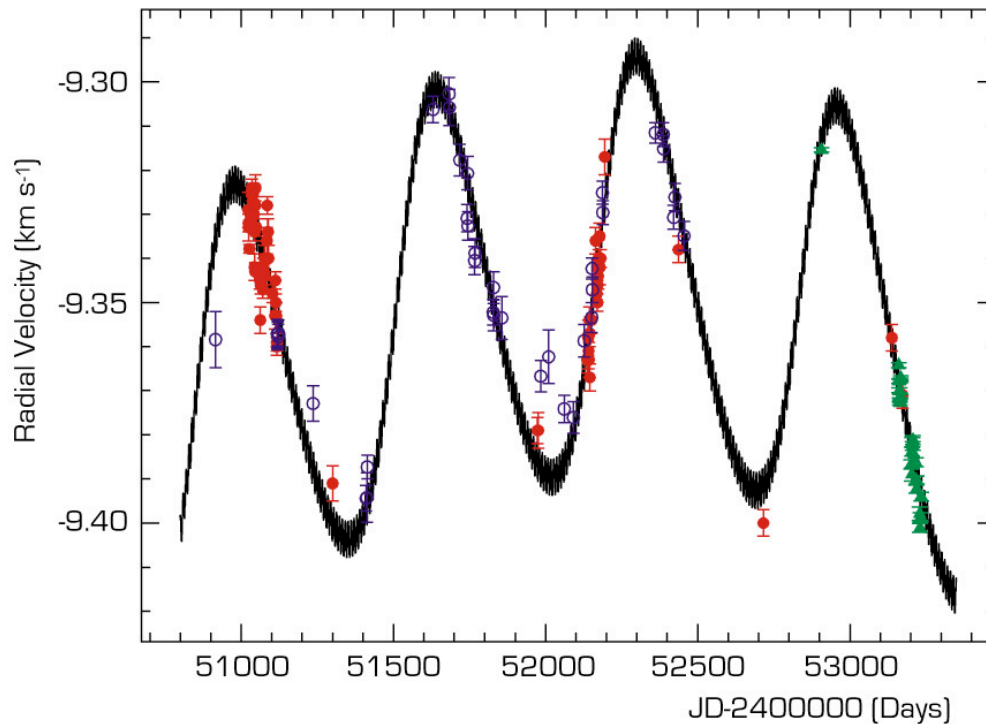
- The main complementary technique ushered in by the **Kepler** mission in 2008 was that of photometric transit determination using eclipsing of stars by planets. Advances were forged by the ability to measure photometric decrements on the order of 1% or less. This can be employed in conjunction with the Doppler approach.

**Plot:** Kepler candidate exoplanet census, circa 2011

- Such **transit shadowing** has led to the discovery of over 4500 exoplanets as of 2025. Since transits constitute a small portion of orbit time, many stars must be monitored simultaneously to provide this detection yield.

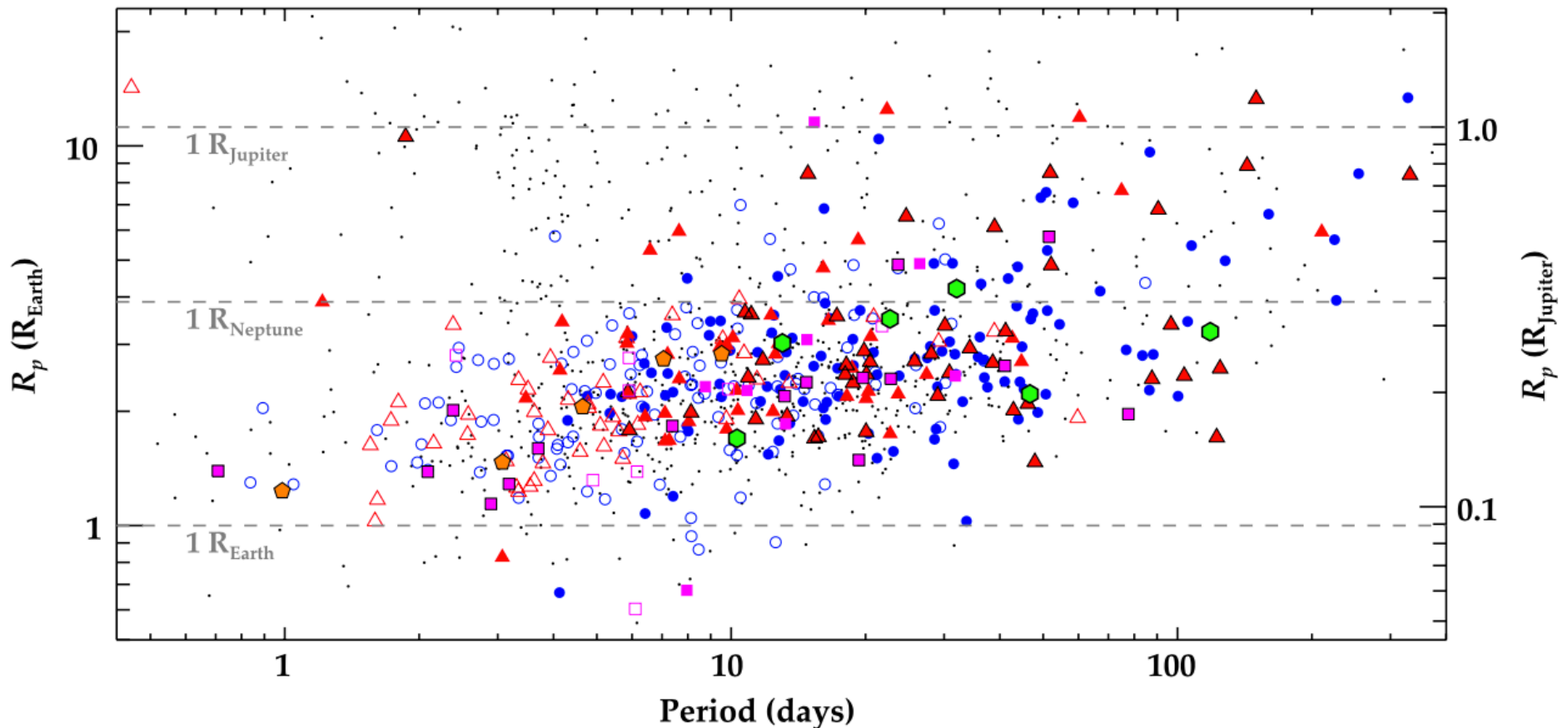
# $\mu$ Arae Velocity Variations

Courtesy: ESO Public Image Archives



- Left panel:* Measurements of the radial velocity of the star  $\mu$  Arae obtained by HARPS (ESO 3.6m telescope at La Silla - green triangles), CORALIE (Euler 1.2m telescope, La Silla - red dots) and UCLES (Anglo-Australian Telescope - blue circles). Best fit solid line assumes the existence of two planets and a long-period companion. The width of the line implies the existence of a short period planet. Data shown span July 1998 to August 2004.
- Right Panel:* The HARPS radial velocity measurements phase-folded with the orbital period of the short period (9.5 days) exoplanet. The semi-amplitude of the curve is less than 5 metre/sec, which at this 9.5 day period implies a minimum mass for the sub-Jupiter planet of 14 times that of the Earth.

# Extrasolar Planets: Kepler Census



- [Lissauer et al. \(2011, ApJS \*\*197\*\*, 8\)](#) lists over 1200 candidate planets from Kepler mission transits, emphasizing the prevalence of multi-planet systems.
- Period range is a few months down to a fraction of a day;
- Radius range is from Jupiters down to sub-Earth scale!

To get a sense of the challenge of observing transits, one needs to estimate both their frequency and their **photometric decrement** on intensity  $\Delta I/I$  from a planet's parent star. We will focus on Jupiter-scale exoplanets for benchmarking detections. The baseline assumption is that the mean mass density of an exoplanet is close to heliospheric ones, i.e.  $\bar{\rho}_J \sim 1 \text{ g cm}^{-3}$ .

- The fiducial radius of an exoJupiter is simply  $R_X \sim [3M_J/(4\pi\bar{\rho}_J)]^{1/3}$ . The typical radius of a solar mass star is similarly defined:  $R_\odot \sim [3M_\odot/(4\pi\bar{\rho}_\odot)]^{1/3}$  with  $\bar{\rho}_\odot \approx 1.41 \text{ g cm}^{-3}$ . Accordingly, one anticipates  $R_X/R_\odot \sim 0.1$ . Assume circularization of the exo-Jupiter's orbit. Then, the duration  $\Delta t$  of the transit for a total eclipse (*draw a diagram*) can be related to the period  $P_X$  and radius  $a_X$  for its orbit:

$$v_X \Delta t \approx R_\odot \quad \text{with} \quad v_X = \frac{2\pi a_X}{P_X} \quad \Rightarrow \quad \frac{\Delta t}{P_X} \approx \frac{R_\odot}{2\pi a_X} \quad . \quad (14)$$

One now eliminates  $a_X$  using Kepler's Third Law:

$$\frac{a_X^3}{P_X^2} \sim \frac{GM_\odot}{4\pi^2} \quad \Rightarrow \quad a_X \sim \left( \frac{GM_\odot P_X^2}{4\pi^2} \right)^{1/3} \quad . \quad (15)$$

Inserting this into the RHS of Eq. (14) and using  $R_\odot \sim [3M_\odot/(4\pi\bar{\rho}_\odot)]^{1/3}$  to replace  $R_\odot$  yields a transit duration to orbital period ratio

$$\frac{\Delta t}{P_X} \sim \left( \frac{8\pi^2}{3} G \bar{\rho}_\odot P_X^2 \right)^{-1/3} \approx \frac{0.00175}{P_{100d}^{2/3}} \quad . \quad (16)$$

This ratio represents a probability of observing a transit at any snapshot observational time. Note the absence of a dependence on  $M_X$ .

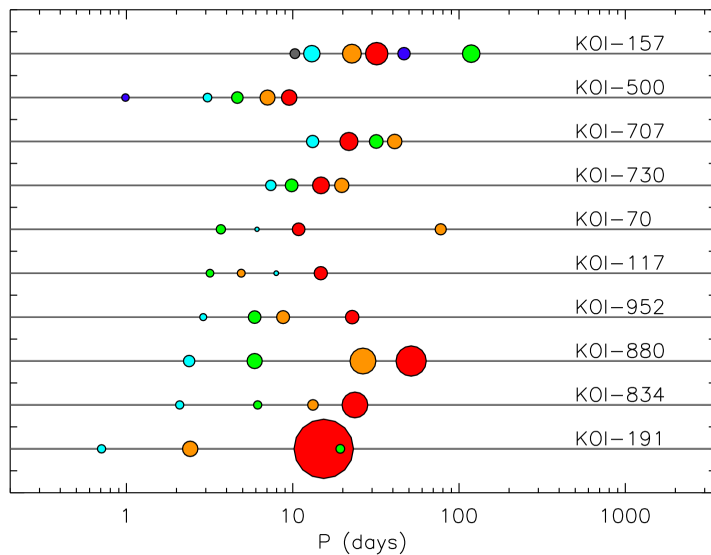
- \* This implies a *strong selection bias* in favor of  $P_X \lesssim 30$  day objects.

- The fractional intensity decrement during such transits, presuming total eclipses ( $\Rightarrow$  an upper bound to the decrement), is simply a ratio of the cross sectional areas of the exoJupiter and the parent star:

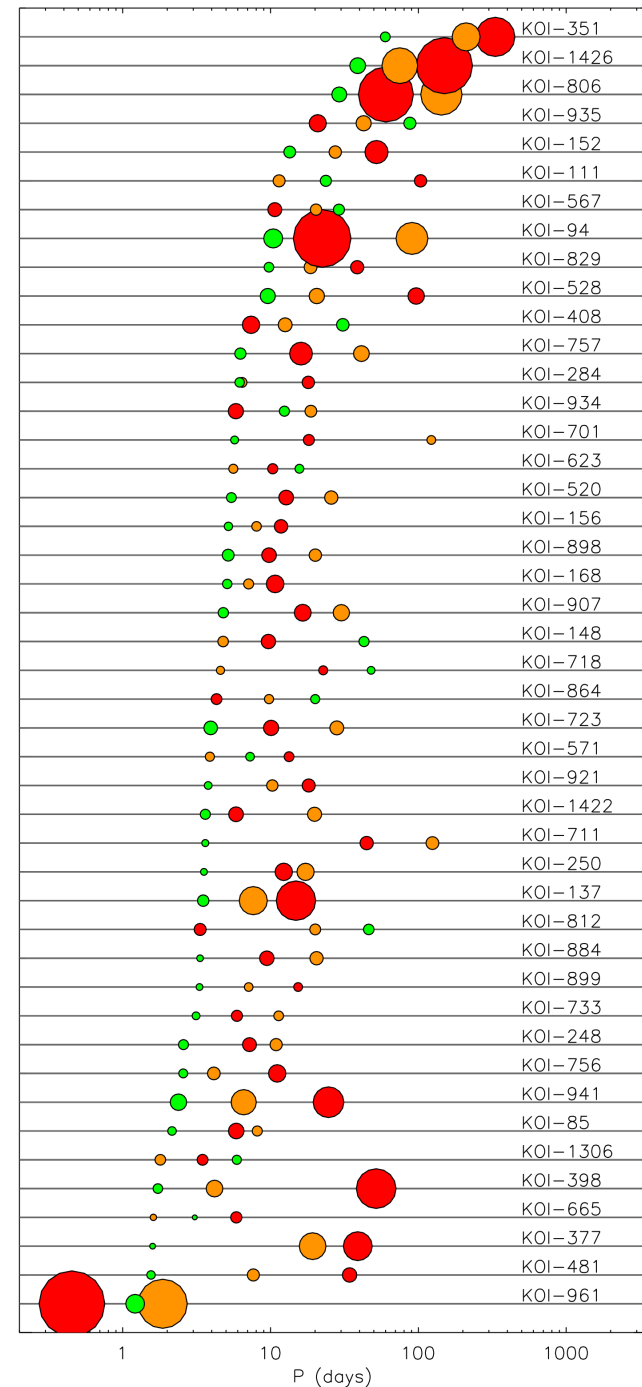
$$\frac{\Delta I}{I} \approx \frac{\pi R_X^2}{\pi R_\odot^2} = \left( \frac{\bar{\rho}_\odot M_X}{\bar{\rho}_X M_\odot} \right)^{2/3} \sim \left( \frac{M_J}{M_\odot} \right)^{2/3} \sim 10^{-2} \quad . \quad (17)$$

This defines general mission technical requirements (e.g., for Kepler, TESS), which become more stringent for exoEarths, for which  $\Delta I/I < 10^{-3}$ .

# Multi-Planet Systems According to Kepler



- Lissauer et al. (2011, ApJS **197**, 8): multi-planet systems are common.
- 4-6 planets (above); 3-4 planets (RHS)
- Planet size  $\propto$  dot size; colors correlate with size and temperature.



### 3 Virial Theorem

For larger ensembles of stars, global averages can be described using an energy theorem, called the **virial theorem**. It is applicable to solar systems, open and globular clusters, galaxies, clusters of galaxies and gas therein.

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- It provides a means for estimating total mass content using determinations of the kinetic energy content. A key example is *the inference of dark matter from galactic rotation curves*.
- First, in two body systems, we have

$$E = \frac{1}{2}\mu v_p^2 - G\frac{M\mu}{r_p} \ , \quad r_p = a(1 - e) \ , \quad v_p^2 = \frac{GM}{a} \left( \frac{1+e}{1-e} \right) \ , \quad (18)$$

which leads to

$$E = \frac{GM\mu}{a} \left\{ \frac{1}{2} \frac{1+e}{1-e} - \frac{1}{1-e} \right\} = -\frac{GM\mu}{2a} = -\frac{Gm_1m_2}{2a} \ . \quad (19)$$

Therefore,

$$\boxed{E = \frac{1}{2} \langle U \rangle \ , \quad \langle U \rangle = -\frac{Gm_1m_2}{a} \ .} \quad (20)$$

This is the 2-body virial theorem, and  $\langle U \rangle$  is the time-averaged potential energy of the system (proof left to the reader).

\* An example of this pertains to near-Earth orbit satellites, where the factor of 2 defines the added kinetic energy required for rockets to escape Earth's gravity relative to those that place satellites in orbit.

\* The utility centers on the kinetic energy, which couple to inferred Doppler speeds through line diagnostics, and the gravitational potential energy being of comparable magnitude. This then enables one to infer the total mass of a star or gas ensemble.

To prove the **general virial theorem**, we form the following momentum moment of particle positions:

$$Q = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \quad . \quad (21)$$

The time derivative of this can simply be recast as

$$\frac{dQ}{dt} = \frac{d}{dt} \sum_i m_i \frac{d\mathbf{r}_i}{dt} \cdot \mathbf{r}_i = \frac{1}{2} \frac{d^2 I}{dt^2} \quad , \quad I = \sum_i m_i r_i^2 \quad . \quad (22)$$

Here  $I$  is the **moment of inertia** of the multi-body system. This differential equation can be formed in an alternative way by retaining the differentiation of momentum so as to capture the force  $\mathbf{F}_i = d\mathbf{p}_i/dt$  explicitly:

$$\frac{dQ}{dt} \equiv \frac{1}{2} \frac{d^2 I}{dt^2} = \sum_i \mathbf{p}_i \cdot \frac{d\mathbf{r}_i}{dt} + \sum_i \frac{d\mathbf{p}_i}{dt} \cdot \mathbf{r}_i \quad . \quad (23)$$

The first term on the right is just twice **the kinetic energy**  $K$ :

$$\sum_i \mathbf{p}_i \cdot \frac{d\mathbf{r}_i}{dt} = \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i = 2K \quad . \quad (24)$$

Accordingly, we have formulated an energy equation, where the force term connects to the gravitational potential energy of the system, and the moment of inertia second derivative term describes the redistribution of rotational kinetic energy in the ensemble of bodies.

Defining  $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$  as the gravitational force on body  $i$  due to body  $j$ , then one can manipulate the force contribution using a double summation trick as follows:

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \sum_i \left( \sum_{j \neq i} \mathbf{F}_{ij} \right) \cdot \mathbf{r}_i \equiv \frac{1}{2} \sum_i \left( \sum_{j \neq i} [\mathbf{F}_{ij} - \mathbf{F}_{ji}] \right) \cdot \mathbf{r}_i \quad (25)$$

Performing a simple index relabeling  $i \leftrightarrow j$  on the second term on the RHS establishes

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = \frac{1}{2} \sum_i \sum_{j \neq i} \mathbf{F}_{ij} \cdot [\mathbf{r}_i - \mathbf{r}_j] \quad . \quad (26)$$

Now, the gravitational force law can be inserted,

$$\mathbf{F}_{ij} = -\frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} [\mathbf{r}_i - \mathbf{r}_j] \quad , \quad (27)$$

so that

$$\sum_i \mathbf{F}_i \cdot \mathbf{r}_i = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} = U \quad , \quad (28)$$

the **total potential energy**. Observe that the factor of  $1/2$  disappears here since each  $i, j$  pair is counted twice.

Collecting results, we have determined that

$$\frac{1}{2} \frac{d^2 I}{dt^2} = U + 2K \quad . \quad (29)$$

For ensembles that are *in kinetic equilibrium on long times (static)*, such as globular clusters, stars themselves, or gas at the centers of clusters of galaxies, the time derivative on the LHS is extremely small and may be neglected. The same can be applied for periodic systems if the equation is integrated over a multiple of the system period. The upshot is then that long term ensemble averages obey

$\langle U \rangle = -2\langle K \rangle \quad \Rightarrow \quad \langle E \rangle \equiv \langle K \rangle + \langle U \rangle = \frac{1}{2} \langle U \rangle \quad ,$

(30)

which is the generalized **virial theorem** for self-gravitating ensembles of bodies or gas particles.

- Its principal use is to forge a direct correlation between the kinetic “temperature” of such a system, i.e.  $\langle K \rangle$ , an observable, and the total gravitating mass of the ensemble, assuming that its mean radial extent is measurable.