

6 Stellar Opacity

Collisional concepts that control stellar opacity can be paralleled to those pertinent to atomic excitation and ionization just explored.

C & O,
Sec. 9.2

The **cross section**, or effective target area for atom or ion collisions is scaled by the Bohr radius: $\sigma = 2\pi a_0^2$. Atoms of speed v sweep out a volume $V = \sigma vt$ in time t , and so encounter $nV = n\sigma vt$ target atoms for collisional excitation or ionization. Here n is the *number* density of atoms.

- Thus the average distance traveled between collisions is

$$l = \frac{vt}{nV} = \frac{1}{n\sigma} \quad ; \quad (20)$$

The distance l is called the **mean free path**. Assuming the Bohr radius of $a_0 = 5.29 \times 10^{-9}$ cm, we arrive at $l = 1.9 \times 10^{-2}$ cm for a solar photospheric density of $n = \rho/m_H = 1.5 \times 10^{17}$ cm⁻³.

* This scale is much shorter than the temperature gradient scalelength in the sun, so excitation/ionization considerations can assume isothermal scenarios, i.e. or *local thermodynamic equilibrium*.

Now turn to radiation. At any given position, we expect the diminution of intensity to be proportional to the density ρ of atoms along a path ds . This coupling leads to the differential form

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds \quad , \quad (21)$$

where κ_λ is called the (mass) **absorption coefficient** or **opacity**.

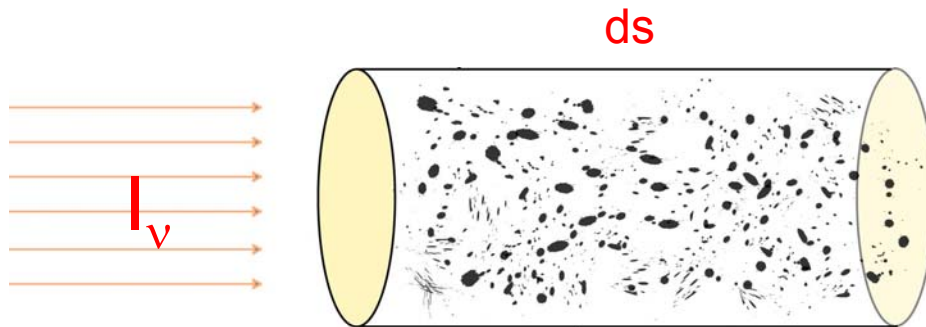
- The attenuation of a light beam can then be determined:

$$I_\lambda = I_{\lambda,0} \exp \left\{ - \int_0^s \kappa_\lambda \rho ds' \right\} \quad . \quad (22)$$

For a uniform opacity and density, this becomes a simple exponential dependence on path length s , as we have already explored when considering ISM and atmospheric extinction.

Interactions between photons and matter

absorption of radiation



$$dI_\nu = -\kappa_\nu I_\nu ds$$

κ_ν : absorption coefficient

$$[\kappa_\nu] = \text{cm}^{-1}$$

microscopical view: $\kappa_\nu = n \sigma_\nu$

loss of intensity in the beam (true absorption/scattering)

Over a distance s :

$$I_\nu^o \xrightarrow{s} I_\nu(s)$$

$$I_\nu(s) = I_\nu^o e^{-\int_0^s \kappa_\nu ds}$$

Slide Courtesy of
R-P Kudritzki, IFA,
U. Hawaii

Convention: $\tau_\nu = 0$ at the outer edge
of the atmosphere, increasing
inwards

$$\tau_\nu := \int_0^s \kappa_\nu ds$$

optical depth
(dimensionless)

or: $d\tau_\nu = \kappa_\nu ds$

- The **optical depth** is then defined to be the measure of exponential attenuation:

$$\tau_\lambda = \int_0^s \kappa_\lambda \rho ds' \quad . \quad (23)$$

The optical depth can be viewed as the number of mean free paths along a given path. Clearly, $\tau > 1$ conditions are quickly achieved as the solar atmosphere is penetrated.

* The narrow band of surface layers where $\tau \lesssim 1$ define the region that we probe spectroscopically: it is called the stellar **photosphere**.

- By analogy with the case of atomic collisions,

$$l = \frac{1}{\kappa_\lambda \rho} \equiv \frac{1}{n\sigma} \quad (24)$$

is the absorption mean free path.

In a stellar context, atmospheric opacity receives contributions from *bound-bound* transitions, *bound-free* absorption (photo-ionization), *free-free* emission (bremsstrahlung), and electron (Compton) scattering. We can write

$$\kappa = \kappa_{bb} + \kappa_{bf} + \kappa_{ff} + \kappa_{es} \quad (25)$$

These depend sensitively on the temperature, density and composition of a stellar atmosphere.

Plot: Opacity in ρ - T Space for White Dwarfs

- Hence usually we employ an average opacity called the **Rosseland Mean Opacity** to describe the effective absorption.

C & O,
pp. 249–50

This mean is a strong function of temperature and density.

Plot: Rosseland Mean Opacity

Opacity in ρ -T Space: White Dwarfs

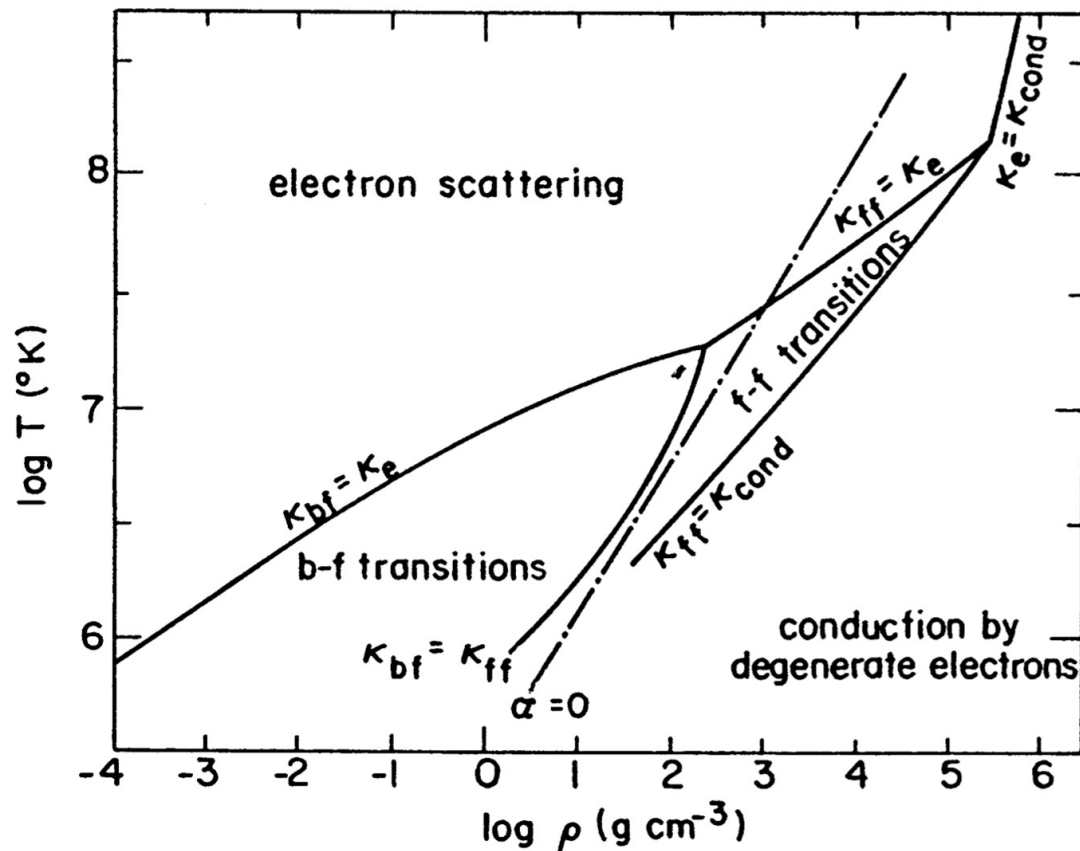
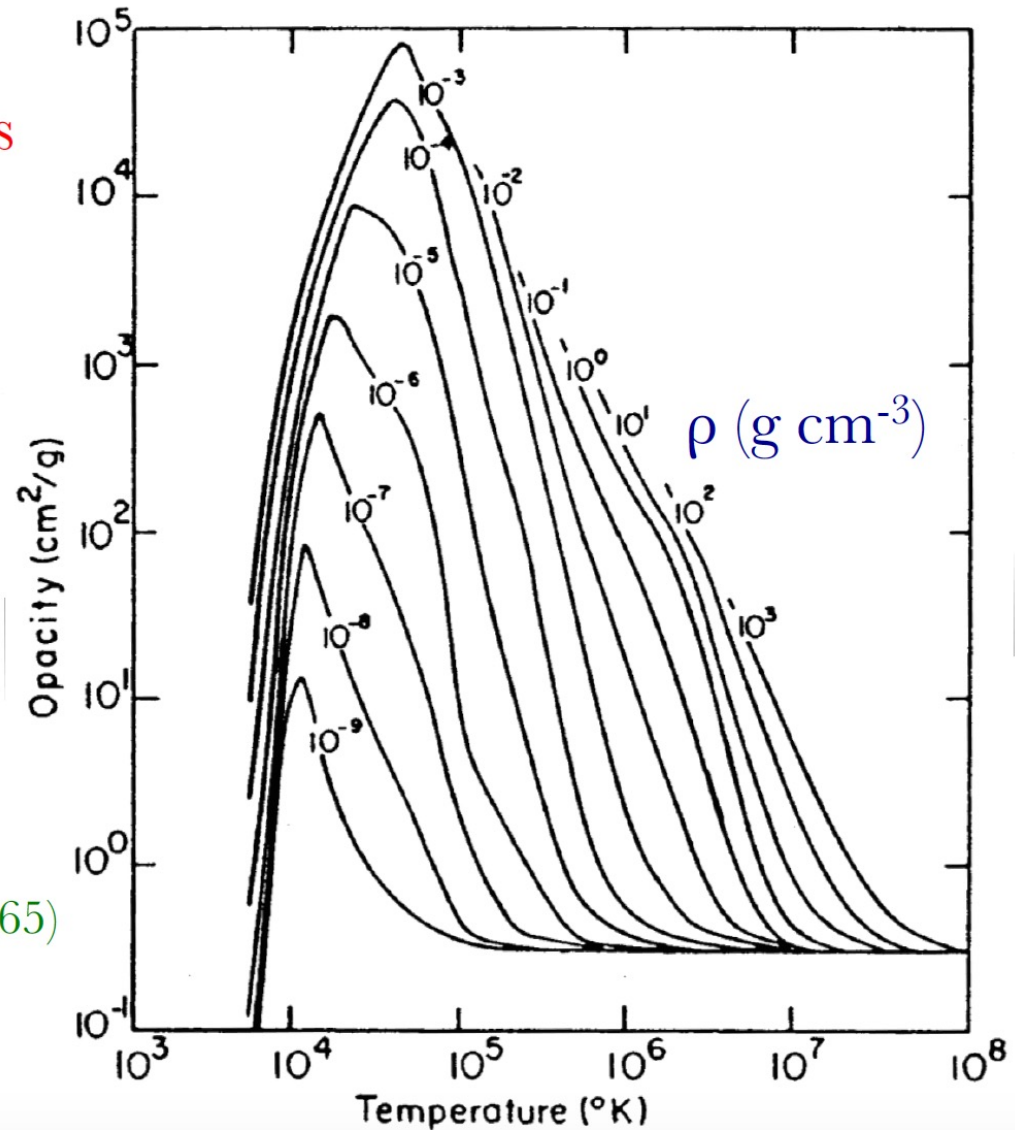


Fig. 8.3 Opacity as a function of density and temperature in a star of population I composition. The diagram is divided into four regions characterized by different mechanism of energy transport. The sources of opacity that dominate these mechanisms are electron scattering, bound-free transitions, free-free transitions, and the effective opacity that would describe the energy transport by degenerate electrons. The dashed line shows where the degeneracy parameter α (see equation 4-93) equals zero (after Hayashi, Hoshi, and Sugimoto, Ha62c).

From: M. Harwit
Astrophysical Concepts

Rosseland Mean Opacity

For solar abundances



From: M. Harwit
Astrophysical Concepts
After: Ezer and Cameron (1965)

7 Radiative Transfer

Since only the outermost layers of the sun and stars in general are optically thin, radiation transport in stellar interiors is generally diffusive on timescales much longer than the free-streaming time, $R_{\odot}/c \sim 2 \text{ sec}$.

C & O,
pp. 252–4

Diffusion can be modeled by a **random walk** of photons out from the stellar interior. The net displacement $\mathbf{d}$ in a large number N of randomly directed steps $\mathbf{l}_i$ is

$$\mathbf{d} = \sum_{i=1,N} \mathbf{l}_i \quad (26)$$

so that the magnitude can be obtained from

$$d^2 = \mathbf{d} \cdot \mathbf{d} = \sum_{i=1,N} \sum_{j=1,N} \mathbf{l}_i \cdot \mathbf{l}_j \equiv Nl^2 + l^2 \sum_{i \neq j} \cos \theta_{ij} \quad , \quad (27)$$

if all the steps are of equal length $|\mathbf{l}_i| = l$. Here $\cos \theta_{ij} = \mathbf{l}_i \cdot \mathbf{l}_j / l^2$ is the angle between two vector steps. The sum over the angle cosines tends to zero as $N \rightarrow \infty$, essentially an integral over angles of the cosine. Hence it follows that the number N of steps to the surface and the optical depth $\tau_{\lambda} = d/l$ satisfy

$$d = l\sqrt{N} \quad , \quad N = \tau_{\lambda}^2 \quad . \quad (28)$$

- Diffusion is an inefficient means of radiative transfer; convection plays an important role in transporting nuclear energy out of the solar interior.
- From the phenomenon of **limb darkening**, namely that the solar extremities appear darker than the center to an observer, one can deduce that the temperature declines with distance from the center of the sun.

We can see typically about $\tau_{\lambda} = 2/3$ deep into the sun. When this is line of sight is tangential to the solar surface, i.e. at the solar **limb**, we can only probe into comparatively shallow layers.

* The darker appearance implies a cooler temperature, and therefore *we infer that there is a negative temperature gradient with photospheric radius.*

8 Structure of Spectral Lines

The shape of a spectral line contains a wealth of information. It consists of a *core* and two *wings*. If F_c is the surrounding continuum flux level, then the quantity $1 - F_\lambda/F_c$ is referred to as the **depth** of the line. The effective strength of the line is called the **equivalent width** W of the line:

**C & O,
Sec. 9.5**

$$W = \int \frac{F_c - F_\lambda}{F_c} d\lambda \quad . \quad (29)$$

This is to be distinguished from the **full width at half-maximum** (FWHM), which is the width where it is at least half as deep as the maximum depth, i.e. side-to-side width, denoted by $(\Delta\lambda)_{1/2}$.

Plot: Typical Spectral Line Shape

There are three main mechanisms for broadening spectral lines; natural broadening, Doppler broadening and pressure broadening.

- **Natural broadening:** spectral lines are not infinitely sharp due to the quantum nature of energy states. Heisenberg's uncertainty principle yields an energy uncertainty $\Delta E \approx \hbar/\Delta t$ in an atomic state that lives for time Δt . *Energy levels are fuzzy.*

Hence, if an atomic electron transitions between states $i \rightarrow f$, which have respective decay times of Δt_i and Δt_f , then the natural width of the line is

$$\Delta\lambda \approx \frac{\lambda^2}{2\pi c} \left(\frac{1}{\Delta t_i} + \frac{1}{\Delta t_f} \right) \quad , \quad (30)$$

which can be derived using $dE/d\lambda \propto \lambda^{-2}$.

* e.g. The lifetime of the first and second excited states of hydrogen is less than about $\Delta t \sim 10^{-8}$ sec. Hence the natural broadening of the H_α Balmer line at $\lambda = 6563 \text{ \AA}$ is around $\Delta\lambda \approx 4.57 \times 10^{-4} \text{ \AA}$.

Spectral Line Equivalent Width

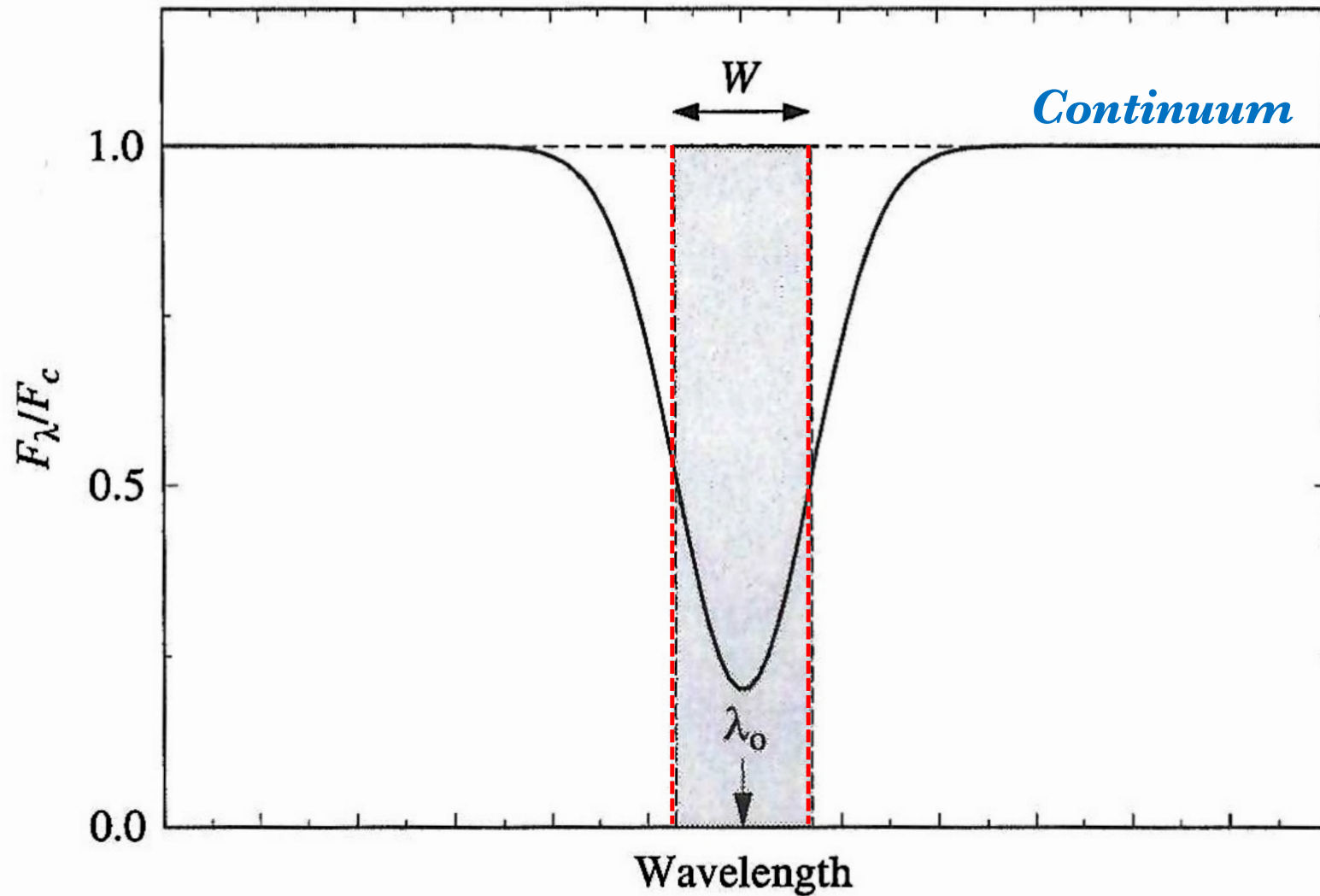


Figure 9.18 of Carroll & Ostlie, *Modern Astrophysics*.

- Let us consider a classical model for natural line widths. We assume that electrons are accelerating in their atomic orbitals, and therefore must radiate. The Larmor power (discussed in Thomson scattering) for an acceleration $\dot{\mathbf{v}}$ can be used to identify a radiative lifetime:

$$P = \frac{2}{3} \frac{e^2 |\dot{\mathbf{v}}|^2}{c^3} \Rightarrow \tau_{\text{rad}} \sim \frac{m_e |\mathbf{v}|^2}{2P} \sim \frac{3m_e c^3}{2e^2} \left(\frac{|\mathbf{v}|}{|\dot{\mathbf{v}}|} \right)^2 . \quad (31)$$

In a classical orbit decay, the lifetime will roughly satisfy $\tau_{\text{rad}} |\dot{\mathbf{v}}| \sim |\mathbf{v}|$. This then establishes the classical radiation timescale for an electron $\tau_{\text{rad}} \rightarrow \tau_{\text{cl}}$:

$$\tau_{\text{cl}} = \frac{2e^2}{3m_e c^3} \equiv \frac{2r_0}{3c} \approx 6.27 \times 10^{-24} \text{ sec}^{-1} . \quad (32)$$

This clearly indicates that purely classical atoms are radiatively unstable.

- Consider now a 1D simple harmonic oscillator with a natural oscillation frequency $\omega_0 \equiv 2\pi\nu_0$. We will incorporate the small damping influence of radiation reaction, assuming that $\omega_0 \tau_{\text{cl}} \ll 1$. Distilling the equation of motion to one dimension, the x -direction, radiation reaction generates

$$\underbrace{-\tau_{\text{cl}} \frac{d\ddot{x}}{dt}} + \ddot{x} + \omega_0^2 x = 0 . \quad (33)$$

For small damping, the underbraced term (not derived here) can be neglected, and the solution is approximately sinusoidal, with $\ddot{x} \approx -\omega_0^2 x$. In general, substituting a trial solution $x \propto e^{\alpha t}$, we have

$$\alpha = \pm i\omega_0 - \frac{\omega_0^2 \tau_{\text{cl}}}{2} + O(\tau^2) . \quad (34)$$

Due to the presence of negative real components in the exponential, this modified SHO solution is a weakly **damped oscillator**:

$$x(t) = x_0 e^{-\Gamma t/2} \cos \omega_0 t = \frac{x_0}{2} \left\{ e^{-\Gamma t/2 + i\omega_0 t} + e^{-\Gamma t/2 - i\omega_0 t} \right\} , \quad (35)$$

where the **damping width** or inverse decay time is

$$\Gamma = \omega_0^2 \tau_{\text{cl}} = \frac{2}{3} \frac{r_0}{c} \omega_0^2 . \quad (36)$$

Lang,
pp. 200–1

To derive the power spectrum of the radiation, we form the Fourier transform of the acceleration:

$$\hat{a}(\omega) = \frac{1}{2\pi} \int_0^\infty \ddot{x}(t) e^{i\omega t} dt = \frac{ix_0}{4\pi} \left\{ \frac{\omega^2}{\omega + \omega_0 + i\Gamma/2} + \frac{\omega^2}{\omega - \omega_0 + i\Gamma/2} \right\}, \quad (37)$$

where the oscillator is switched on only at times $t \geq 0$. This transform becomes very large (**resonant**) near $\omega = \pm\omega_0$, producing a spectral line. Since we can restrict considerations to $\omega > 0$, we can neglect the first term, so that the Fourier power in Larmor formalism is encapsulated in

$$P(\omega) = \frac{2}{3} \frac{e^2 |\hat{a}(\omega)|^2}{c^3} \propto \left(\frac{x_0 \omega_0}{4\pi} \right)^2 \frac{\omega_0^2}{(\omega - \omega_0)^2 + (\Gamma/2)^2}. \quad (38)$$

This is a **Lorentz profile** for the resonance, and is the characteristic form of line profiles when damping/decay of states is involved.

- Since $\Delta\omega = \Gamma$, the FWHM in wavelength for the line is

$$\Delta\lambda = 2\pi c \frac{\Delta\omega}{\omega_0^2} \approx 2\pi c \tau_{\text{cl}} = \frac{4\pi}{3} r_0 \approx 1.18 \times 10^{-12} \text{ cm}, \quad (39)$$

so that the **natural line width** in this classical electromagnetic description of a radiating oscillator model for atoms is intrinsically small.

If, instead, we insert the Bohr model value $\omega_0 = \Delta E/\hbar$ into Eq. (36), we do not actually change the value of $\Delta\lambda$. This achromatic result for natural line wavelength widths is a failing of the classical picture. However, using the fact that $\Delta E \sim e^2/(2a_0)$ so that $\omega_0 \sim \alpha_f c/(2a_0) = \alpha_f^3 c/(2r_0)$, with $a_0 = r_0/\alpha_f^2$, we do discern that the semi-classical lifetime of excited atomic states is

$$\tau_{\text{scl}} \sim \frac{2\pi}{\Gamma} = \frac{3\pi}{\omega_0^2} \frac{c}{r_0} = \frac{12\pi}{\alpha_f^6} \frac{r_0}{c} = \frac{18\pi}{\alpha_f^6} \tau_{\text{cl}} \approx 2.35 \times 10^{-9} \text{ sec}^{-1}, \quad (40)$$

orders of magnitude larger than the purely classical result. Yet, still the atom is radiatively unstable!