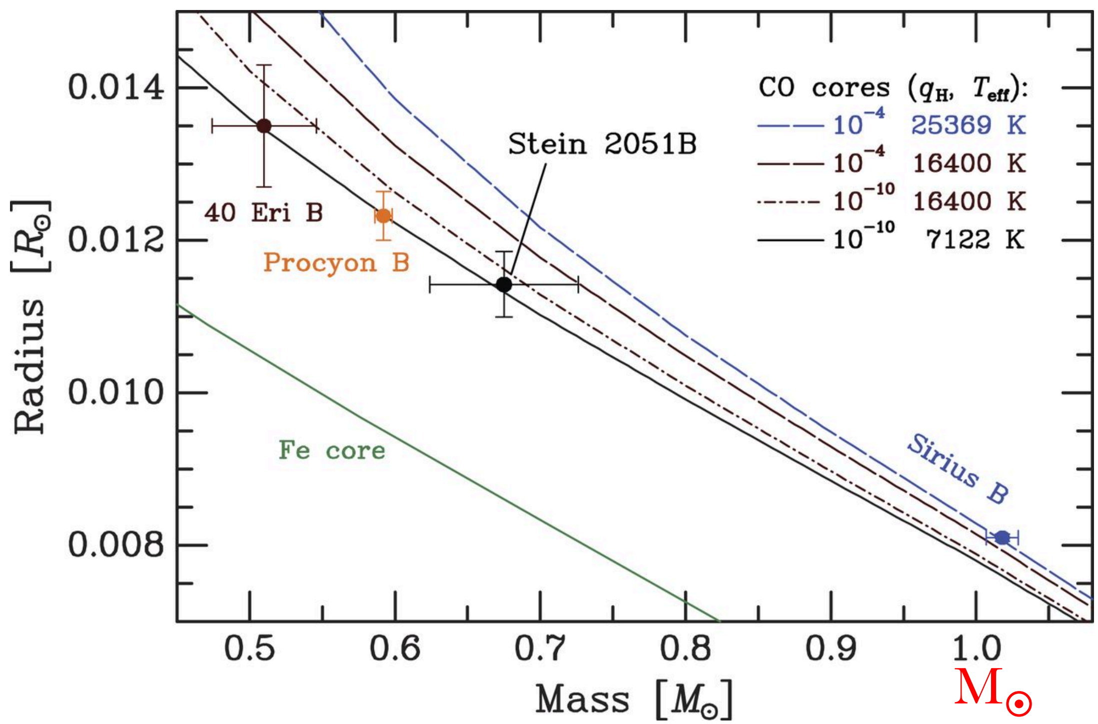


White Dwarf Mass-Radius Relation

- Sahu et al. (Science **356**, 1046, 2017)

Table 1: Mass/Radius Determination for Nearby White Dwarfs in Binaries

White Dwarf	Astrometric Mass ($M_\odot$)	Radius ($R_\odot/100$)	$v_g = GM/Rc$ (km sec^{-1})[c]	Redshift[d] (km sec^{-1})
Sirius B [a]	1.000 ± 0.016	0.84 ± 0.02	75.7 ± 3.0	80.4 ± 40.8
40 Eridani B [a]	0.50 ± 0.02	1.36 ± 0.02	23.4 ± 1.3	26.5 ± 1.5
Procyon B [a]	0.60 ± 0.03	0.96 ± 0.04	39.8 ± 3.6	–
Stein 2051 B [b]	0.675 ± 0.051	1.14 ± 0.04	37.7 ± 4.2	–

Notes: [a]Provencal et al. (1998, *ApJ*) for Sirius B, 40 Eri B & Procyon B. [b]Sahu et al. (2017, *Science*) for Stein 2051 B and astrometric microlensing. [c]The gravitational redshift formula is $v_g = 0.636(M/M_\odot)(R_\odot/R)$ km/sec. [d]The Doppler redshift for Sirius B is from Barstow et al. (2005, MNRAS), for 40 Eridani B from Koester & Weidemann (1991, AJ).

2 The Chandrasekhar Mass Limit

For the non-relativistic mass-radius relations, in principle the star can be infinitely massive and arbitrarily small. What happens as $M_{\rm wd}$ increases?

Relativistic effects limit the mass a white dwarf can possess. The equation of state $P \propto n_e^{5/3}$ in Eq. (7) is no longer viable. We can use the pressure integral to determine that $P = n_e p v/3$ for $v \leq c$. Asserting that $\Delta x \sim n_e^{-1/3}$, we can use Heisenberg's uncertainty principle to establish

$$p \sim \frac{\hbar}{\Delta x} \approx \hbar n_e^{1/3} \quad . \tag{10}$$

This can be arbitrarily high. The deduced non-relativistic speed would be $v = p/m_e \approx \hbar n_e^{1/3}/m_e$, leading to the equation of state in Eq. (7).

- When the electron density exceeds around $(m_e c/\hbar)^3$ (the inverse of the cube of the Compton wavelength), the degenerate electrons are relativistic, and we set $v \sim c$. It then follows that

$$P = \frac{1}{3} n_e p_{\rm F} c = \frac{(3\pi^2)^{1/3}}{4} \hbar c \, n_e^{4/3} \quad , \tag{11}$$

a truly relativistic equation of state. For $\mu_c = A/Z$ as the **mean molecular weight** of the WD core, we can write this in **polytropic form**:

$$P = \mathcal{K} \rho^{\Gamma} \quad , \quad \mathcal{K} = \frac{(3\pi^2)^{1/3}}{4} \frac{\hbar c}{(\mu_c m_p)^{4/3}} \quad , \tag{12}$$

with $\Gamma = 4/3$. $\mathcal{K}$ is now specified for the relativistic case $n_e \gtrsim \lambda\!\!\!^{-}_{\rm C}{}^{-3}$ only. We can ascertain the general pressure scale, noting that for hydrogen, $\rho \sim 10^6$ g cm^{-3} gives $n_e \sim 6 \times 10^{29}\,$cm^3. Accordingly,

$$P = \frac{1.23 \times 10^{23}}{\mu_c^{4/3}} \left(\frac{\rho}{10^6 \text{ g cm}^{-3}} \right)^{4/3} \text{dyne cm}^{-2} \quad . \tag{13}$$

This is a bit lower than the scale for the non-relativistic degenerate electron gas, implying that the relativistic EOS only arises in white dwarfs at densities a fair bit higher than $\rho \sim 10^6\,$g cm^{-3}, i.e. generally in the stellar interior.

The gravitational pressure GM^2/R^4 in Eq. (1) intrinsically scales as $M^{2/3}\,\rho^{4/3}$ also (since $R \propto (M/\rho)^{1/3}$) . Hence balance can only be achieved if the degeneracy pressure is sufficiently great, or equivalently if the white dwarf mass is sufficiently small. This leads to

$$\boxed{M_{\rm wd} \;\lesssim\; M_{\rm Ch} \;\sim\; \left(\frac{Z}{Am_p}\right)^2 \left(\frac{\hbar c}{G}\right)^{3/2} \quad ,} \tag{14}$$

or an exact result of $M_{\rm Ch} = 1.44 M_{\odot}$. This is the famous **Chandrasekhar mass limit** of white dwarfs, discovered by Chandrasekhar in 1931. It assumes $Z/A = 0.5$ for C+O, and is *independent of the electron mass*.

Plot: White Dwarf Mass and Radius Dependence on Density

∗ Note that $(\hbar c/G)^{1/2} \approx 2.18 \times 10^{-5}\,$g is the **Planck mass**.

• This fundamental mass can be increased somewhat by permitting the star to rotate as a Maclaurin spheroid: angular momentum provides additional support against the pull of gravity.

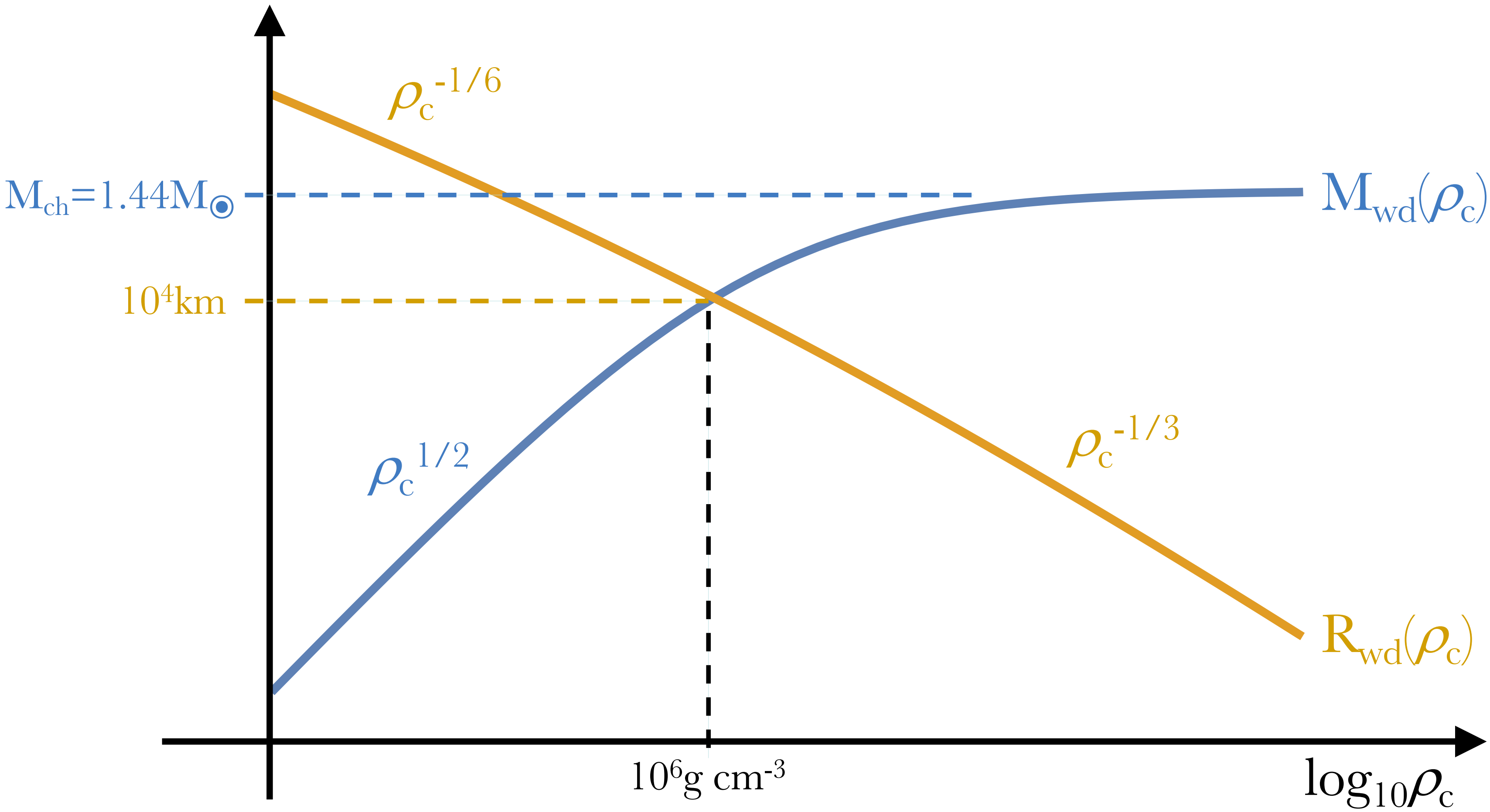
Plot: Rotational Increase of White Dwarf Mass Limit

• Note also that magnetic fields can increase the buoyancy of outer layers of white dwarfs by a few percent, although not leading to appreciable increases in masses. WD fields can be measured via the Zeeman effect on hydrogen line splitting and also frequency shift (when $B \gtrsim 1\,$MGauss).

∗ Observations of this are easiest in white dwarfs of stronger magnetization, leading to a population range of $10^4 \lesssim B \lesssim 10^9\,$Gauss for 600 WDs in the Sloan Digital Sky Survey (SDSS). The highest of these are larger than flux freezing arguments indicate, suggesting that some dynamo action in enhancing $\mathbf{B}$ is at play during dwarf formation.

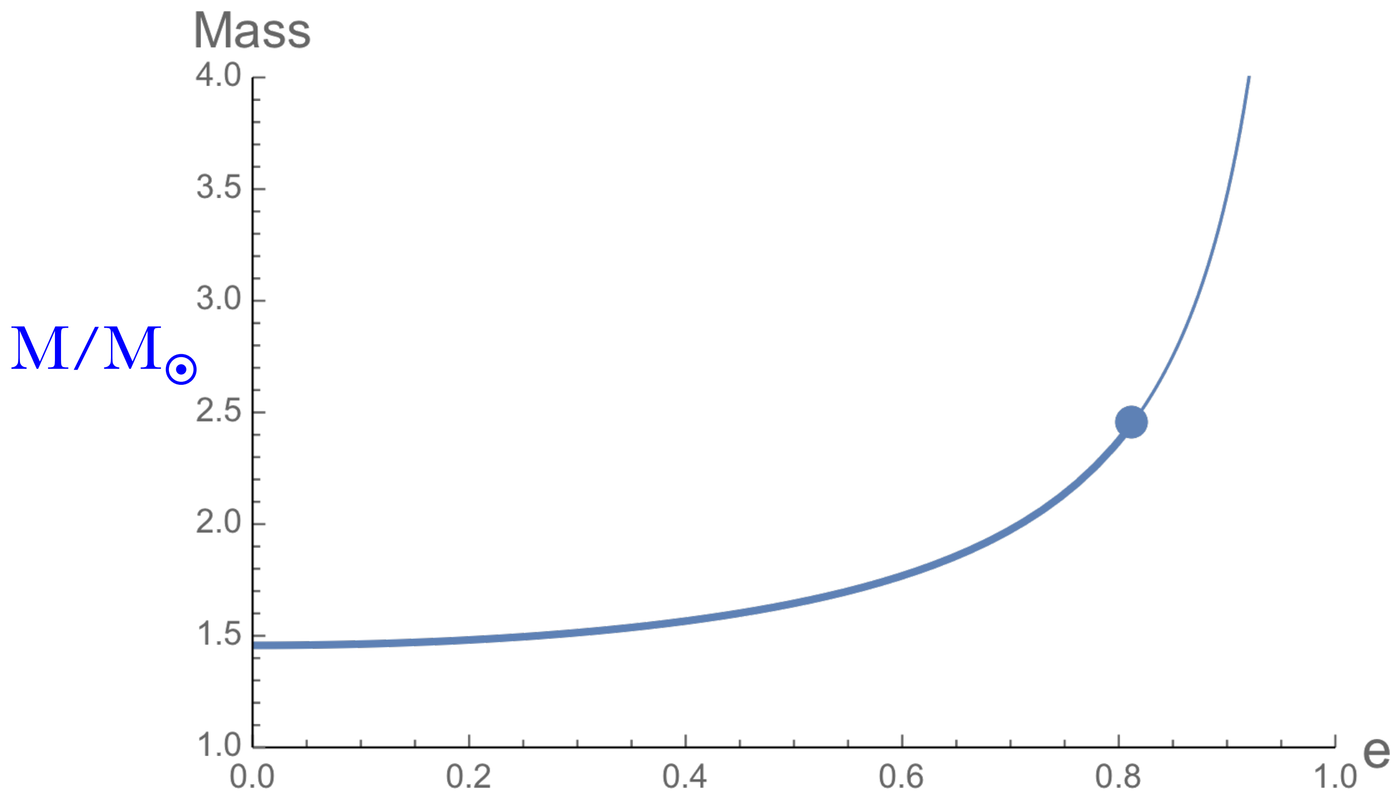
Plot: White Dwarf Magnetic Fields

White dwarf mass and radius dependence on mass density ρ_c



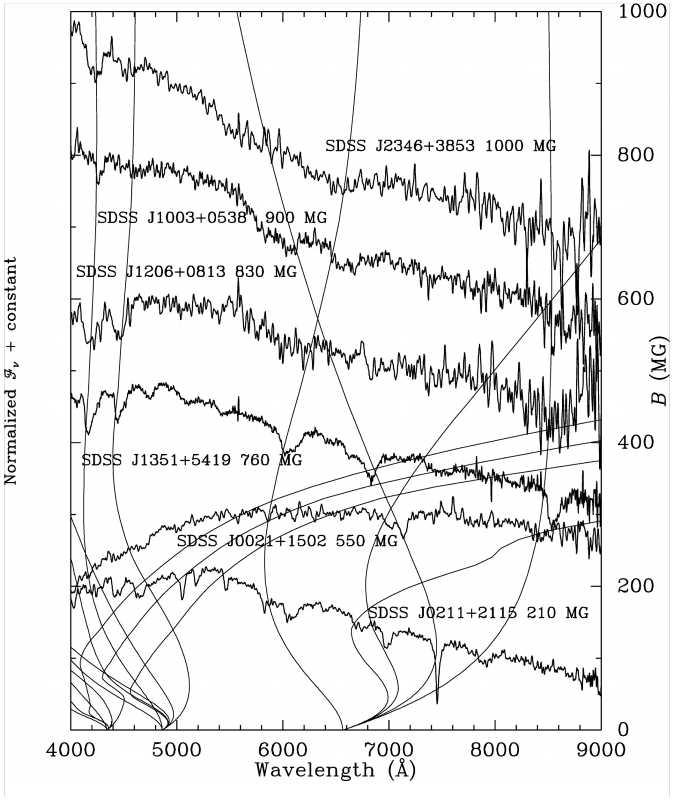
- Dependence of stellar mass M and radius R on central density ρ_c.

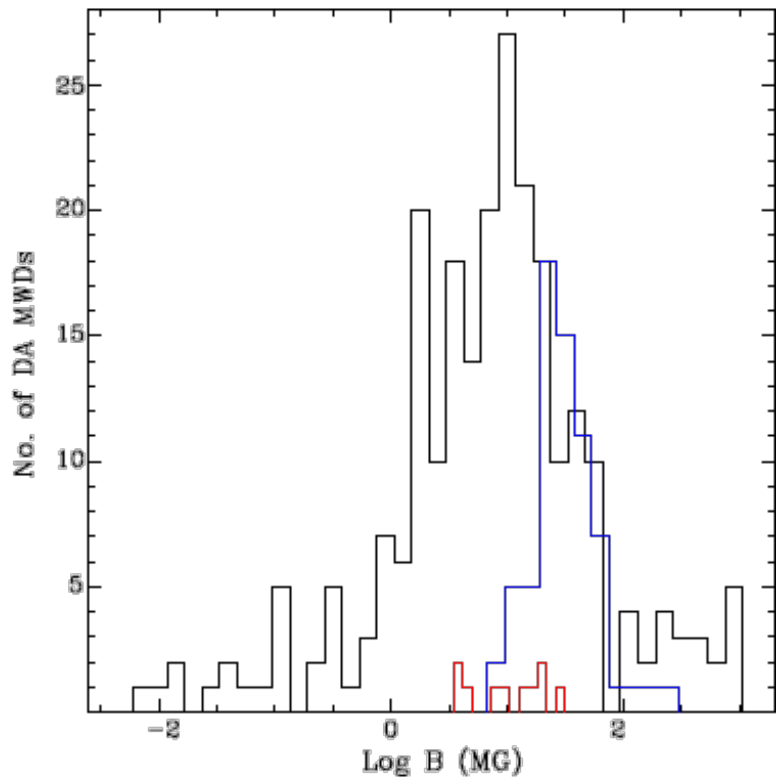
Rotational increase of White Dwarf Mass beyond the Chandrasekhar Limit



- Mass increase of Chandrasekhar white dwarfs that are Maclaurin spheroids of oblateness eccentricity e, that is uniquely coupled to the rotation parameter $\Omega/[2\pi G\rho]^{1/2}$.
- Only eccentricities less than e=0.8216 (blue dot) are secularly stable (heavy curve), and this ultimately limits the mass enhancement to $2.45M_{\odot}$ for a C/O white dwarf.

White Dwarf Magnetic Fields





- *Left panel*: Optical spectra for six WDs with Hα and Hβ lines split by the Zeeman effect, which is used to measure B. Curves are wavelength variations of split lines as a function of field strength (right axis). Fig. 1 from Vanlandingham et al. (2005, AJ **130**, 734).
- *Right panel*: Magnetic field distribution of ~600 magnetic white dwarfs in the Sloan Digital Sky Survey (SDSS). Black histogram is for isolated WDs, and blue is for polars. Fig. 8 from Ferrario, de Martino & Gaensicke (2015, SSRv **191**, 111).

3 White Dwarf Cooling

Because there is no normal thermonuclear burning in their interiors, white dwarfs essentially cool via surface thermal radiation without altering their radius; the hydrostatic balance is not altered during their luminous lifetimes. **C & O, Sec. 16.5**

In conventional thermonuclear reactions, the thermal energy of the gas overcomes nuclear Coulomb repulsions via quantum tunneling to seed nuclear transitions that are predominantly exothermic. This is true for main sequence stars, and for primordial nucleosynthesis.

• In condensed, white dwarf interiors, the finite value of the zero-point Fermi energy $\varepsilon_{\rm F}$ at $T \approx 0$ can also permit nuclear reactions; these are termed **pyconuclear reactions**, and their burning rate is slow.

- ∗ derived from the Greek word **pyknos**, which means dense.
- ∗ an example is the familiar hydrogen fusion reaction $p+p \to D+e^{+}+\nu$.

In the layers below the *nondegenerate* photosphere and crust, heat is transported by **electron conduction**, since such conductivity is high when the electrons are degenerate. The efficiency of this is high, so that the stellar interior is *effectively isothermal.* [`Sketch this.`]

• Convection is not important in white dwarfs. The outer non-degenerate layers are in "radiative equilibrium," with energy flux mediated by **photon diffusion** instead of electron conduction. The governing heat transport equation is

$$L \;=\; -4\pi r^2 \, \frac{c}{3\kappa\rho} \, \frac{d}{dr} \left(aT^4 \right) \quad . \tag{15}$$

Here $\kappa(r)$ is the **opacity**, in units of cm^2 g^{-1}, of the outer layers. The quantity $1/\kappa\rho$ is an estimate of the diffusive mean free path. At typical white dwarf *near-surface temperatures*, opacity is governed by Compton scattering, photoelectric/ionization events and bremsstrahlung. The photo-ionization contribution is dominant, and so one can adopt **Kramer's opacity**

$$\kappa \;\propto\; \kappa_0 \rho T^{-7/2} \quad , \tag{16}$$

to describe the diffusion. The temperature dependence traces the rough frequency dependence of the photo-ionization cross sections. **Kramer's opacity** is similar to the Rosseland mean opacity.

Using the Stefan-Boltzmann law, the heat transport equation is readily converted into an equation for the temperature gradient:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2} \quad . \tag{17}$$

As temperature declines outward from the degeneracy interface, the rate of decline drops precipitously with radius. Accordingly, the non-degenerate surface layers are extremely thin, with a large mean temperature gradient.

The radial gradient of the blackbody flux is sufficiently small that it is approximately constant on the diffusion length scale. Hence one can infer that

$$L \propto \frac{1}{\kappa} \propto T^{7/2} \quad . \tag{18}$$

This then tells us that the temperature gradient in Eq. (17) is mostly described by the steep density gradient in the outer non-degenerate near-surface layers. Detailed derivations of the equation of state and the hydrostatic balance in the surface layers yield the couplings

$$\rho \propto T^{13/4} \quad \text{and} \quad P \propto T^{17/4} \quad . \tag{19}$$

The radial dependence can then be found using Eq. (17).

The full crustal solution can then be expressed using the temperature T_c of the crust/degenerate interior interface, i.e. the **core temperature**. This then gives a white dwarf luminosity of

$$L_{\rm wd} \sim C\, T_c^{7/2} \quad , \quad C = 7.3\times 10^5 \left(\frac{M_{\rm wd}}{M_\odot}\right)\frac{\mu}{Z} \quad . \tag{20}$$

Here C has units of erg sec^{-1} K$^{-7/2}$. Since the thermal energy of the stellar interior is

$$U = \frac{M_{\rm wd}}{Am_H}\frac{3}{2}kT_c \quad , \tag{21}$$

then a rough estimate of the cooling timescale (the high conductivity redistributes the thermal energy rapidly through the star) is

$$\tau_c \sim \frac{U}{L_{\rm wd}} = \frac{3\, M_{\rm wd} k}{2 A m_H C T_c^{5/2}} \quad . \tag{22}$$

Note that since $C \propto M_{\rm wd}$, this timescale is roughly independent of the white dwarf mass!

• Solving the cooling equation $-dU/dt = L_{\rm wd}$ then gives

$$T_c(t) = \frac{T_0}{(1 + t/\tau_0)^{2/5}} \quad , \quad L_{\rm wd}(t) = \frac{L_0}{(1 + t/\tau_0)^{7/5}} \tag{23}$$

for the evolution of the core temperature and the WD luminosity, where

$$\tau_0 = \frac{3\, M_{\rm wd} k}{5 A m_H C T_0^{5/2}} \quad . \tag{24}$$

Furthermore, since all white dwarfs start with similar central temperatures, this cooling curve is more or less a standard "decay," with $\tau_0 \sim 1.5 \times 10^8$ years.

• Deviations from the standard cooling curve are expected, and are observed, at late stages of evolution (typically $\sim 5 \times 10^9$ years) due to **crystallization**.

∗ Cooling lowers $T_{\rm wd}$ to the point where dense C and O form a lattice structure, from inside first, where P is high, progressing towards the outside. This structure is like diamond formation under extreme pressure.

∗ Crystallization is a phase transition that *releases* latent heat – consequently slowing the cooling, and generating a "bump" in the cooling curve.

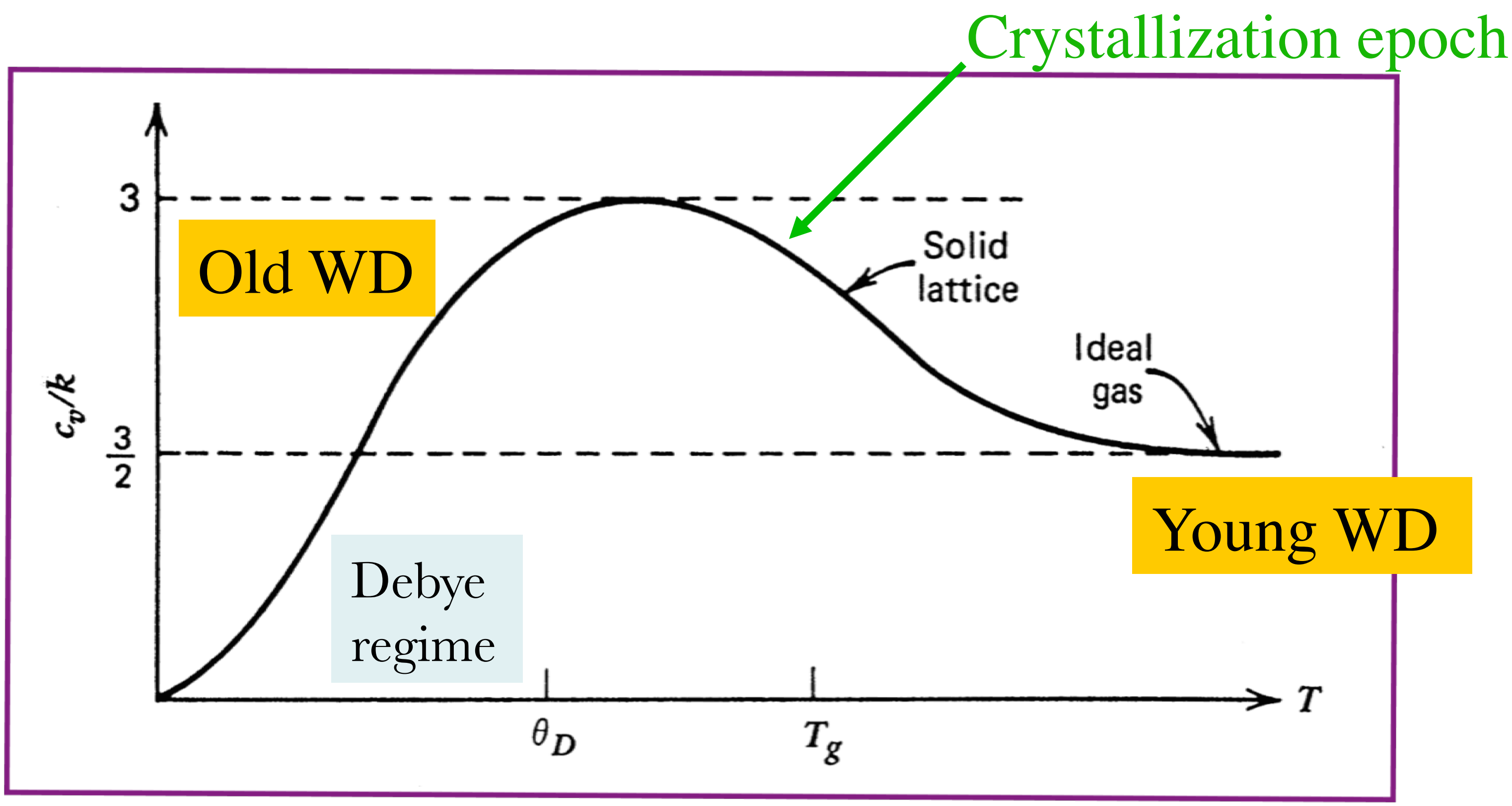
Plot: White Dwarf Specific Heat Capacity

• The robustness of the cooling curve leads to a usefulness of white dwarfs as age calibrators in the universe.

Plot: White Dwarf Luminosity Distribution

The Milky Way white dwarf population suggests an age of $\gtrsim 9.0 \pm 1.8 \times 10^9$ years. To this must be added the contribution for main sequence evolution prior to the white dwarf phase.

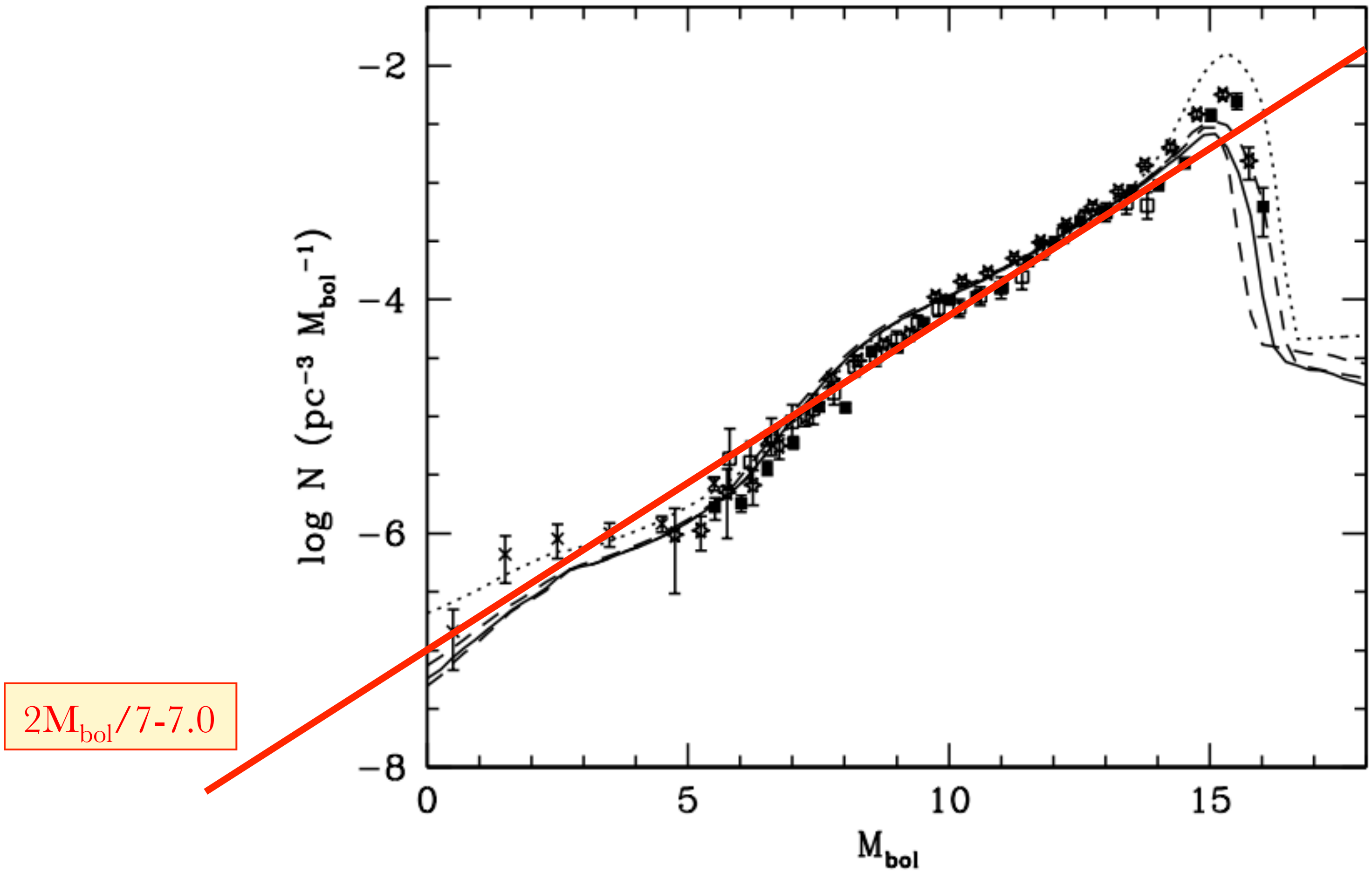
Specific Heat Capacity: Debye Regime



- Specific heat capacity as a function of temperature – schematic diagram for ions only. At high temperature, the lattice melts, forming an ideal gas with $c_V=3k/2$. At modest temperatures, crystallization results, and c_V increases to 3k. When $T < \Theta_D$, collective influences on vibrational modes (Debye screening) lower $c_V \propto T^3$.

White Dwarf Luminosity Function

(Isern, Artigas & García-Berro, EPJ 43, 05002, 2013)



- Large survey data for the white dwarf luminosity function. Models for curves comprise different Galactic disk descriptions and star formation rates.