2.1 The Schönberg-Chandrasekhar Limit

• The mass of the isothermal core at the tip of the subgiant branch can be simply related to the ratio of the mean molecular weights in the envelope (μ_e) and the core (μ_c), and is called the **Schönberg-Chandrasekhar limit**:

$$\frac{M_c}{M} \;\approx\; 0.37 \left(\frac{\mu_e}{\mu_c} \right)^2 \quad . \tag{18}$$

Carroll and Ostlie detail an estimation of this limit, distilled as follows.

Plot: Sketch two-zone Structure in Sub-Giants

The stellar interior satisfies (longish derivation) a modified virial theorem:

$$4\pi R_c^3\, P_c - 2K_c \;=\; U_c \quad , \tag{19}$$

where the core kinetic energy is $K_c = 3N_c kT_c/2$ for a number of particles $N_c = M_c/(\mu_c m_H)$ in the core, and the gravitational potential energy of the core is $U_c \sim -3GM_c^2/(5R_c)$.

The modification is that we no longer have a zero pressure boundary condition at the core/envelope interface (as opposed to the stellar surface), so we have an energy term $3P_cV_c$ appearing.

• We now solve for P_c. For $\lambda_c = GM_c\mu_c m_H/(kT_c)$, this yields

$$P_c \;=\; \frac{3}{4\pi R_c^3} \left(\frac{M_c kT_c}{\mu_c m_H} - \frac{GM_c^2}{5R_c} \right) \;=\; \frac{3}{4\pi}\, \frac{GM_c^2}{R_c^4} \left(\frac{R_c}{\lambda_c} - \frac{1}{5} \right) \quad . \tag{20}$$

To provide maximal support at the onset of the red giant phase, for fixed core radius and temperature, we can differentiate w.r.t. M_c to yield

$$\frac{\partial P_c}{\partial M_c} \;=\; 0 \quad \Rightarrow \quad R_c \;\sim\; \frac{2\lambda_c}{5} \quad \text{for} \quad \lambda_c \;=\; \frac{GM_c\mu_c m_H}{kT_c} \quad . \tag{21}$$

Then the scaling

$$P_c \;\approx\; \frac{3}{20\pi}\, \frac{GM_c^2}{R_c^4} \;\approx\; 1.86\, \frac{GM_c^2}{\lambda_c^4} \tag{22}$$

quickly follows. This can be equated to the pressure exerted by the overlying material in the envelope, which is the force per unit area, i.e.

$$P_e \;\sim\; \frac{1}{4\pi}\,\frac{GM^2}{R^4} \;\sim\; \frac{\rho_e}{\mu_e m_H}\,kT_c \quad ; \tag{23}$$

see details of the derivation in C&O. Note that $M \gg M_c$ for the star's mass. The first equality is essentially the virial theorem applied to the envelope, with zero outer pressure. The second equality asserts that the equation of state in the envelope is set by the temperature at the core-envelope interface.

Setting $\rho_e \sim 3M/(4\pi R^3)$, we can solve for R to yield

$$R \;\approx\; \frac{GM\mu_e m_H}{3kT_c} \;\equiv\; \frac{1}{3}\,\frac{M}{M_c}\,\frac{\mu_e}{\mu_c}\,\lambda_c \quad . \tag{24}$$

This derives the envelope pressure:

$$P_e \;\sim\; \frac{GM^2}{4\pi}\left(\frac{3kT_c}{GM\mu_e m_H}\right)^4 \;\approx\; 6.45\left(\frac{M_c}{M}\,\frac{\mu_c^2}{\mu_e^2}\right)^2\frac{GM_c^2}{\lambda_c^4} \quad . \tag{25}$$

This is very similar to Eq. (22) when setting $\mu_e \to \mu_c$ and $M \to M_c$. It follows that the solution of $P_e = P_c$ then generates the solution

$$\frac{M_c}{M} \;\approx\; 0.54\left(\frac{\mu_e}{\mu_c}\right)^2 \quad . \tag{26}$$

Information on density stratification throughout the star, and temperature gradients in the envelope reduce the numerical coefficient significantly.

• Note that detailed treatment of the coefficients and radial gradients beyond the 2-zone simplification is somewhat superfluous, since we have ignored quantum **electron degeneracy pressure**.

∗ This fermionic property spawned by the Pauli exclusion principle, provides a zero-point temperature-independent pressure that provides considerable contribution to the subgiant phase of solar mass stars.

• The importance of the **Schönberg-Chandrasekhar limit** is that it helps identify the end state of evolution of a particular star; low mass cores will probably end up as white dwarfs. This connects to the Vogt-Russell Theorem through the envelope/core metallicity ratio μ_e/μ_c.

10. STELLAR EVOLUTION II

Matthew Baring – Lecture Notes for ASTR 350, Fall 2025

1 Late Stages of Stellar Evolution

Later on in the evolution of a main sequence star, there is the appearance of multiple shells of thermonuclear burning of different isotopes, accompanied by intense convective behavior and extensive mass loss. Key elements of the evolutionary sequence can be summarized as follows. **C & O, Sec. 13.2**

Plot: Post Main Sequence Stellar Evolution

• At the end of main sequence phase the core has evolved mostly to helium and shrinks. This heats the surrounding hydrogen shell, which ignites and pushes the star through the subgiant phase with expansion of the envelope.

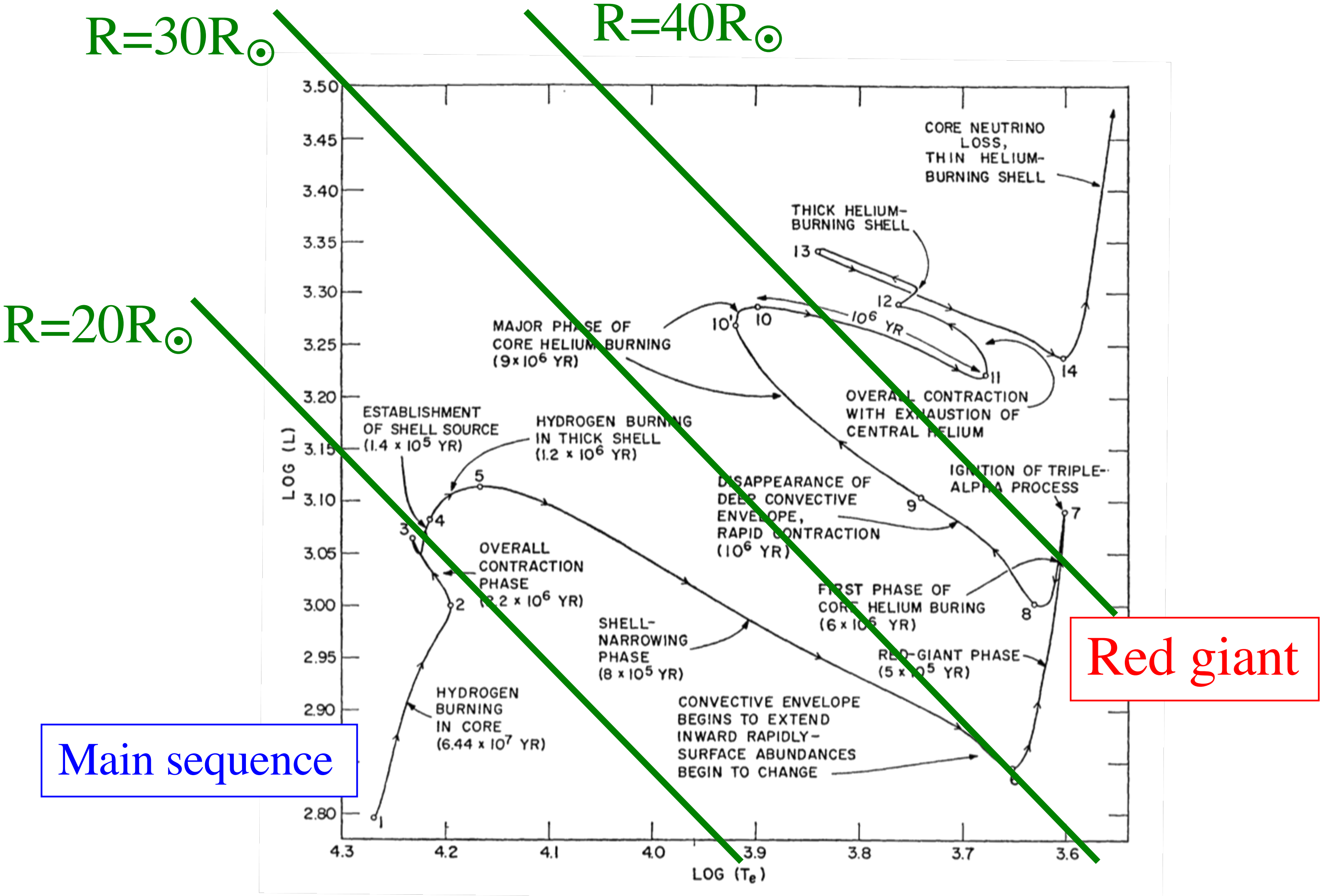
∗ Once the Schönberg-Chandrasekhar limit is reached (locus point 5), the core collapses further on a (short) Kelvin-Helmholtz timescale $t_{\rm KH} \sim \Delta E_g/L \sim GM^2/(LR)$. The core is no longer approximately isothermal, and the hydrogen shell is heated and narrowed significantly.

∗ The increased energy generation is not converted into luminosity, but is absorbed by *envelope expansion*, with active convection penetrating down towards the core. The star moves to the red with a drop in luminosity and effective temperature with significant contribution from H^- to opacity.

∗ *This implies the existence of spectroscopic signatures of convective action,* distinguishing such stars from solar-type abundances.

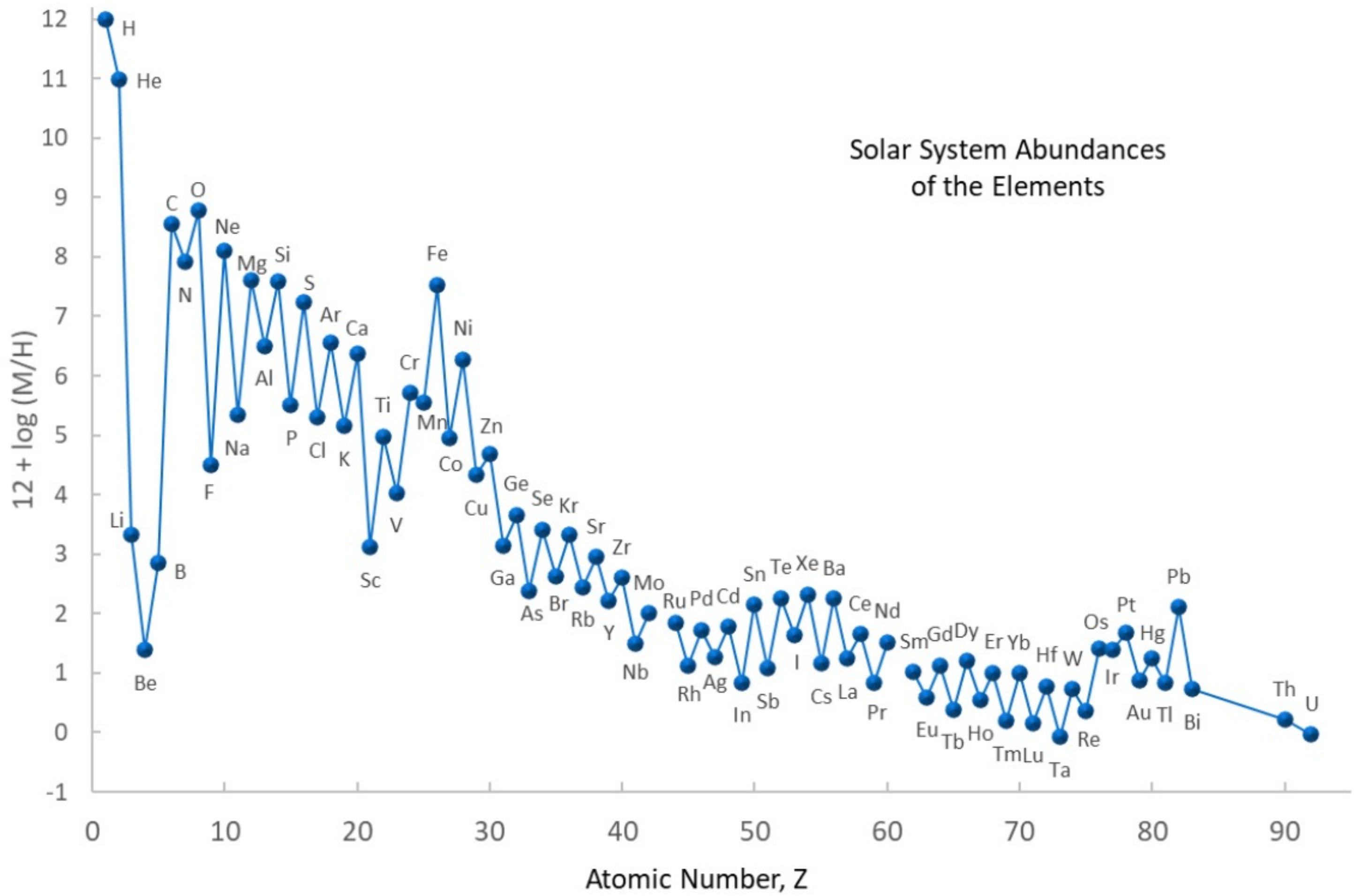
Plot: Benchmark: Solar Abundances

Evolution of a Massive Star



- Post main sequence evolution of a $5M_{\odot}$ star. Iben, I., ARAA **5**, 571 (1967).

Solar System Elemental Abundances



- From Lodders, K. 2020, Solar Elemental Abundances (Oxford University Press) [arXiv.org/1912.00844]

• Convective action can mix elements so that abundances of rarer elements such as lithium can change near the surface.

• The star subsequently becomes fully convective, transporting large amounts of energy outwards. This causes a rapid expansion of the envelope, and an increased luminosity accompanied by greater core production rates. This is the onset of the **red giant phase** (locus point 6), where the star moves vertically in the HR diagram.

Plot: Structural Evolution Diagram of Kippenhahn

• At the peak of the red giant branch (locus point 7), the temperature is hot enough to seed the triple alpha process in a short-lived, explosive **helium core flash**. The star then evolves its helium core into carbon and oxygen as it progresses blueward along the **horizontal branch** (locus points 7–11).

∗ The convection zone retreats to nearer the surface, and the outer envelopes can develop instabilities leading to pulsations; such variables are discussed in the next Chapter.

• After helium burning is exhausted, the evolution replicates the subgiant phase for hydrogen burning depletion, and the star expands and reddens, eventually moving up the **asymptotic giant branch** (AGB).

∗ AGB stars are inherently unstable to helium shell burning, leading to so-called helium-shell flashes and epicyclic stellar pulsation.

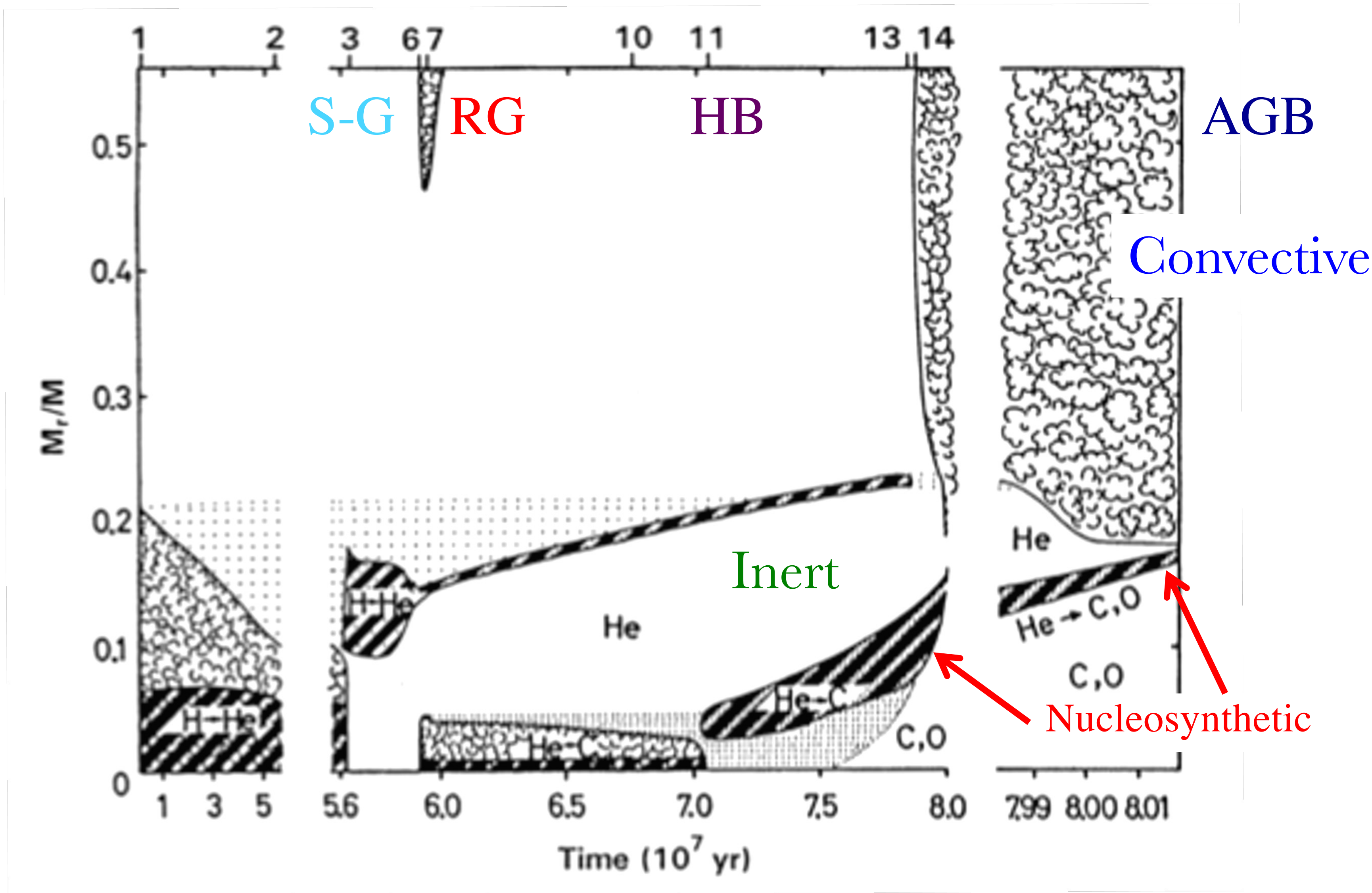
∗ Sequential heating and expansion of the helium shell followed by cooling to partial degeneracy and contraction accompanies the flashes.

Plot: Helium Flash Lightcurves

• Stellar pulsation can lead to extensive mass loss via stellar winds. These winds can be highly non-uniform, and be lit up by the radiation from the dying star, which is the phenomenon known as a **planetary nebula**.

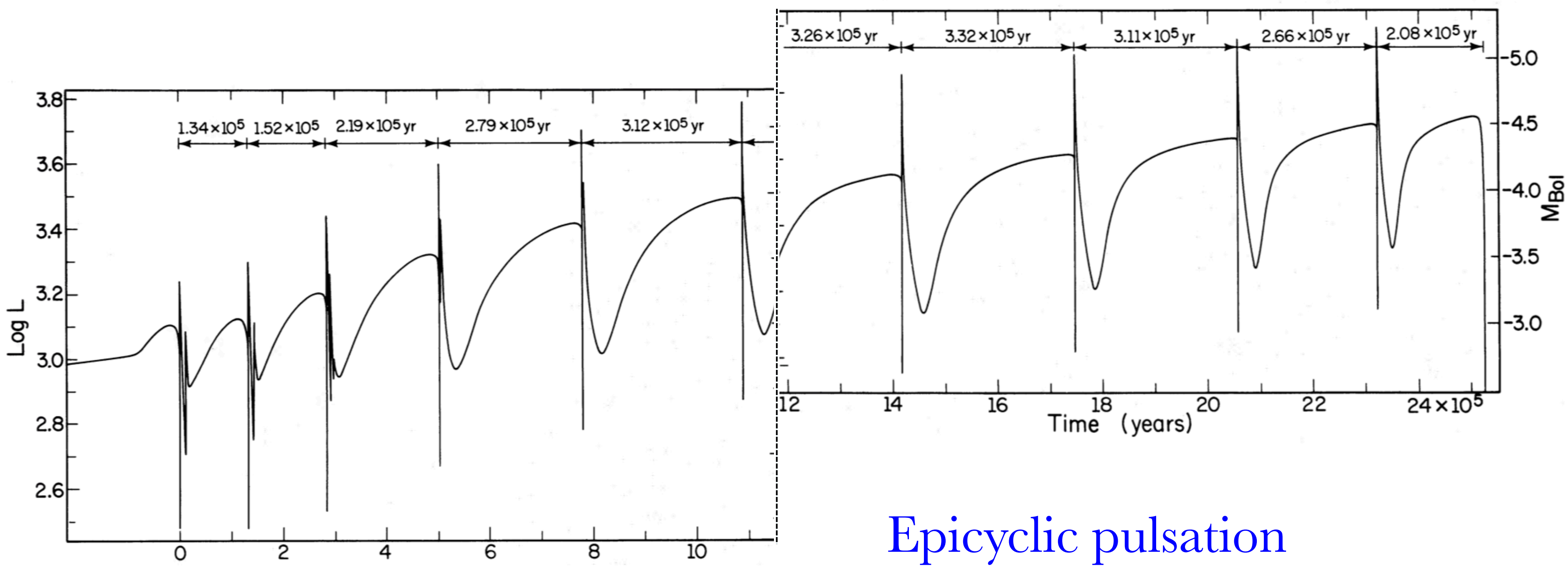
Plot: Planetary Nebula NGC 6543

Structural Evolution of a Massive Star



- Evolution of a $5M_{\odot}$ star. Kippenhahn, R., et al., Z. Astrophys. **61**, 241 (1965).

Helium Flash Light Curves



Epicyclic pulsation

- Surface luminosity as a function of time for a $0.6M_{\odot}$ AGB star undergoing helium shell flashes. Iben, ApJ 260, 821 (1982)

Planetary Nebula NGC 6543

• Helium burning in the shell deposits more carbon and oxygen onto the core until eventually electron degeneracy pressure can no longer support the core. It collapses catastrophically, the resulting explosion is called a **supernova**.

∗ If the core is less than $1.4M_\odot$ (the Chandrasekhar limit) in mass, it can form a **white dwarf** star; if it is more massive, it will form a **neutron star** or a **black hole**.

1.1 Massive Stars

Stars greater than about 8–10 $M_\odot$ suffer a different evolutionary sequence, due largely to their greater core temperatures. Core shrinking is more impressive, leading to temperatures far in excess of 10^8 K. This permits burning of more massive elements. **C & O, pp. 530–3**

• Oxygen will readily burn after the subgiant phase, and in quick succession, neon, sodium, magnesium can burn as the temperature rises. Then the temperature can reach $\gtrsim 2 \times 10^9$ K and silicon and sulphur burning can proceed, in a sequence all the way to the $^{56}_{26}Fe$ peak:

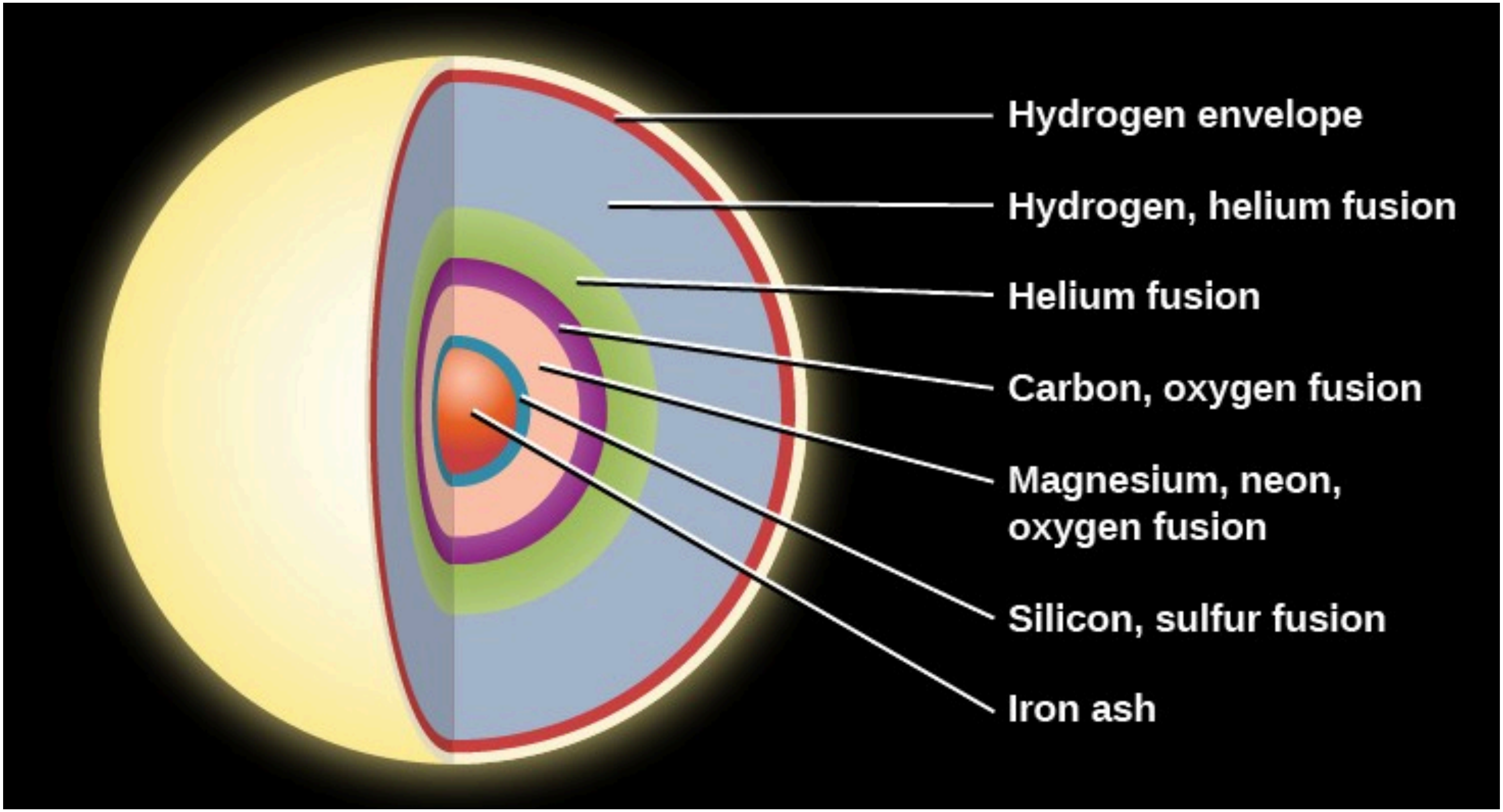
$$\begin{aligned} ^{28}_{14}Si + ^{4}_{2}He &\leftrightarrow ^{32}_{16}S + \gamma \\ ^{32}_{16}S + ^{4}_{2}He &\leftrightarrow ^{36}_{18}Ar + \gamma \\ &\vdots \\ ^{52}_{24}Cr + ^{4}_{2}He &\leftrightarrow ^{56}_{26}Fe + \gamma \end{aligned} \tag{1}$$

∗ Since burning to greater masses is endothermic, the fusion process stops here. A sequence of *onion shells* of burning of successively heavier elements forms, with an iron core.

Plot: Onion-Shell Interior of an Evolved Massive Star

∗ Sequential burning toward the iron peak becomes rapidly faster: for a $20M_\odot$ star, hydrogen burning takes 10^7 years, helium burning takes 10^6 years, carbon burning 300 years, oxygen burning 200 days, and silicon burning just 2 days!

Onion Shell Model of Massive Star Interiors



Credit: Lumen Learning

• Note that at such temperatures, the gamma-rays can seed **photodisintegration of the nuclei**, generating copious neutrons.

$$\begin{aligned} {}^{56}_{26}Fe + \gamma \ &\leftrightarrow \ 13\,{}^{4}_{2}He + 4n \\ {}^{4}_{2}He + \gamma \ &\leftrightarrow \ 2\,{}^{1}_{1}H + 2n \end{aligned} \tag{2}$$

∗ Such endothermic processes are in detailed balance indicated in the reactions of Eq. (1). These rapidly cool the core and hence soon cease.

Plot: Woosley & Weaver Abundance/Radius Model for a Massive Star

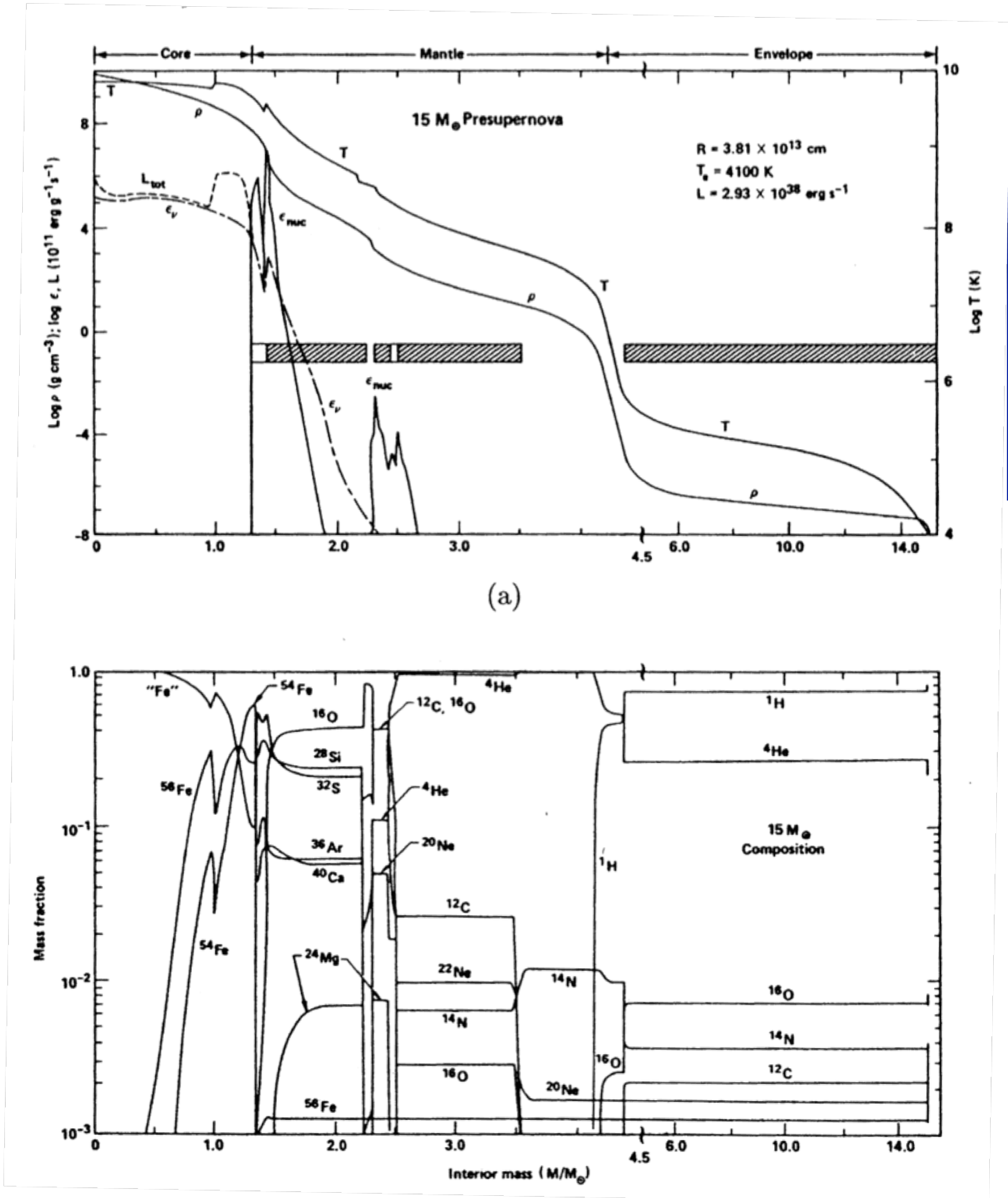
• Electron capture in the iron core plus photodisintegration readily removes electron degeneracy pressure to the point that the core can no longer support itself. The collapse is rapid and **homologous** (i.e. with virtually no mixing of layers), reaching speeds of around $c/5$.

∗ The outer layers are temporarily "suspended" above the supersonically collapsing core, and the core reaches supernuclear densities in about a second.

∗ The repulsive degeneracy pressure associated with neutrons kicks in and causes the core to *bounce*, sending a shock wave out into the overlying shells before they have had time to collapse.

• Photodisintegration robs the shock of its energy. Depending on the size of the core, the shock can stall and become an **accretion shock**. Neutrino heating of the shocked material can eventually push the shock outwards again.

∗ Simulations are extremely sensitive to assumed boundary conditions such a dimensionality (1D codes often failed to explode).



Interior Structure of a 15M$_\odot$ Star

- Woosley & Weaver ARAA 24, 205, (1986)